8. DEGENERATE FERMION SYSTEMS

(Book Chapters 12 and 13)

GOAL: To discuss Fermion systems in the low-temperature regime, where $kT << E_F$, ie., the degenerate fermion case. We say a little bit about the real world, in systems including metals, semiconductors, neutron stars and white dwarfs, and 3He liquid.

In this section we go into more detail on the behavior of a set of fermions – but we specialize now to the case where the temperature T is low, so that $k_BT << E_F$, so that the deviation in the distribution function from the simple T = 0 case is concentrated in a small energy region around E_F, of width ~ $O(k_BT)$.

Just as in the case of a set of bosons, we start by looking at the case where there are no interactions, ie., at the idea Fermi gas. We will look at how one can write expressions for the various thermodynamic quantities in this case in the form of simple analytic expansions around the $T = 0$ case. We will then move on to discuss what happens when there are interactions between the fermions. Just as with bosons, this does change things. However the changes are somewhat different from those in bosons (where the main change was in the low-energy excitations, replacing free particle excitations with phonons, and thereby allowing superfluidity).

In the case of fermions one also gets phonons – but otherwise the low-energy excitations are not radically changed, and this allowed Landau in 1956-59 to give a description of the low-T properties of a set of interacting fermions in terms of his "Fermi liquid" theory, which I will say a little about. The quasiparticles in a Fermi liquid turn out to be related in a fairly direct way to the free particle excitations of the Fermi gas. However there was one key thing that Landau missed, and which was properly discussed first by Bardeen, Cooper and Schrieffer in the "BCS theory" of superconductivity. As already noted previously, attractive interactions in a Fermi liquid can cause "Cooper pairing", in which pairs of fermions undergo Bose condensation. I will also say a little about this.

8(a) <u>Degenerate Fermi Gas</u>

Recall that we found general results for the Fermi gas in Chapter 6; in section 6(a) we derived a result (eqtn. (6.11)) for the particle number N which implicitly gave the behavior of the chemical potential as a function of temperature T. In section 6(c) this equation was solved numerically, so we know roughly how $\mu(T)$ behaves. What we would like to do now is go to the degenerate regime, where we can actually derive analytic results.

(i) <u>Sommerfeld Expansion:</u> The degenerate regime is, by definition, the regime for which $K_BT << E_F$. Since in this case the Fermi function only varies in an energy region of width ~ $O(k_BT)$ around the Fermi energy, this means that everything is determined by this small region. In particular, the expression we have for the particle number, given by eqtn (6.11), viz., by

$$N = V\int_0^\infty f(E)g(E)dE \tag{1}$$

which we use to find $\mu(T)$, can be evaluated in the degenerate regime by focusing only on the narrow energy regime within k_BT of the Fermi energy.

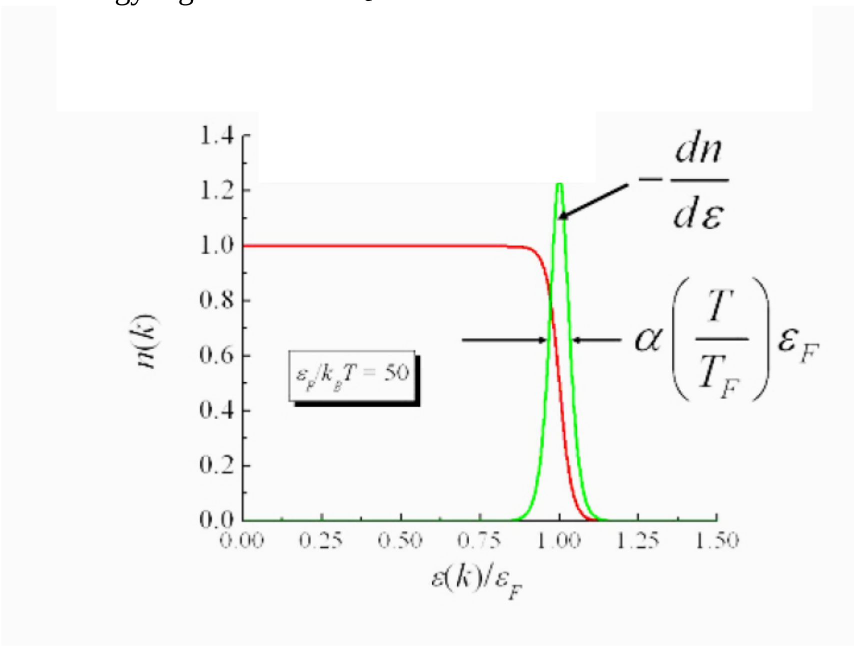
The way to do this is of course by integrating by parts. Suppose we have to do some integral of form

$$I = \int_0^\infty H(E)f(E)dE \tag{2}$$

where f (E) is the Fermi function, and we are in the degenerate regime. We then integrate this by parts to get

$$I = \int_0^\infty H(E)f(E)dE = \int_0^\infty h'(E)f(E)dE = \left[f(E)h(E)\right]_0^\infty - \int_0^\infty f'(E)h(E)dE = -\int_0^\infty f'(E)h(E)dE \tag{3}$$

where we have written the integral of $H(\varepsilon)$ as $h(E) \equiv \int_0^E H(\varepsilon)d\varepsilon$, so that $h'(E) = H(E)$ and $h(0) = 0$; it then follows that $\left[f(E)h(E)\right]_0^\infty = 0$ since both $f(\infty) = h(0) = 0$, which gives us the final result in (3). We have now isolated that part of f (E) which actually varies on energy, for the derivative f '(E) is very sharply peaked around E_F, in a region of width ~ O($2k_BT$), and decays rapidly to zero elsewhere – it acts as a 'filter function', picking out the energy regime around E_F.



Derivative dn/dε of the Fermi-Dirac function, in green. It has a width α roughly equal to $6k_BT$. It is shown here for k_BT/E_F = 0.02

We can see this clearly if we plot the negative of the derivative of the Fermi function, as shown. Note that the width of this Fermi-Dirac filter function" is actually $\alpha \sim 6k_BT$, which is larger than one might have guessed.

Now, armed with these insights, we can do a Taylor-MacLaurin expansion of the integral in (3). To do this we write

$$h(E) = h(\mu) + (E-\mu)h'(\mu) + \frac{1}{2}(E-\mu)^2 h''(\mu) + ... \tag{4}$$

and insert this into eqtn. (3), to get

$$I = h(\mu)I_0 + h'(\mu)I_1 + h''(\mu)I_2 + \dots \tag{5}$$

where :

$$I_0 = -\int_0^\infty f' dE = 1 \tag{6}$$

$$I_1 = -\int_0^\infty (E-\mu) f' dE \approx -\int_{-\infty}^\infty (E-\mu) f' dE = 0 \tag{7}$$

since $f'(E)$ is an even function of $E-\mu$; and finally

$$I_2 = -\frac{1}{2}\int_0^\infty (E-\mu)^2 f' dE \approx -\frac{1}{2}\int_{-\infty}^\infty (E-\mu)^2 f' dE = \frac{1}{2}(k_B T)^2 \int_{-\infty}^\infty \frac{x^2 e^x}{\left(1+e^x\right)^2} dx = \frac{\pi^2}{6}\left(k_B T\right)^2 \tag{8}$$

so that the original integral becomes

$$I \approx h(\mu) + \frac{\pi^2}{6}\left(k_B T\right)^2 h''(\mu) \quad + \text{ etc.} \tag{9}$$

We can now use this kind of expansion to do integrals around the Fermi surface; this expansion is called a Sommerfeld expansion, since Sommerfeld first used it shortly after the discovery of QM, to look at degenerate Fermi gases.

