7. REAL BOSONS - from ^{4}He to PHOTONS

GOAL: To discuss non-interacting Bose gases, both in the massive case where the chemical potential is non-zero except at low T, and for massless bosons like photons, where it is always zero and photons can be created in arbitrary numbers.

In this section we go into more detail on the behavior of a set of bosons. The properties can be very surprising. If the bosons are massive, then we cannot create or destroy them willy-nilly; the only way to change the particle number N is by having the particles move in and out of a surrounding reservoir or bath. However we shall see that the chemical potential still can go to zero at finite temperature – the consequence of this is rather drastic since it then means that there is no cost to moving particles in from the bath. We will see that what then happens is what is called "Bose-Einstein condensation" or BEC (often just called Bose condensation).

On the other hand in the case of bosons that are already massless – like photons – the chemical potential is always zero. So photons can be multiplied or destroyed at will, and indeed classical EM fields can contain enormous numbers of photons in the same state. When the photons in the same state can also be made "phase-coherent", so that they all have the same phase, we then get coherent light, as in a laser or maser. Under some circumstances we can get the same sort of behavior with other massless bosons like phonons or magnons.

We begin by going back to the massive case, already discussed in the last chapter – we then move on to discuss massless bosons like photons.

7(a) Massive Bosons & Bose Condensation

Suppose we have a set of bosons of rest mass m. Then in the ordinary course of events a set of N such bosons in a closed contained will have their particle number conserved – the only way to change this number would be if the temperature was high enough to disintegrate the bosons (in which case our original description would be invalid).

We already treated this system under the circumstances that the mean occupancy of the boson states was small, ie, that $f(E, \beta) << 1$. However we also saw that this could not be the case at all energies when the temperature T was low. Recall that the Bose distribution function diverges when $E\text{-}\mu/kT << 1$ (cf graph on page 8 of chapter 6). If the chemical potential u is negative this is not a problem, but we were not able to establish the behaviour of μ because we could not properly solve the equation - ie., eqtn. (6.21) of the last chapter – which determined μ for us. Recall that we found that the particle number was given in a ***continuum approximation*** for the density of states by

$$N = \frac{V}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{x^{1/2}}{\exp[x-\beta\mu]-1}dx \qquad (1)$$

But that no solution to (1) exists if N exceeds a certain critical value N_{cr},, which happens when $\mu \to 0$. (recall we have just seen that the Bose distribution is only defined for $\mu < 0$). We also recall that this breakdown of the continuum approximation at a critical particle number can be written in terms of a critical density, given by $\rho_{cr} = 2.612\rho_q$ where as usual, we have

$$\rho_q = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \qquad (2)$$

To get past the continuum approximation, we have to go back to our original derivation of the density of states, and take account of the discrete nature of this density of states at low energy. Let us begin by assuming the gas is contained in box of with volume $V = L^3$. The single particle energies are as before:

$$E_{nlm} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n^2 + l^2 + m^2) \qquad (3)$$

where *n, l* and *m* are positive integers. The ground state then has energy:

$$E_{111} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 3 \qquad (4)$$

Let us define our energy scale such that this ground state has zero energy, so that the Bose distribution is then well defined for $\mu < 0$ (rather than for $\mu < E_{111}$). Now, the separation between the ground state and first excited state is

$$E_{211} - E_{111} = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (4+1+1-3) = \Delta \qquad (5)$$

This is typically a very small energy compared to k_BT. For example, in the case of 10^{22} atoms of ^4He (where $m = 6.6\times10^{-27}kg$), contained in a 1 cm^3 box, we have $\Delta / k_B = 1.8\times10^{-14}K$. At the much higher temperature of 1K, the number of states within k_BT of zero energy (ie., up to an energy kBT above the ground state) is just

$$\delta N = V\int_o^{k_BT} g(E)dE = V\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\left(k_BT\right)^{3/2} = V\frac{1}{6\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2} = V\frac{4}{3\pi^{1/2}}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \qquad (6)$$

which if we put in the numbers becomes

$$\delta N = V\frac{4}{3\pi^{1/2}}\rho_q \approx 0.75\times10^{-6}m^3\left(\frac{6.6\times10^{-27}kg\times1.38\times10^{-23}J\cdot K^{-1}1K}{2\pi(1.05\times10^{-34}J\cdot s)^2}\right)^{3/2} = 1.3\times10^{21} \qquad (7)$$

Now if we were to distribute all of the 10^{22} atoms evenly over all of these 10^{21} states the average occupation number would be a little bit larger than 1 at this temperature of 1K. However this is not a very good approximation, since we have already violated the condition that the average occupancy be << 1. So how do we handle this?

The problem can be seen to arise from the continuum approximation since this approximation assumes that the number of states in a small energy interval $g(E)dE$ goes to zero as $E \to 0$. However, what of the ground state? We have seen that for the Bose distribution, if $\mu = 0$, the occupation number for this state must diverge as T goers to zero. At finite *T* we have an occupation of this ground state given naively by

$$f(0) = \frac{1}{\exp[-\beta\mu]-1} \approx \frac{1}{-\beta\mu} = \frac{k_B T}{|\mu|} \tag{8}$$

which diverges at any *T* if $\mu = 0$ (and which will tend towards *N* if the particle number is conserved). On the other hand for the first excited state we have, if the chemical potential $|\mu| << \Delta$, the result

$$f(\Delta) = \frac{1}{\exp[\beta(\Delta-\mu)]-1} \approx \frac{1}{\beta\Delta-\beta\mu} \approx \frac{k_B T}{\Delta} \tag{9}$$

so that as $T \to 0$ $f(\Delta)$ decreases, which is in contrast with $f(0)$ which increases at $T \to 0$. This is not to say $f(\Delta)$ of the first excited state is not large. In fact it can be a very large number as indicated above. Nevertheless it is always microscopic compared to *N*, whereas $f(0)$ is macroscopic and tends to *N* as $T \to 0$.

Thus in the sum over the states, we must isolate the ground state in the sum. So we write, instead of eqtn (1), the revised equation

$$\begin{aligned} N &= \sum_i f(E_i) = f(0) + \sum_{i\neq 0} \frac{1}{\exp[\beta(E_i-\mu)]-1} \approx \frac{1}{\exp[-\beta\mu]-1} + \frac{V}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}}{\exp[x-\beta\mu]-1} dx \\ &\approx \frac{1}{\exp[-\beta\mu]-1} + 2.612V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \end{aligned} \tag{10}$$

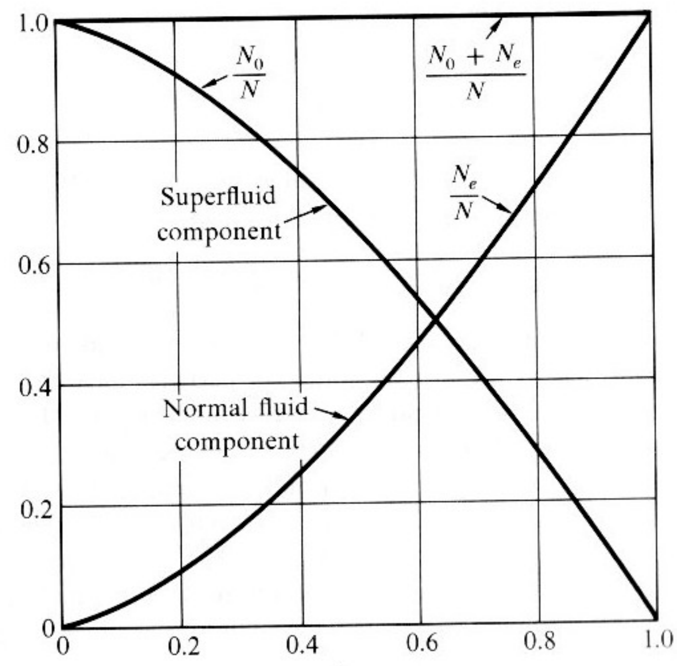
where the first term $N_0 = \frac{1}{\exp[-\beta\mu]-1}$ is the number of particles in the ground state and the second term $N_e = 2.612V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$ is the number of particles in all the excited states. Dividing by N and rearranging the fraction of particles in the ground state, we then get

$$\frac{N_0}{N} = 1 - \frac{2.612V}{N}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} = 1 - \left(\frac{T}{T_{BE}}\right)^{3/2} \tag{11}$$

where the Bose Einstein condensation (BEC) temperature is

$$T_{BE} \equiv \frac{2\pi\hbar^2}{mk_B}\left(\frac{\rho}{2.612}\right)^{2/3} \tag{12}$$

We can show a picture of what these results look like as a function of the temperature ratio x = T/T_{BE}, as shown in the figure:



Note that in the figure, the "superfluid component" denotes the fraction of particles in the ground state. This is actually misleading, as I will discuss below, but this error very common in textbooks. The reason for the error is that most systems that have BEC also show superfluidity – I will explain what this is and why it happens.