(ii) **Chemical Potential**: Let's first use the Sommerfeld expansion to extract the low T behavior of the chemical potential. The integral in (1) is of the kind in (2), with

$$h(E) = \int_0^E g(\varepsilon) d\varepsilon = \frac{2}{3} c E^{3/2} \tag{10}$$

where the constant $c = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$. Since $h'(E) = cE^{1/2} = g(E)$ and $h''(E) = \frac{1}{2}cE^{-1/2} = \frac{1}{2}\frac{g(E)}{E}$, we then find from (9) that

$$\frac{N}{V} = \int_0^\infty f(E)g(E)dE \approx \int_0^\mu g(\varepsilon)d\varepsilon + \frac{\pi^2}{12}\left(k_B T\right)^2 \frac{g(\mu)}{\mu} \tag{11}$$

Now, since

$$\int_o^{E_F} g(E)dE = \int_0^\mu g(E)dE + \frac{\pi^2}{12}\left(k_B T\right)^2 \frac{g(\mu)}{\mu} \tag{12}$$

and since
$$\int_{\mu}^{E_F} g(E)dE = \frac{\pi^2}{12}(k_B T)^2 \frac{g(\mu)}{\mu} \tag{13}$$

and finally since μ is very close to E_F at low temperatures, and $g(E)$ varies slowly with energy, we can then write with accuracy up to order $(k_BT)^2$, that
$$(E_F - \mu)g(E_F) = \frac{\pi^2}{12}(k_B T)^2 \frac{g(E_F)}{E_F} \tag{14}$$

Thus we finally get, up to $\sim O(k_BT)^2$, the result we want for $\mu(T)$, viz.,

$$\mu = E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right] \quad + \quad O(k_BT)^4 \tag{15}$$

so that the chemical potential for a Fermi gas in 3 dimensions falls off initially like T^2.

(iii) **<u>Energy & Heat Capacity</u>**: We can use the Sommerfeld expansion to calculate any low-T property of the Fermi gas. Consider, eg., the energy, given by
$$U = V\int_0^{\infty} Eg(E)f(E)dE \tag{16}$$
Which has the same form as before (see eqtn. (2); we define:

$$h(E) \equiv \int_0^E \varepsilon g(\varepsilon)d\varepsilon = c\int_0^E \varepsilon^{3/2}d\varepsilon = \frac{2}{5}cE^{5/2} \tag{17}$$
so that
$$h'(E) = cE^{3/2} \qquad \text{and} \qquad h''(E) = \frac{3}{2}cE^{1/2} \tag{18}$$
And the energy is then given by the expansion

$$\frac{U}{V} \approx h(\mu) + \frac{\pi^2}{6}(k_B T)^2 h''(\mu) = \frac{2}{5}c\mu^{5/2} + \frac{3}{2}c\mu^{1/2}\frac{\pi^2}{6}(k_B T)^2 = \frac{2}{5}c\mu^{5/2} + \frac{3\pi^2}{12}c\mu^{1/2}(k_B T)^2 \tag{19}$$

Using our approximate expression in (15) for $\mu(T)$, and the expansion $(1-x)^{5/2} \approx 1 - \frac{5}{2}x$, we can approximate the first term in (19) by
$$\mu^{5/2} = E_F^{5/2}\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right] \tag{20}$$

In the second term $\mu^{1/2} \approx E_F^{\ 1/2}\left[1-\frac{1}{2}\frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2\right]$ but we can drop the T^2 correction since this leads to a term ~ $(T/T_F)^4$, which can be neglected. Collecting terms in (19) we then have

$$
\begin{aligned}
\frac{U}{V} &= \frac{2}{5}c\mu^{5/2} + \frac{3\pi^2}{12}c\mu^{1/2}(k_BT)^2 \\
&\approx \frac{2}{5}cE_F^{\ 5/2} - c\frac{\pi^2}{12}(k_BT)^2E_F^{1/2} + \frac{3\pi^2}{12}c(k_BT)^2E_F^{1/2} \\
&\approx \frac{2}{5}cE_F^{\ 5/2} + \frac{\pi^2}{6}c(k_BT)^2E_F^{1/2} \\
U &= \frac{2}{5}cVE_F^{\ 5/2} + \frac{\pi^2}{6}cV(k_BT)^2E_F^{1/2}
\end{aligned}
\tag{21}
$$

Let's write this result in terms of the particle number N, using $N = \frac{2}{3}VcE_F^{3/2}$. Then we have

$$U = \frac{3}{5}Nk_BT_F + \frac{\pi^2}{4}N\frac{(k_BT)^2}{k_BT_F} \tag{22}$$

where the first term is the temperature independent term at T=0. The heat capacity at constant volume is then given, to lowest order in T, by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2}{2}Nk_B\frac{T}{T_F} = \gamma T \tag{23}$$

The physical interpretation of this result is illuminating. We have already seen that the specific heat is, roughly speaking, a measure of "how many degrees of freedom are active" in the system at some given temperature. Recall, eg., the case of a two-level system (TLS) which we discussed in chapter 4 (see eq. (4.35), and the graph of this function shown there); the specific heat shows a maximum (the Schottky anomaly) at a temperature of order the TLS splitting. This is because around that temperature, excitations are populating both states roughly equally, and a change in T can cause a large shift in occupation between the states.

If we now consider the Fermi gas, it is clear that the number of "active" degrees of freedom appearing in the result for C_V is proportional to T/T_F. However this is exactly what we would expect – the "Fermi window" function shown earlier tells us that only fermions in a region of width ~ O(T) near the Fermi energy can move from one state to another. This number of states is of course very different from what prevails in the opposite high-T limit, where all the degrees of freedom are active – we then just get the classical ideal gas result for $T >> T_F$, where

$$U = \frac{3}{2}Nk_BT \qquad \text{(classical)} \tag{24}$$

and

$$C_V = \frac{3}{2}Nk_B \qquad \text{(classical)} \tag{25}$$

Thus in the degenerate case the heat capacity is reduced, as compared to the classical case, by the factor $\pi^2 T/3T_F$. This reduction of the specific heat and of the number of active states is sometimes called "Pauli blocking." Sommerfeld was the first to derive this result in 1928.

(iv) <u>Pressure in degenerate Fermi Gas</u>: We are familiar with the idea that as one goes to zero temperature, the pressure in a gas also goes to zero – simply because the particles stop moving. Thus, for an ideal gas we recall that the pressure is

$$pV = Nk_BT \qquad (26)$$

with corresponding internal energy

$$U = \frac{3}{2}Nk_BT \qquad (27)$$

so that both $\to 0$ as $T \to 0$ with a constant ratio:

$$\frac{pV}{U} = \frac{2}{3} \qquad (28)$$

These results are consistent with the fact that in an ideal gas, the energy comes in the form of the kinetic energy of the particles, and since the temperature is simply a manifestation of this energy, U must be proportional to T, and goes to zero with T. However, as we have already seen, in a degenerate Fermi gas the energy is non-zero at $T=0$ (compare eqtn. (22)); indeed we have

$$U(T=0) = U_0 = \frac{3}{5}NE_F = \frac{3}{5}N\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \qquad (29)$$

We can determine the zero temperature pressure in the Fermi gas directly from this. Recall the pressure is given from the free energy by

$$p = -\left(\frac{\partial F}{\partial V}\right)_T \qquad (30)$$

and since we are at $T=0$, the free energy $F = U - TS$ and the internal energy U are the same, ie, $F(T=0) = F_0 = U_0$, and so the zero T pressure is

$$p_0 = -\left(\frac{\partial U_0}{\partial V}\right)_T = \frac{2}{3V}U_0 = \frac{2}{3V}\frac{3}{5}NE_F = \frac{2}{5}\frac{N}{V}E_F \qquad (31)$$

the pressure and internal energy can be very large but the ratio

$$\frac{p_0V}{U_0} = \frac{2}{3} \qquad (32)$$

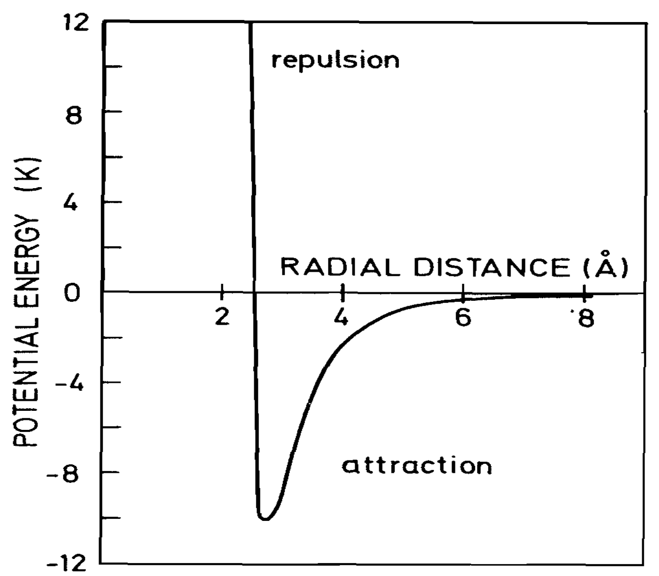
is the same as in the ideal gas (cf eqtn. (28)). .