7(b) Superfluids – Basic features

Actually it turns out that Nature is full of superfluids. This can happen in two ways, viz.,

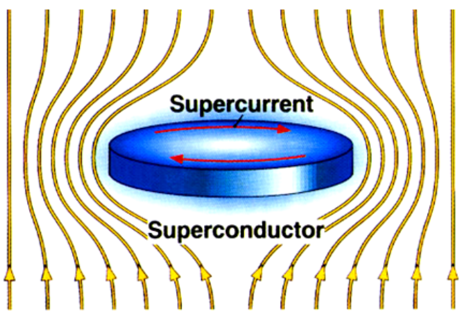
(i) Particles which behave as bosons at sufficiently low T can condense into a BEC
(ii) Particles which are actually fermionic at low T can nevertheless form "Cooper pairs" of fermions which behave like bosons, and show BEC.

In what follows I will explain both what superfluidity is, and why systems with BEC can show it – and I will give various examples so you can see how the theory works out.

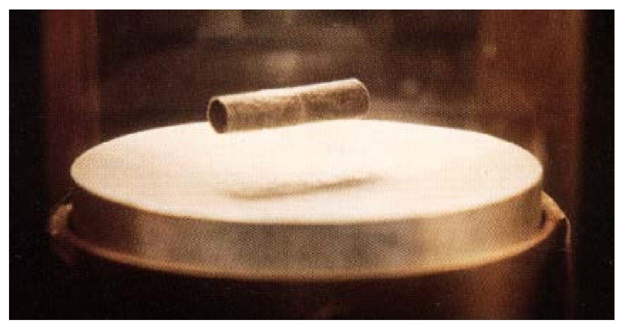
(i) <u>Superfluidity & Superconductivity</u>: These phenomena are amongst the most spectacular examples of collective quantum behavior in Nature. Their fundamental properties are often widely misunderstood, even by physicists, so I will do this carefully. It is important to distinguish the defining property of a superfluid (appearing in its rotational properties) with other spectacular properties like zero viscosity, which are consequences of the defining property and which may not always occur.

Note that if the particles that carry the superfluidity are charged (as happens, eg., for the protons in a neutron star, or for the electrons in metals at low T), then we get a superconductor. So I will go through some key properties for both neutral superfluids and charged superconductors together.

(a) **<u>Meissner effect/Hess_Fairbank effect</u>:** Let us first consider a superconductor, which undergoes BEC below a certain temperature Tc. Above Tc the system will typically (but not always) be a metal, with a finite electrical resistance (caused by the coupling of the electrons to impurities, phonons, etc.). Below Tc the system will usually be superconducting (unless, eg.,there are too many magnetic impurities in it). If it is superconducting, it will show the Meissner effect – if we apply a static magnetic field to the system, then above Tc this will have only a small effect (weakly magnetizing the system) but if we then cool below Tc, the field will be expelled from the system – the superconductor develops a spontaneous electrical current which then acts against the external field to screen the interior of the superconductor from the external field, and thereby expels the field.



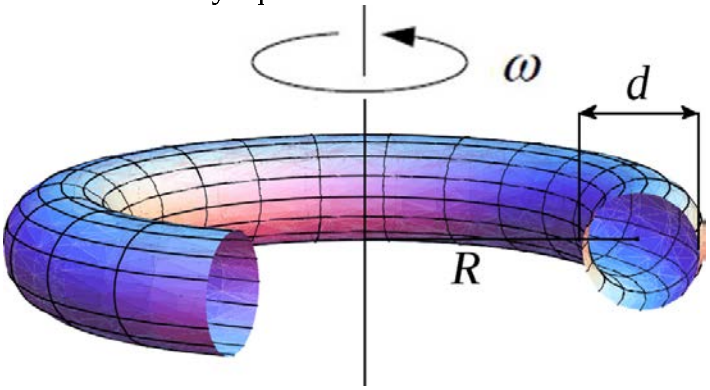
Flux expulsion by Type-I superconductor



Superconducting levitation of a magnet

The figure above left shows what we are talking about here – we show the field lines as they arrange themselves once the system goes superfluid. It clearly takes energy to push the field lines out like this – I will discuss below where this energy comes from. If we attempt to put a magnet on top of a superconductor, the superconductor will push it away, even levitating it as shown above – this is because the circulating current of the superconductor generates a 'back filed' which counters the field from the magnet.