8(b) Examples of Degenerate Fermi Systems

The Fermi energy $E_F = k_B T_F$ varies enormously in Nature, depending on both the mass of the particles and, more importantly, their density. The most important examples of degenerate Fermi systems on earth are metals and semiconductors – these are important at least to us because they are the essential ingredients of much of modern technology. Note that I am assuming, when I say this, that we deal with "mobile" fermions – the fermions in insulators are not mobile and have no Fermi surface.

(i) Normal liquid 3He: In real degenerate Fermi systems, the interparticle interactions are usually quite strong. The best known and most thoroughly studied case is that of liquid ^3He, which like its Bose counterpart ^4He has acted as something of a "Rosetta stone" for tests of the theory – this is because both of these systems are physically very simple, and so the theory can be tested in a very unambiguous way. The results are interesting.

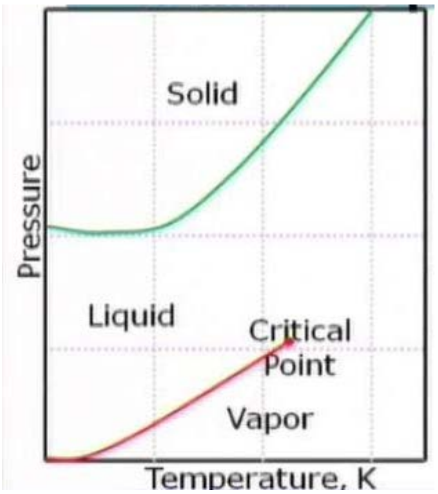
It is useful to first look at the energetics of the system. The most revealing way to do this is to look first at the interatomic interaction potential between two atoms of ^3He; this is shown below. Note that this graph only shows the potential energy, but the total ground state energy also includes the zero point kinetic energy, which adds a term proportional to $1/r^2$, where r is the separation between the atoms – this makes the potential well a lot more shallow.

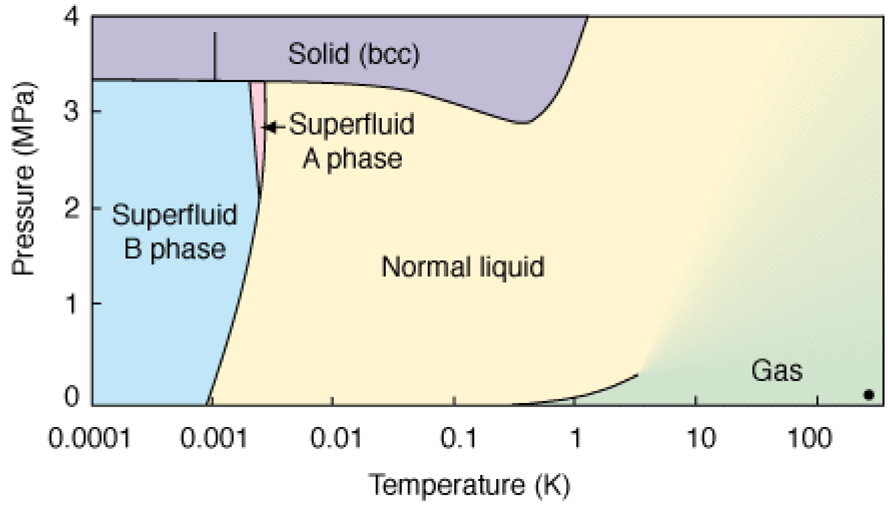


Potential energy between two 3He atoms

The form of this potential comes from 2 sources. The attractive part is from van der Waals interactions – this is just the name we give to the force generated when the electrons on one atom polarize those on the other, repelling them, so that the electrons on each atom are pushed somewhat to the side facing away from the other atom. The net result is that each atom now acts as a weak dipole, and we get an interaction $V(r) \sim 1/r^6$ between them. On the other hand the short-range repulsion is what is called a "Pauli exchange coupling" – the electrons on the 2 atoms begin to overlap, the exclusion principle forces them into higher energy states. These states are at energies ~ eV, so this repulsion is extremely strong.

The potential well is so shpllow that the system can only liquefy at temperatures of a few Kelvin, just like ^4He. It is then interesting to look at the phase diagram of the system. I show this in 2 different ways below:





The first figure, at left, is a linear plot in pressure P and temperature T; it shows the phase diagram for pressures $0 < P < 60$ atmospheres, and temperatures $0 < T < 3.5$K. The second plot shows the same results but now the temperature is plotted logarithmically. We see from the first graph that apart from the absence of a Bose superfluid phase, the system looks similar to ^4He, in that it only solidifies at high pressure – otherwise it remains liquid down to $T = 0$.

The second graph is more interesting. Remarkably, at a much lower temperature the "normal" liquid goes into a superfluid phase – in fact there are 2 such superfluid phases, called ^3He-A and ^3He-B (in fact, if we add a 3rd axis which refers to the applied magnetic field, there is even a 3rd superfluid phase). There are also 3 different solid phases, which have different spin structures. This brings us to 2 key differences between the ^3He and ^4He systems, viz.,

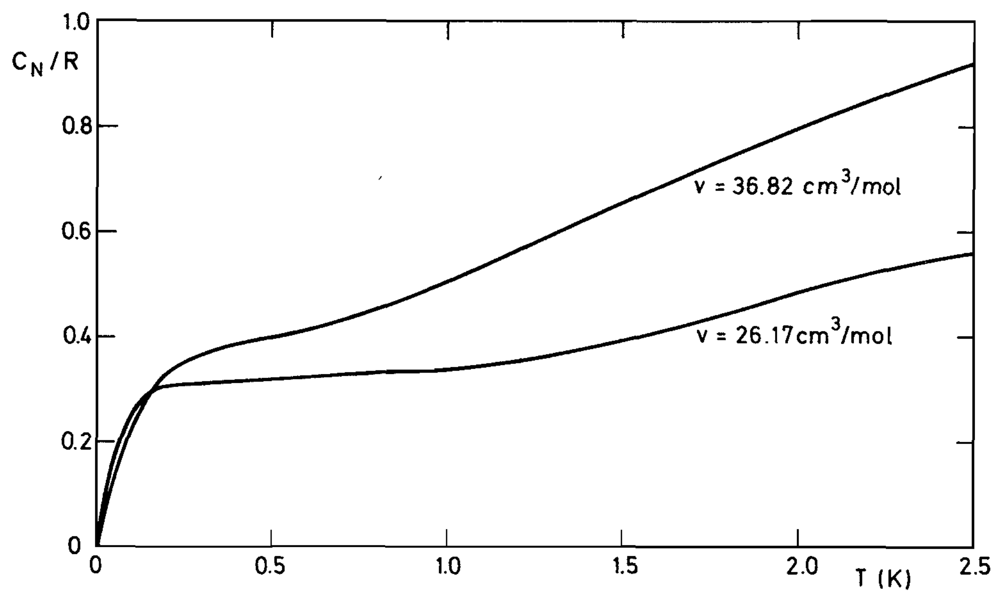
(i) The 3He atoms have a spin. This is not an electronic spin – the 2 electrons in the 3He atom are an s-state, with total spin zero – but rather a nuclear spin (the nucleus has an odd number of nucleons). Thus we get nuclear magnetism, and the various solid and superfluid phase all differ in their spin structures.

(ii) Because the 3He atoms are fermions, we expect to get degenerate fermions at low T, behaving in some way like a Fermi gas when $T << T_F$. Actually, if one looks at the density of the liquid state, we calculate that $T_F \sim 5$K (depending on the pressure). However at very low T we get superfluid phases – this is again because the fermions can form Cooper pairs which Bose condense.

The superfluid phases of 3He are actually extraordinarily interesting. The reason is that the internal spin structure of the Cooper pairs can take various forms, and this is then reflected in the superfluid properties – in essence we have different kinds of "spin superfluid". We can imagine, eg., that the nuclear spins form a spin singlet state, with total spin zero; but they could also form a triplet state, with total spin 1, and various other states as well. The complexity of the superfluid properties then comes because the superflow properties become all mixed up with the magnetic properties.

There is no space in this course to go into all these details – it is simply interesting to note that any Fermi system with an attractive part ot the interaction between the fermions will eventually form Cooper pairs at sufficiently low T, and so the resulting superfluidity is a generic phenomenon amongst fermions. In ^{3}He the transition temperature Tc to the superfluid phases depends on pressure – when P ~ 34 bar (ie., along the solid-liquid phase boundary) we have Tc ~ 2.7 mK (ie, 2.7 x 10^{-3} K), whereas when P ~ 0 bar, Tc ~ 1mK.

Let us now turn to the "normal phase" of the liquid, existing for Tc < T < 2K. We can ask whether it really does behave like a degenerate Fermi gas for $T << T_F$ ~ 2K. To find this out we have to look at experiment. The first obvious thing to do is look at the specific heat – and compare with eqtn (23). The experimental results are as follows:



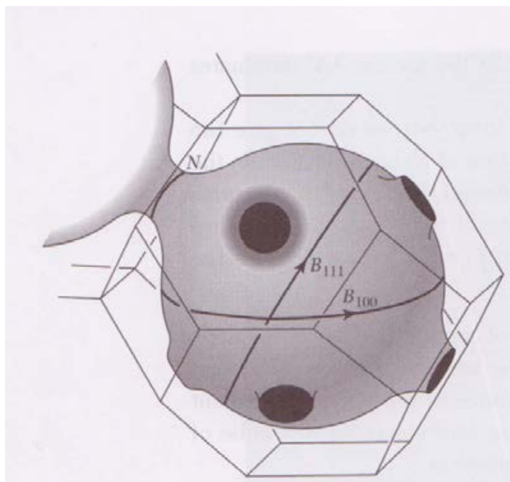
The specific heat of liquid ^{3}He, taken for 2 different molar volumes, corresponding to pressures P = 0 atm and P = 30 atm respectively.

We see that the specific heat is linear in T in liquid ^3He below a temperature ~ 0.2K; however the coefficent of the linear term is actually much larger than that in eqtrn (23). Above this temperature we get a gradual turning over, and eventually come to a new linear regime above about 1K. If we determine the slope in this latterregion, we actually find that this is similar the true Fermi gas behaviour, although the value for T_F we extract form it os stil less than 5K (it is actually more like 1.5K).

These remarkable results were actually predicted in the famous "Fermi liquid theory" of LD Landau (derived in the period 1956-59). The resason for the large enhancement in the slope at low T comes from the interactions between the quasiparticles in the liquid, which also behave as fermions and so are subject to Pauli blocking, but which have a much higher effective mass than the ^3He particles themselves (each quasiparticle involves the collective motion of many particles). To show that the quasiparticles must be fermions, and behave like "renormlized" versions of the original non-interacting fermions, needs quite subtle arguments which were given by Landau.

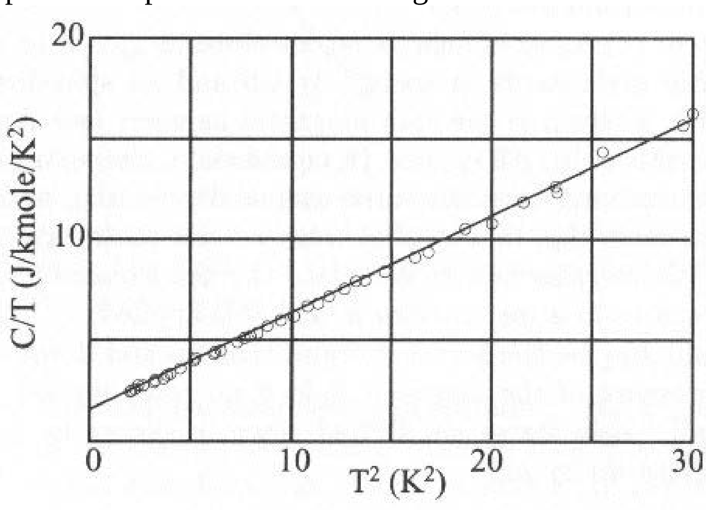
(ii) <u>Conducting Metals</u>: By far the most important example on earth of Fermi-Dirac statistics is that of conducting systems of electrons – what we call "metals". In real metals the surface is not spherical – this is because the electrons are not free, but subject to the potential from the ions in the solid. In a perfect crystal this potential is periodic in space, and so we can actually classify the electron states by their energy and their "quasi-momentum – we do not go into this here. There are also interactions between the electrons, so the electron system essentially behaves as a Landau Fermi liquid in a periodic potential.

The effect of the lattice potential is to distort the shape of the Fermi surface. For metals like the alkali metals (Na, K, etc.) or the noble metals (Au, Ag, etc.) and for simple transition metals like Cu, these distortions are not too large. As an example we show the Fermi surface of Cu. It is interesting to note the presence of 8 "necks" in the [111] directions where the Fermi surface extends into the next "Brillouin zone".



Fermi surface of Cu metal

The detailed discussion of metals is quite complicated – in real metals it is complicated not only by the electron-electron interactions, but also by the interaction between the electrons and the phonons in the solid. However the specific heat at low *T* is still linear in *T*; the phonons give a contribution ~ T^3, as we would expect from a massless bosonic gas, and the linear contribution dominates at low *T*. We can see this in experiments – a typical example is shown in the figure:



Plot of $C_V(T)/T$ as a function of T^2 for *Cu* metal

The plot is designed to separate the phonon and electron contributions – if we plot $C_V(T)/T$ then the electronic term is just the constant γ, and is then the intercept as T goes to zero. We expect the low T specific heat to take the form

$$C_V(T) = \gamma T + \beta T^3 \qquad (33)$$

where $\gamma \sim O(1/T_F)$ is the coefficient of the electronic contribution, and $\beta \sim O(1/\theta_D^3)$ is the coefficeint of the phonon contribution (and θ_D is the Debye temparature). Thus if we plot $C_V(T)/T$ against T^2 instead of T, then since the phonon contribution to $C_V(T)/T$ is proportional to $T^{2,}$ the slope of the graph just gives us the coefficient β.

One last word about the fermion specific heat. Let us notice from eqtn. (23) that the linear term in $C_V(T)$ can also be written, for the free Fermi gas, as

$$\gamma = V\frac{\pi^2}{3}g(E_F)k_B^2 \quad . \qquad (34)$$

Thus γ is a measure of the density of states at the Fermi energy. This is also true for interacting Fermi liquids like liquid ^{3}He; in fact, the increased mas of the quasiparticles id directly reflected in the increased density of states at the Fermi energy, as compared to the gas.

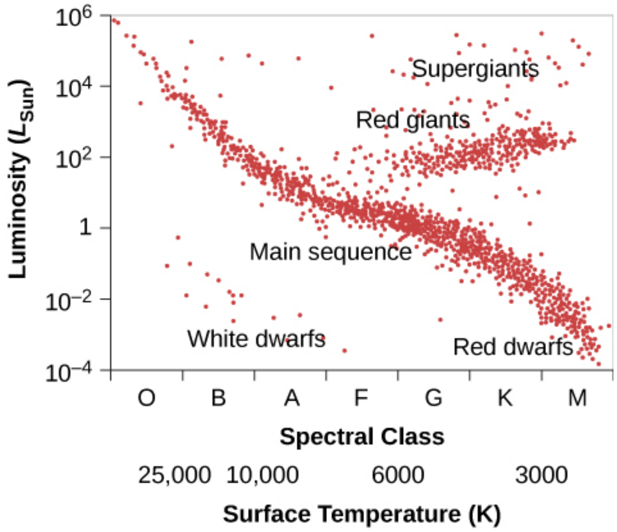
Finally, let us go back to the pressure existing in the electron gas. In a metal such as *Cu* this "Fermi degeneracy pressure" is ~300 kbar, ie., roughly 300,000 times atmospheric pressure. The reason the metal does not explode is of course that this pressure is counterbalanced by the attractive electrostatic force of the positive ions in the metal – the total system is electrically neutral, and the net force is zero. Note however that you still have to do work to get an electron out of the solid – the net effect of all the different forces actoing on the electrons is to leave them sitting in a potential well encompassing the whole solid, out of which the electrons have to climb to escape. The energy required to get out is called the "work function"; it is typically several *eV*.