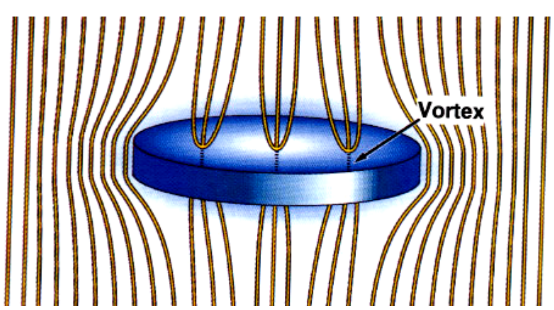
Neutral superfluids do not of course interact very much with a magnetic field. The analogue to the Meissner effect here is the Hess-Fairbank effect – we imagine setting a container holding a system like ^{4}He liquid into rotation. Now above Tc the liquid will rotate with the container (this is the famous "Newton rotating bucket experiment", by means of which Newton tried to define absolute space). However if we cool through Tc, the superfluid spontaneously stops rotating, giving its angular momentum up to the bucket (which then starts to rotate faster!). We can say that the system has “expelled rotation”, or expelled the effect of an external torque. The Meissner and Hess-Fairbank effects two effects are theoretically equivalent.



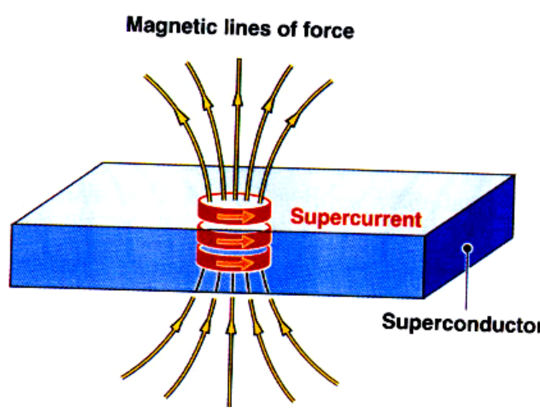
This ‘expulsion of rotation” can lead to remarkable effects. One consists in filling the above torus with both the superfluid and with packed jeweler’s rouge (an extremely fine powder, which looks almost like lipstick – the particle size may be only 300-500 nm, and it is used to polish metals, mirrors and even precious stones). If one rotates the torus when T > Tc, the fluid is completely locked to the close-packed jeweler’s rouge, because of the fluid viscosity. But if we cool below Tc, the superfluid unlocks from the rotating ring and can come to a complete stop, in an inertial reference frame, even if the torus is rotating.

Both of these experiments (which have been done in systems like superfluid ^{4}He) also illustrate another key property of most superfluids, viz., the completely lack of viscosity or electrical resistance. Actually the key point here is that the non-rotating superfluid, or the screened superconductor, are actually ***equilibrium states*** for the system (and at $T = 0$, they are the ground states). However the lack of any dissipation is actually a property, not of the equilibrium state, but rather of the non-equilibrium state – I discuss this below.

There is a catch here. In reality there are 2 kinds of superconductor – Type I and Type II – and in a type II superconductor, what happens is that above a critical field, the system will allow “quantized flux lines” to penetrate through the bulk superconductor. In a neutral superfluid, above a critical rotation velocity, the system allows quantized vortex lines to penetrate. In the regions away from the flux tubes, or away from the vortex lines, the superfluid behaviour is maintained – the flux is confined to the flux lines, and the vorticity to the vortex lines – but this is a way for the system to lower its energy.



Flux lines penetrating a superconductor



close-up of a flux line

Now an absolutely fundamental property of these flux and/or vortex lines is that they are ***quantized***. By this we mean that the amount of flux in each line, or vorticity in each vortex line, can only take discrete values governed by quantum mechanics. This is because the motion of electrons around a flux line, or neutral particles around a vortex line, will accumulate a phase which has to be a multiple of 2π. Since this phase is associated with the momentum, and hence the velocity of the particle or electron, this means that, eg., the velocity of the particles around a vortex line is quantized. Since the strength of the vortex line is given by the total "circulation" of the fluid around it, this means that the circulation is quantized; likewise for the electrical current around a flux line, and hence the flux through it. We find that:

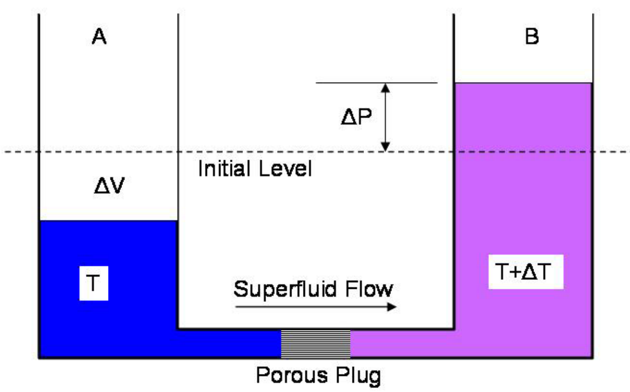
$$\begin{aligned} \text{Flux} &= n\Phi_0 \qquad \text{where the "flux quantum"} \quad \Phi_0 = h/2e \\ \text{Circulation} &= n\kappa_0 \qquad \text{where the "circulation quantum"} \quad \kappa_0 = h/m \end{aligned} \tag{13}$$

Where the circulation is defined as the line integral of the velocity on a circuit around a vortex line, where e is the electron charge, and *m* the mass of the neutral particles involved in superfluidity. The reason we have 2e instead of e is because the superconducting current (the "supercurrent) is carried by Cooper pairs of electrons.

(b) **Dissipationless Flow**: This is a property, not of the equilibrium state, but of how the state reacts when we apply a force to it, in order to get it to change. This it has to do with the non-equilibrium response of the system to change.

We are accustomed, when dealing with any normal fluid or gas, and with any solid, to seeing a resistance to such a change. This typically appears in the form of friction – energy is dissipated, into a huge number of microscopic degrees of freedom – it manifests itself then as heat, ie., frictional heating. There is a consequent frictional force resisting the change. In a fluid or a gas this manifests itself as "***viscosity***", which describes the way in which collisions between the fluid or gas particles rapidly causes any collective motion in the fluid to be dissipated into an enormous number of different microscopic motions.

This is indeed what happens in any superfluid system above Tc. However one can do a "capillary flow" experiment like that show here – below Tc the results are dramatic.

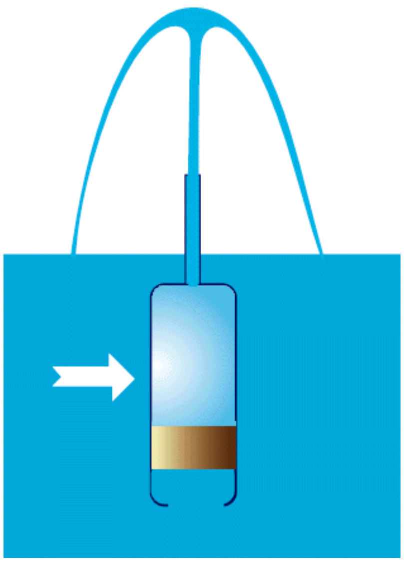


Imagine that in the figure we have a plug made of, eg., jeweler's rouge or even clay. Above Tc, no fluid can flow through this (imagine water trying to flow through the bottom of a clay mug – it would happen extremely slowly or not at all.

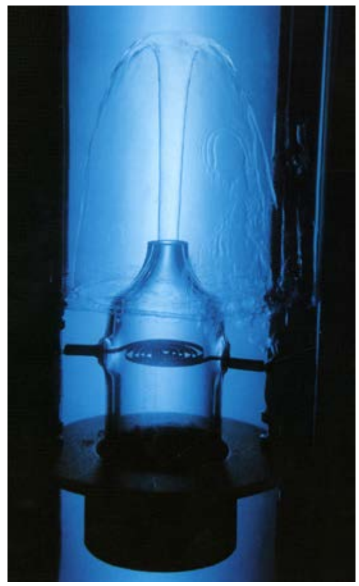
However below Tc, the superfluid flows with no resistance AT ALL, at least up to a certain critical velocity. The level of the superfluids on either side of the plug will then rapidly adjust to be the same. If you poured superfluid into a clay coffee mug, it would simply go straight through the bottom!

Just as dramatic is the "fountain effect" in which one again plugs the bottom of an open container with something porous at the atomic scale. We have superfluid both inside and outside the container. Now we shine a light on the fluid in the inside of the container, which heats it very gently (to do this in practice, one puts something dark inside so that it absorbs the radiant energy).