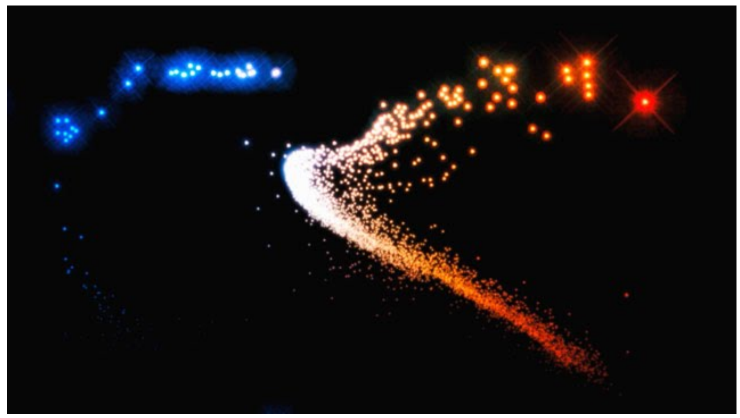
(iii) White Dwarfs and Neutron Stars: We have already met neutron stars in the discussion of superfluidity. However it is important to realize that in almost all fermionic superfluids and superconductors, where the superfluidity is caused by Cooper pairing, the pairing only involves fermions very close to the Fermi energy. Those fermions that combine into Cooper pairs then behave essentially like bosons. At the same time, all the other states in the system behave just like ordinary fermions in a degenerate fermion system.

The net result is that those bulk thermodynamic properties that involve all of the particles in the system will hardly be affected. Good examples of this are the total energy in (29), and the Fermi pressure in (31); these are produced by summing contributions from al the particles in the system. On the other hand those properties (like the specific heat), that involve only the fermions in the "Fermi energy window" produced by the derivative $df(E)/dE$, ie., the thin band of fermions within roughly k_BT of the Fermi energy, will be completely changed by Cooper pairing.

With this in mind we now look in more detail at white dwarfs and neutron stars, this time taking account of the fact that they are basically Fermi liquids like ^3He or metals, but with far higher densities.

Stellar Evolution: To see why white dwarfs and neutron stars exist, we fist need to understand the enormous variety of different stars and what it is that regulates their lives. It was actually realized over a century ago that stars can be classified fairly simply, in a way which reflects their mass, chemical composition, and age. Later, after the discovery of quantum mechanics and general relativity, it became possible to understand the regularities that exist.





LEFT: The Hertzprung-Russell diagram, classifying star according to their temperature and luminosity. RIGHT: a more schematic diagram which shows the colours of the various stellar types

One finds that the vast majority of stars lie on the main sequence – the cooler they are, the more numerous they are. Where they lie on the main sequence depends on their mass and age – cooler type-M dwarfs will have masses roughly between 0.08 - 0.3 solar masses, whereas a type-G dwarf like the sun has a solar mass, and a type-A star like Sirius may have 2-3 solar masses. Hotter type B and O stars can have masses of 5-20 solar masses – in very rate cases the mass can go as high as 40-100 solar masses.

One sees that the higher temperature, more massive stars are far more luminous – this is because stars are black bodies and so obey the Stefan-Boltzmann law (cf. eqtn. (7.36)), according to which the radiative power emission from a black body is proportional to T^4, where T is the its temperature. If the surface area of the black body is A, then clearly the power emission is also proportional to A, so we can say that the total luminosity of a star will go like

$$L = (R/R_{sol})^2 \, (T/T_{sol})^4 \; L_{sol} \qquad (35)$$

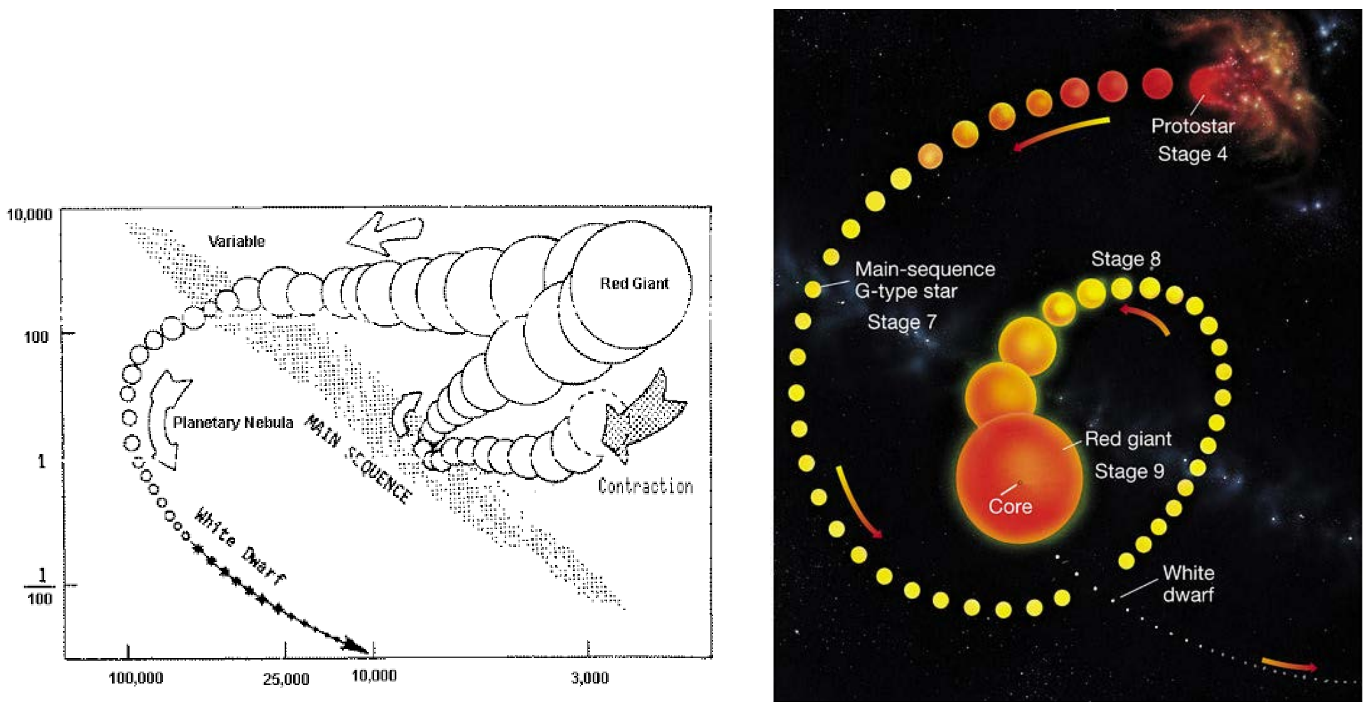
where R is the radius of the star, R_{sol} the solar radius, T the stellar surface temperature and T_{sol} ~ 6000K the surface temperature of the sun, and L_{sol} the solar luminosity.

What is holding these stars up, ie., what stops them from collapsing? The answer is a combination of electron degeneracy pressure, to be discussed below, and the radiation pressure from the thermonuclear fusion – this pressure is, by the Stefan-Boltzmann law, proportional to T^4, and the pressure on a unit element of the star's surface is also inversely proportional to the star's surface area A (because the total radiative force is spread over this entire area). Thus the radiative pressure is proportional to T^4/R^2.

The radius of the stars thus also increases with their luminosity – even though the mass is also increasing, thereby increasing the inward gravitational attraction, the radiative pressure from the vastly increased luminosity increases much more quickly with the mass. This then causes the star to expand (which also cools it because the same radiative power from the central core is going through a larger area). The final radius is then a balance between the outwards radiative pressure and inward gravitational attraction. For stars with mass > roughly 100 solar masses it is however impossible to find equilibrium – such a star will simply blow off matter until its mass is low enough to be stable.

As a consequence of this the sizes of stars vary enormously. The small red dwarfs may have a diameter ~ 0.1 that of the sun, ie., roughly 130,000 km, about the size of Jupiter. A white dwarf may have a diameter roughly that of the earth (ie., 12,500 km). On the other hand a blue supergiant can have a diameter ~ 40 solar diameters (or 55 million km) and a red supergiant may be as large as 2-3,000 solar diameters (ie., up to about 4 billion km), so that if placed at the position of the sun, its surface would be at the orbit of Uranus. In light units, the diameter of a white dwarf is about 0.05 light seconds, whereas a red supergiant can have a diameter of 4 light hours – differing by a factor of ~ 300,000.

Evolution of low mass stars: Now let us see why we have such a peculiar stellar classification. The easiest way to do this is to start with the example of the sun, whose life cycle is shown here:



LEFT: evolution of the sun on the Hertzprung-Russell diagram, form proto-star to white dwarf.
RIGHT: the same evolution, with images spaced 250 million years apart.

Currently the sun is some 5 billion years into its lifespan, with another 6 billion to go before it begins to expand into a red giant. This late expansion, with a large rise (by a factor of 50-100) in luminosity, will come because whereas the core of the sun is currently burning H at a temperature of 14.7 million K, it will eventually start to run out of *H*, and need to start burning *He*, at a higher core *T*; the increase in *T* will, following the Stefan-Boltmann law, lead to a large increase in luminosity (and a lowering of the surface *T*), by the mechanism described above.

The red giant phase passes once the *He* is used up – the sun will not be able to go on to burn heavier nuclei at higher core T because it is insufficiently massive (any further rise in core temperature would cause shedding of the outer regions, because of the radiation pressure, and the gravitational attraction is not enough to stop this – moreover, this shedding would simply decrease the mass further). Thus the sun at this stage runs out of fuel – and it then collapses under its own gravity until it reaches a new state, the white dwarf state, where it is very compact and extremely dense (with a mean density ~ 10^6 g/cm^3). At this point it is no longer radiation pressure that is stopping further collapse (white dwarf stars are hardly radiating), but degeneracy pressure – see below.

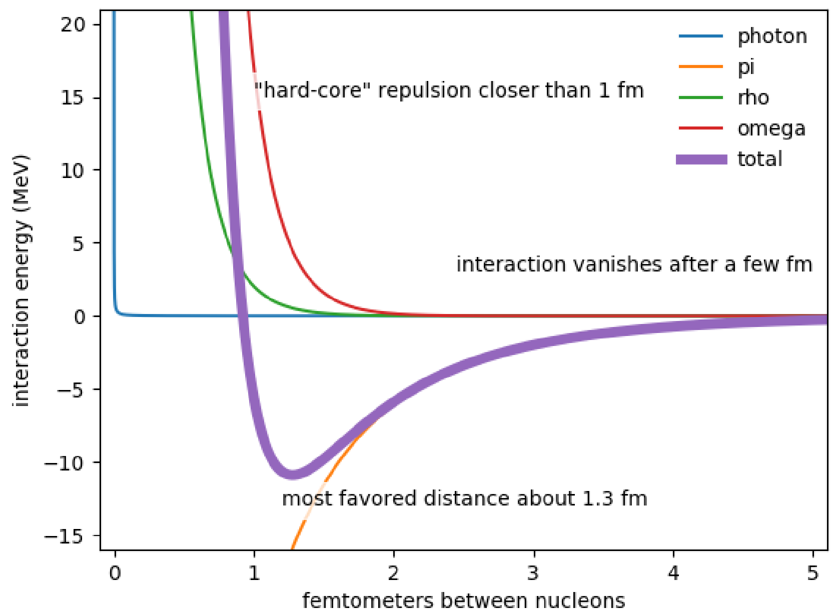
Now all of this tells us how the sun will evolve – but what of a star having a different mass? Consider, eg., a star of initial mass 2-3 suns. Its evolution is very similar, but as it collapses from the initial protostar, it will arrive at a point on the main sequence considerably to the left of the sun, as an A-type dwarf, with surface temperature from 8,000 - 10,000K, and luminosity from 10-100 suns. On the other hand the far more common stars of initial mass around 0.1-0.3 masses will evolve down to the main sequence as M-type dwarfs, much the right of the sun, and with surface temperatures ~ 2,500 - 3,000K and luminosities ranging from 10^{-6} - 10^{-2} suns. Naturally these dim red dwarfs have much longer life expectancies than the sun; their luminosities are so low that it will take many trillions of years for the coolest dwarfs to use up their *H* fuel. But – and this is the key point - all of these stars will eventually evolve into white dwarfs.

Evolution of high mass stars: Much rarer are the initially massive stars, with masses > 8 solar masses. Their life cycle is very different. These massive stars evolve initially into hot blue main sequence stars (spectral type B or even W), and burn very brightly until they use up their *H*. For a star of 50 solar masses this will take only a few million yrs, but during this time it will shine with a luminosity exceeding 100,000 suns. Once the H is exhausted, it will start to burn first *He*, then *C*, then *Si*, and so on until the most stable nucleus is reached, viz., the *Fe* nucleus. Each of these fusion reactions requires an ever higher core burning temperature, and as a result the star balloons out to form a red supergiant of truly enormous size. The only reason that these higher T fusion processes are possible is that the star is sufficiently massive to stay intact, with the aid of its powerful gravitational attraction, even in spite of the great luminosity.

However, all good things come to an end, and once the core is transformed into *Fe* nuclei (with the final burning taking place at a core temperature ~ 3 billion K), fusion can no longer proceed. The final burning processes exhaust themselves incredibly fast at this temperature – the entire *Si* burning cycle takes only a day or so. By this time the core density is approaching 10^{10} g/cc; the core mass is now roughly 1.5 - 2.5 solar masses. It is essentially a very high temperature plasma made up of negatively charged electrons and positively charged *Fe* nuclei.

The ensuing story is both dramatic and takes place incredibly quickly. In the absence of any further possibly thermonuclear fusion processes, the star begins to contract, further raising the temperature, to ~ 8 x 10^9 K. At this point, 2 things begin to happen, viz., (i) the photons in the core at this high T now possess enough energy to ***photodisintegrate*** the *Fe* nuclei, breaking them up into neutrons and ^{4}*He* nuclei, and further breaking the 4He nuclei into neutrons and protons; then (ii) the protons begin to combine with all the free electrons in the core, via the process $p^+ + e^{--} \rightarrow n + \nu_e$ (mediated by the weak interaction).

The result of these 2 processes is that very suddenly, huge amounts of energy are being sucked out of the photon bath to disintegrate the Fe nuclei (all of the energy gained during the fusion of the star during its lifetime is now being paid back, in this extremely endothermic process), that almost as much energy is also being radiated away by the neutrinos, and that the electrons, which had up to this point provided the fermionic degeneracy pressure required to oppose the massive gravitational attraction, begin to disappear. Thus, in the space of seconds or minutes, the core has nothing further to support itself except an electron degeneracy pressure which is rapidly disappearing.



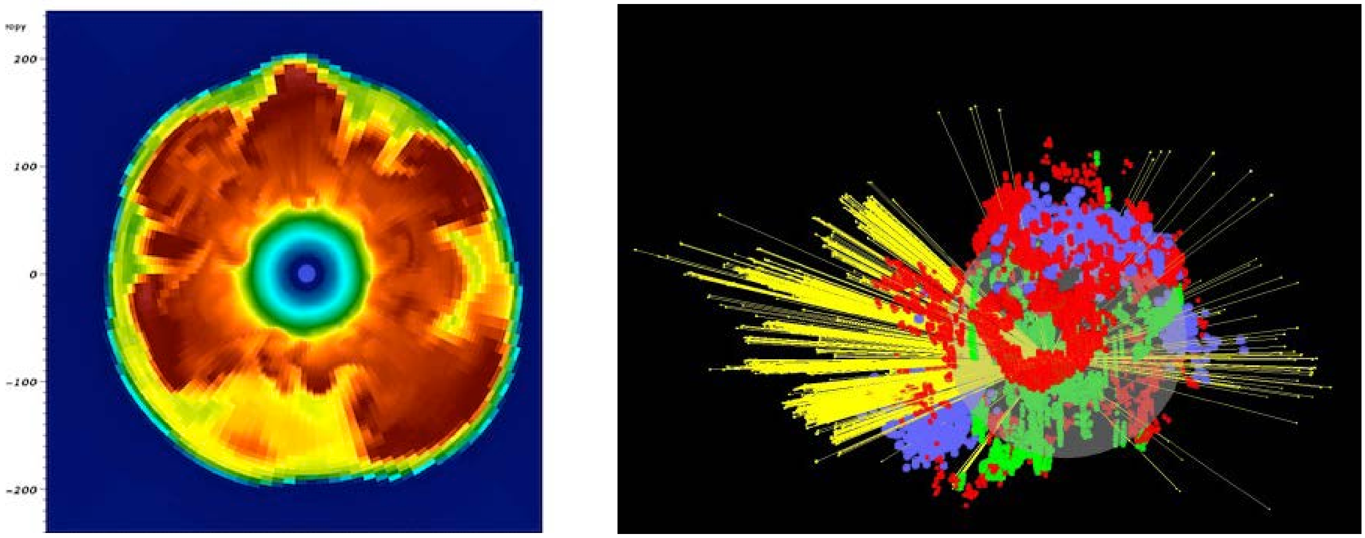
Different contribution to the total inter-nucleonic potential

To understand what happens now we need to know a little about the potential between two nucleons; as we see above it is very strongly repulsive at short range, for distances less than 1 fm = 10^{-15} m.