The result is spectacular – a jet of superfluid shoots out of the top of the container in a fountain, up to 20 cm high, and this continues for as long as the light shines – superfluid continuously pours in through the plug at the bottom to replace the superfluid leaving the top.



The fountain effect (schematic)



Experimental demonstration of the effect

What is going on here? The answer is that by heating the superfluid inside the container, we are slightly raising its entropy density as compared to the outside. Now any system tries to reach equilibrium – entropy will flow between sub-systems in order to equalize their entropy density. Here, the only way the system can do this is by having superfluid flow into the system from below. But, as I will discuss, the superfluid component of the system ***carries no entropy***! Thus no matter how fast it flows, it can never equilibrate then interior part of the container with the exterior.

Similar experiments show that the superfluid can climb walls unde r the influence of capillary (surface tension) forces – see, eg., the video at https://www.youtube.com/watch?v=2Z6UJbwxBZI

The fundamental reason for the non-viscous flow is that there is an energy gap preventing the production of any kind of microscopic motion in the system which could take energy away from the collective motion (which in this case is the bulk superflow). This quantum energy gap becomes quite large in the superfluid – it can be as large as Tc. Basically, for there to be any friction, some sort of excitation must be created, and if the energy to do this is not available, no friction can result.

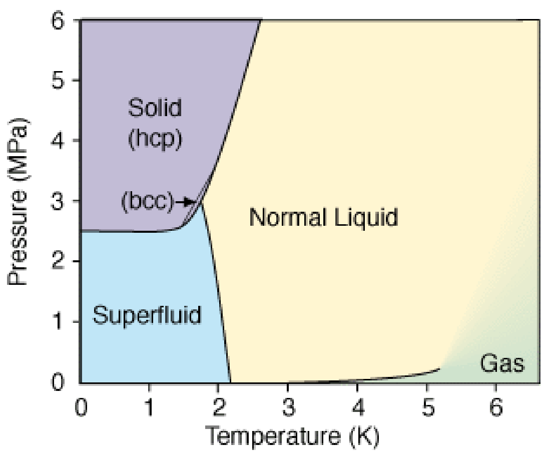
Now, all of this is clearly associated with Bose condensation. However in real superfluids the connection is not entirely simple, because most real superfluids have strong interactions between the bosons. In fact, without these interactions the energy gap just mentioned actually goes to zero – so an ideal Bose gas shows no superflow properties at all!

To see how all of this works, let's not look at some examples.

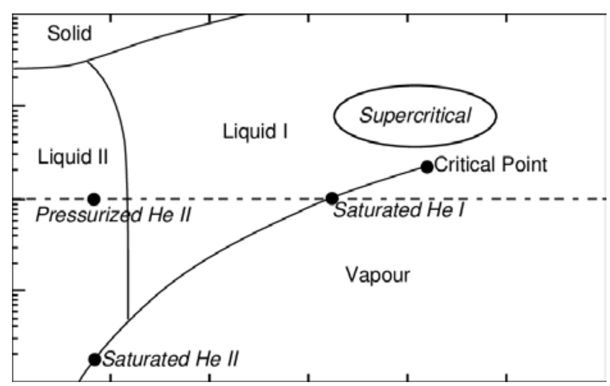
7(c) Examples of Superfluids

As noted above, Nature abounds with superfluids. So I will take just a few examples, and draw some general conclusions from them.

(i) Superfluid 4He: This is the best known example, because it was the first to be discovered, back in 1937-38. The discoverer was made more or less simultaneously, by Allen and Misener at Cambridge (UK), and by Kapitza in Moscow. Let's look very briefly at some of its properties.



Phase diagram of ^{4}He (P vs T)



Same diagram, but now plotting log P vs T)

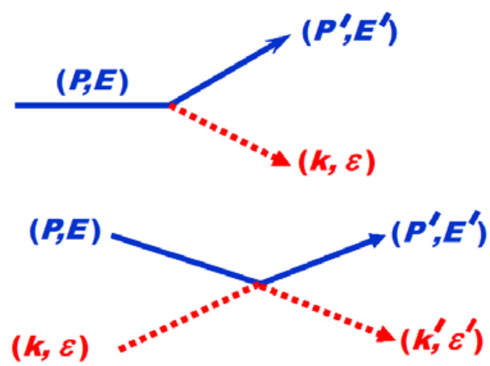
(a) **Phase Diagram**: We first look at the thermodynamic properties – the low-*T* phase diagram is shown above (note that 1 MPa is roughly 10 atmospheres). For a system with the density of liquid ^4He, and having an atomic mass of 4, one finds from equation (12) that $T_{BE} = 3.1K$. However the interaction between He atoms is not negligible; its strength is actually ~ 2K (except at short distances, where it is much stronger and repulsive). Thus the interaction strength is comparable to T_{BE}, and so the liquid is not in any way an ideal gas.