The reason for the very strong hard-core repulsion is that in spite of the very strong attractive forces coming from exchange of pi mesons, as well as other particles (including, at very short range, gluon exchange), the overlap between the nucleons brings in even stronger repulsive exchange forces, coming from the overlap of the nucleons (and eventually the quarks in the nucleons). These exchange forces come from the Pauli exclusion principle. Notice the scale of energies here – instead of a few K (ie., 10^{-4} eV), we deal with energies ~ 10 MeV (ie., 10^{11} K). Otherwise the potential looks remarkably like that between He atoms at very low T !

In any case, the result of this is that the entire core collapses, very fast, under the massive gravitational attraction – this collapse can take literally only a second or two, until the core reaches the far higher nuclear densities at which everything has been squashed into a nuclear fluid, primarily formed of neutrons – at this point the core may be only 20 km across. Then, equally suddenly, the massive core repulsive forces set in, causing a rebound of the outer core, which then meets the rest of the infalling star. The result is a massive shock wave, which stalls the infall of the rest of the star as it proceeds slowly out through it.

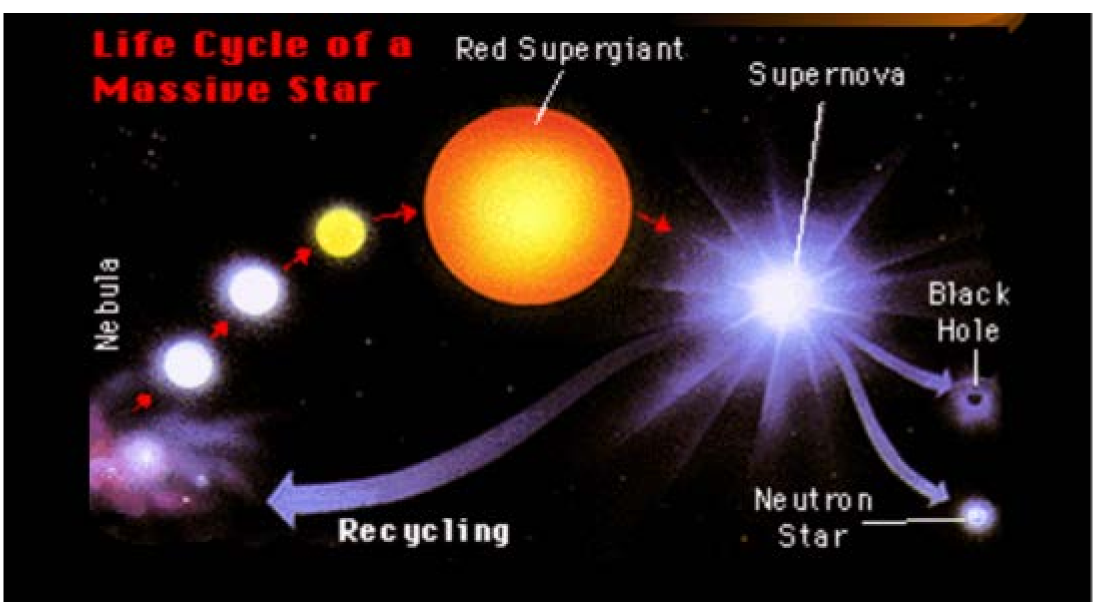
At this point the only signal of what is happening which can reach the outside world is the massive burst of neutrinos (as well as gravitational waves) occurring at the time of the infall and core rebound (these can carry away energy equivalent to a solar mass in seconds). Later, when the shock wave reaches the surface of the infalling star, the rest of the universe sees it – the luminosity of the supernova can reach 10 billion suns, outshining a small galaxy.



LEFT: The core region just after rebound – the figure is 500 km on a side
RIGHT: The core at the moment of rebound – the yellow depicts the neutrino outflux

In reality the processes involved in the supernova are fantastically complex – we have talked up to now as though everything was spherically symmetric, but in reality hydrodynamic instabilities quickly develop, and the whole thing "squirts" jets of matter and radiation in all directions. Some of the most powerful computer simulations ever done have been on these processes – results of this kind are shown above.

After the rebound, what then happens to the inner core? There are 2 possibilities. If the initial mass of the star does not exceed 20-25 solar masses, then the core remnant will settle down as a neutron star – we already gave a brief description of these above. However, if the initial mass of the star exceeds ~25 solar masses, the result is very different – the repulsive forces cannot in principle hold up the star, no matter how strong they are- and it collapses to a black hole. We thus finally arrive at the 2 possible scenarios for the life cycle of a massive star, depicted below.



Life cycle of a star with initial mass > 10 solar masses

The first case, where we get a neutron star, is by far the most common of the two – stars with masses > 25 solar masses are not common. The second scenario, formation of a black hole, occurs because the extra potential energy involved in the short range repulsion of nucleons must also, according the general theory of relativity, act as a source for an extra contribution to the gravitational field generated by these nucleons. For a large enough mass, this extra contribution leads to a "vicious circle" runaway process in which the energy associated with this extra field generates a further contribution to this field, and so on – the result blows up and generates a spacetime singularity.

Although the large mass stars leading to such black holes are rare, they have nevertheless left hundreds of thousands of stellar mass black holes throughout the galaxy.These cannot be seen directly, but they give their presence away if they are part of a binary system in which case they are typically surrounded by an "accretion disc" of material which they have slowly stripped off their companion. As the material in this disc slowly spirals into the black hole, intense X-rays are emitted, signaling the presence of the black hole.

Structure of White Dwarfs: Now that we have seen how thse objects come to be in Nature, we can take a closer look at their structure. In a white dwarf the Fermi pressure of the electrons is much larger than in a metal. The chemical composition of a typical white dwarf is a mixture of fully ionized *C* and *O* nuclei, some *He* nuclei, plus protons and neutrons; this comes along with a very dense gas of electrons, which leaves the system electrically neutral. The core temperature of the star is still roughly 10^7 K, and the surface temperature is typically between 8,000-30,000 K (see the Hertzprung-Russell diagram above). Given the size and mass of the white dwarf, the combination of thermal and radiation pressure from such an object is far less than what would be needed to support the star against gravitational collapse. However, as shown by RH Fowler in 1926, in a very early application of QM, the electron degeneracy pressure can do the job.

We can see this in a simple approximate calculation in which we assume a uniform mass density ρ for the white dwarf, and give it a radius R, and a mass $M = 4\pi R^3 \rho/3$. To examine its stability we employ the standard trick of slightly perturbing it and look to see if it is stable. We look at the 2 contributions to the energy, one gravitational and the other from the degeneracy pressure.

We begin with the gravitational energy. Thus, consider the energy required to move an infinitesimally thin shell at a radius r , and of thickness dr, out to infinity; this is just

$$dE = \frac{G\left(\rho 4\pi r^2 dr\right)\left(\rho \frac{4}{3}\pi r^3\right)}{r} = G\frac{\left(\rho 4\pi\right)^2}{3}\, r^4 dr \qquad (36)$$

It follows that the total energy required to peel off ALL of the mass of a star with radius R, and then move it to infinity one layer at a time, must be

$$E = \int_R^0 dE = G\frac{\left(\rho 4\pi\right)^2}{3}\int_R^0 r^4 dr = G\frac{\left(\rho 4\pi\right)^2}{3}\frac{1}{5}R^5 = \frac{3G}{5R}\left(\rho \frac{4}{3}\pi R^3\right)^2 = \frac{3}{5}\frac{GM^2}{R} \qquad (37)$$

Thus we have deduced that the total gravitational potential energy for a star with uniform density ρ and with radius R is given by

$$U_g = -\frac{3}{5}\frac{GM^2}{R} = -C_g \frac{GM^2}{R} \tag{38}$$

where the constant $C_g = 3/5$ will be somewhat different in a more realistic model in which the density $\rho(r)$ is a function of the radial distance.