There is a gas-liquid phase transition at 4.2K at a pressure of 1 atm. The liquid-gas phase curves at very low T until the temperature gets up to 4-5K. The atomic interactions in He are weak compared to almost all the other elements – this is one reason why He is the only element that remains a liquid down to T=0K at atmospheric pressure. The solid phase exists but only at pressures greater than 25 atm.

There is a critical point in the liquid-vapour phase curve at (T=5.2K and p~2atm) such that no phase transition between the liquid and gas can be seen above 5.2K. Just below 4.2K, at 1 atm the liquid exists in the He I phase which behaves like a strongly interacting and very dense gas. The viscosity, thermal conductivity and heat capacity of the He I liquid are all similar to what is predicted for an ideal gas. One reason for this is that the zero point motion of the atoms in the weakly attractive potential keeps the atoms relatively far apart. As a result the density is about 3.1 times less than one would expect in the absence of zero point motion effects.

Below a temperature of 2.17K, called the lambda point, the liquid enters a new phase called He II which is superfluid. At finite temperatures between 0 and 2.17K, one can define "normal" and "superfluid" components in the motion of the fluid where the normal fraction goes to zero as $T \rightarrow 0$. All of the properties we described above for a superfluid were first seen in the superfluid He II phase a long time ago (in the 1930s and early 1940s), so I do not describe them further here. It is however interesting to look at two other properties of a neutral Bose superfluid that were not described above, and which have been seen in He II more recently:

(b) **Quasiparticle Excitations:** Suppose we try to move a solid body through the superfluid. According to what was said above, we would expect no resistance to the motion of this body in an experiment. The first such experiments demonstrating this were done in the period from the 1960s-80s. They verified very nicely a simple argument made by Landau in the 1940s, which goes as follows:



An external object (in blue) interacts with quasiparticles (in red)
TOP: quasiparticle creation. BOTTOM: quasiparticle scattering

When an object travels in a superfluid it can only lose energy and momentum by creating excitations, or scattering off them (see Figure above). These excitations were called ***quasiparticles*** by Landau. Because the only quantum numbers that one can have in a translationally invariant system like a liquid or gas are momentum and energy, the quasiparticles must be defined by their energy and momentum, just like real particles (and in a non-interacting gas, the quasiparticles would in fact be just the particles themselves – the only way to quantum-mechanically excite the gas would be to give some momentum and energy to the different particles). However in the liquid we expect that the excitations will involve a complex collective motion of many different particles together - because of the interactions, we cannot move different particles independently.

There must then be a specific relationship between the energy and momentum of a given quasiparticle; but it will ***not*** be of the form $\varepsilon^0_k = p^2/2m$ (as it would be for a free atom). The external object will also have some dispersion relation E_P for its energy as a function of P.

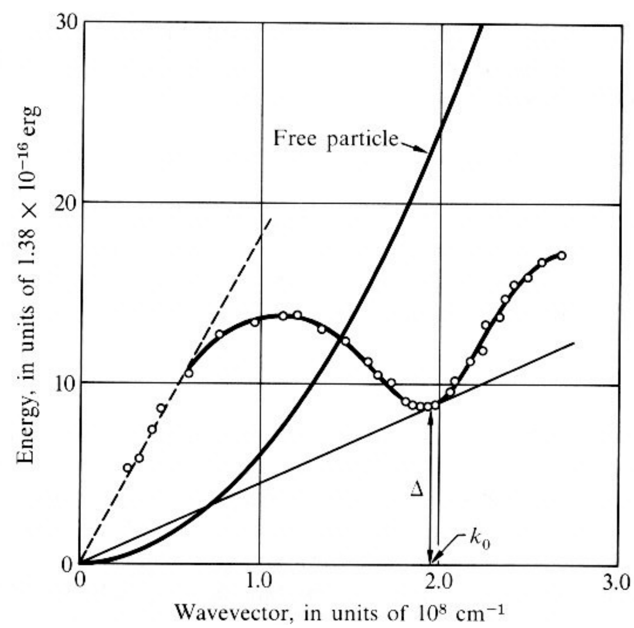
Now a key requirement in any process of the kind shown in the figure on the last page is that energy and momentum be conserved. Thus we have