Turning now to the degeneracy pressure, we know that this imply comes from the internal energy of the degenerate gas of electrons, given for a non-relativistic Fermi gas, as we have already seen, by

$$U_e = \frac{3}{5}NE_F = \frac{3}{5}N\frac{\hbar^2}{2m_e}\left(3\pi^2\frac{N}{V}\right)^{2/3} = \frac{3^{5/3}\hbar^2\pi^{4/3}}{10m_e}\frac{N^{5/3}}{V^{2/3}} \tag{39}$$

As noted above, the actual composition of white dwarfs is mostly a mixture of C and H nuclei, mixed in protons and neutrons. Let us here solve for a somewhat simplified model in which the hadrons are simply an equal mixture of protons and neutrons (with the number chosen to be equal since light nuclei have about equal numbers of neutrons and protons). Then the star will have a mass $M \approx 2Nm_p$ where m_p is the proton mass.

Using this and the fact that

$$V^{2/3} = \left(\frac{4}{3}\pi\right)^{2/3} R^2 \tag{40}$$

we can rewrite (39) as

$$U_e = C_e \frac{M^{5/3}}{R^2} \tag{41}$$

where

$$C_e = \frac{3^{5/3}\hbar^2\pi^{4/3}}{10m_e(2m_p)^{5/3}\left(\frac{4}{3}\pi\right)^{2/3}} \tag{42}$$

Thus the total energy of the system, given by the sum of the gravitational and electron degeneracy energies, is finally given by:

$$U = U_g + U_e = -C_g G\frac{M^2}{R} + C_e\frac{M^{5/3}}{R^2} \tag{43}$$

Now let us return to the question of stability of the system – this will allow us to determine the radius which minimizes the energy of the system. In order to be stable, we require that

$$\frac{dU}{dR} = C_g G\frac{M^2}{R^2} - 2C_e\frac{M^{5/3}}{R^3} = 0 \tag{44}$$

and thus the equilibrium radius is given by

$$R_{eq} = \frac{2C_e}{C_g G} M^{-1/3} \qquad (45)$$

which is the result we wanted. We have thus shown how the degeneracy pressure can successfully oppose the gravitational attraction, and leave the system stable at the radius given by (45). The key to this result is that the positive electron degeneracy energy rises faster, as we decrease the radius, then the modulus of the negative gravitational self-energy – this is clear from (43). In fact, we see from (45) that the mass is inversely proportional to the volume of the white dwarf – its radius decreases as its mass increases.

In a famous chapter in the history of astrophysics, it was noticed in 1930 by a then 19-yr old Chandrasekhar, on his voyage to the UK from India to start his graduate studies, that this argument no longer works if we take into account special relativity. Notice that the degeneracy energy in (39) increases with the density, and moreover in (45) we saw that the radius decreases as we increase the mass of the star. Thus once the mass of the star becomes large enough the Fermi energy can exceed the rest mass of the electron and one must use the relativistic expression for the Fermi energy.

To solve this in the crossover regime where the Fermi energy is of the same order as the electron rest mass is messy. However in the extreme relativistic limit the equations again become simple – we get

$$E_F = \sqrt{(\hbar c k_F)^2 + (m_e c^2)^2} \approx \hbar c k_F = \hbar c \left(3\pi^2 \frac{N}{V}\right)^{1/3} \qquad (46)$$

which now leads to an internal energy for the star given by

$$U_e = C_2 \frac{M^{4/3}}{R} \qquad (47)$$

Chanrasekhar's striking result was then that in this case there is no stable radius for the system, since now both the energy due to electron Fermi pressure and the gravitational self-energy have the same 1/R dependence of radius and we have instead of (44) the result

$$\frac{dU}{dR} = \frac{1}{R^2}(C_g G M^2 - C_2 M^{4/3}) \qquad (48)$$

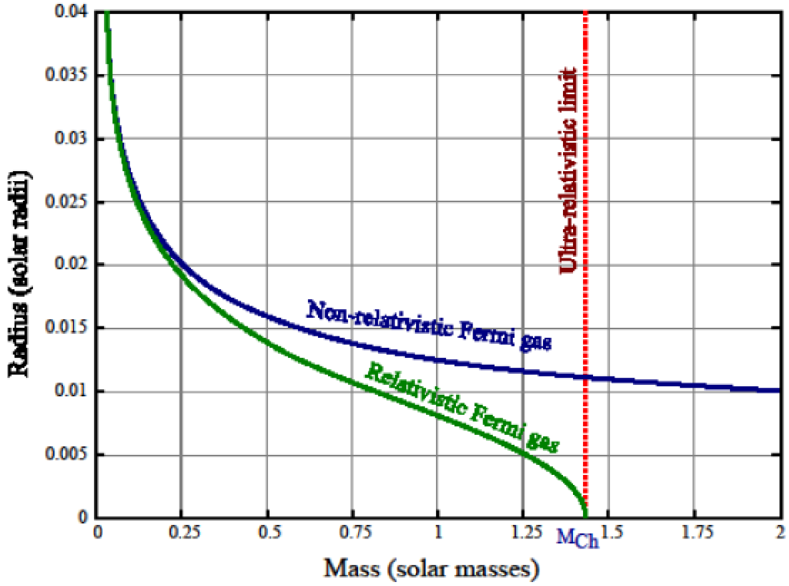
which tells us that the only stable radius is zero. We are thus led to the striking conclusion that there is a critical mass, the "Chandrasekhar mass", for the system, given by

$$M_{Ch} \equiv \left(\frac{C_2}{C_g G}\right)^{3/2} \qquad (49)$$

for which we first find that $\frac{dU}{dR} = 0$.

More realistic calculations, which include a radial dependence to the density, find $M_{Ch} \approx 1.4 M_{sun}$. If $M > M_{Ch}$ the internal energy continues to decrease with decreasing size, leading to a collapse of the star.

Thus the relativistic calculation gives a very different result from the non-relativistic one. The difference between the two is clearly seen in the figure below:



At first this result did not worry astrophysicists too much – they realized that when a white dwarf of mass greater than this Chandrasekhar mass reached the density of a neutron star, the collapse would be halted by powerful nuclear repulsive forces, as we have just seen. The first to realize is, and to discuss the idea of a neutron star, was the then 22 yr old LD Landau. He realized that at a nuclear density of about 10^{15} g/cm^3 the electron Fermi energy is so high that all the protons in the nuclei undergo inverse beta decay, and the star becomes a dense gas of neutrons, to which he gave the name *neutron star*. We then have both electrons and neutrons – but in the core of the star it is neutrons that predominate.

We see that it is then the Fermi pressure of the *neutrons*, plus their very strong repulsive interactions, that stabilize the radius of a neutron star against further collapse. One can work out the same sort of theory as above to treat this – to do a proper job is complicated, but the results are what we saw in the last chapter.

However this was not the end of the story. Continuing his work, Chandrasekhar went on apply ideas from General Relativity (as opposed to special relativity) to the problem. This led him to the conclusion that, as discussed above, nothing could then stop the further collapse of the system once the mass exceeded a somewhat higher limit.As noted above, the key to this result is that the strong spacetime curvature caused by the very dense mass acts as a source for further spactime curvature, and the result is a "runaway" process in which a spacetime singularity is generated. In 1935 the very well-known astrophysicist Eddington publicly ridiculed this idea at a meeting in front of Chandrasekhar. This led Chandrasekhar to leave the UK, and find his future in the US (eventually at the Univ of Chicago); this was neither the first nor the last time that the British establishment succeeded in alienating an outsider. It was to the loss of the UK - the work of Chandrasekhar had a massive influence on 20th century astrophysics, more than that of any other single astrophysicist, as well as on many other areas of physics.