$$P' - P = k \qquad E' - E = \varepsilon \qquad \text{(quasiparticle creation process)} \qquad (14)$$

$$P + k = P' + k' \qquad E + \varepsilon = E' + \varepsilon' \qquad \text{(scattering process)} \qquad (15)$$

and, as we will see, these equations allow us to determine the kinematics of these processes once we know the relation between energy and momentum for the two objects (the external particle and the quasiparticle).

For real ^{4}He Landau used a combination of deductions from experiment and general theoretical arguments in 1941 to guess the correct form for the quasiparticle dispersion ε_k . Many years later, in the 1960s, this was first measured directly by neutron scattering measurements. The results are shown below.



Dispersion relation for 4He quasiparticles

The circles represent measurements for He II at 1.2K made with neutron scattering. Note the linear behaviour at low k (dashed line) which is quite distinct from the free particle dispersion, which goes as k^2. This linear region is actually the region where the quasiparticles are actually just sound waves; the slope of the straight line gives the sound velocity c, since in this region $\varepsilon_k = ck$. As we increase k, the dispersion turns over and descends to a minimum – Landau gave the name "rotons" to the quasiparticles in this part of the spectrum.

As I will show immediately below the implication of the peculiar shape of the He II quasiparticle dispersion is that an object (e.g. a heavy ion) moving in He II below some critical velocity is unable to create quasiparticles and therefore travels unimpeded as if the superfluid wasn't there. By the same argument, the superfluid can flow unimpeded down a capillary provided the fluid velocity is not too high (we treat the capillary walls as though they were an external object interacting with the superfluid.

To show this, let us consider an object with a mass M which is very large compared to the ^{4}He mass. We assume it is traveling at velocity $\vec{V}$ and that it creates a quasiparticle with energy $\varepsilon_{\vec{k}}$ and momentum $\vec{k}$ (we now use vector signs above the momentum and velocity because we are in 3 dimensions). Then conservation of energy and momentum require, from eqtns (14) and (15) above, that

$$\frac{1}{2}MV^2 = \frac{1}{2}MV'^2 + \varepsilon_{\vec{k}} \tag{16}$$

$$M\vec{V} = M\vec{V}' + \hbar\vec{k} \tag{17}$$

where $M\vec{V}$ and $M\vec{V}'$ are the initial and final momentum of the object, and we note that in (16) we simply assume that the large object has a free particle dispersion. We rewrite eqtn. (17) as $M\vec{V} - \hbar\vec{k} = M\vec{V}'$, and then square this and divide by $2M$. We then get:

$$\frac{1}{2}MV^2 - \hbar\vec{V}\cdot\vec{k} + \frac{\hbar^2k^2}{2M} = \frac{1}{2}MV'^2 \tag{18}$$

Subtracting this from eqtn. (16) we immediately find that:

$$\hbar\vec{V}\cdot\vec{k} - \frac{\hbar^2k^2}{2M} = \varepsilon_{\vec{k}} \tag{19}$$

Now, since $\varepsilon_{\vec{k}}$ must be positive there can only be a solution if $\hbar\vec{V}\cdot\vec{k}$ is positive and sufficiently large. For example consider a specific quasiparticle with wave vector $\vec{k}$ and energy ε_k . In order to excite this particular quasiparticle $|\vec{V}|$ must exceed some critical value. The maximum value in $\hbar\vec{V}\cdot\vec{k}$ occurs when $\vec{k}$ is parallel to $\vec{V}$ (quasiparticle is emitted in the direction of motion of the external body). The magnitude of the velocity required to create an excitation in the He with energy ε_k and momentum magnitude $\hbar k$ is then:

$$V = \frac{\varepsilon_{\vec{k}} + \frac{\hbar^2k^2}{2M}}{\hbar k} \approx \frac{\varepsilon_{\vec{k}}}{\hbar k}\,. \tag{20}$$

Suppose in this equation we drop the small correction coming from the $\frac{\hbar^2k^2}{2M}$ term (this term is often called the "recoil term", and it disappears for very large M). Then we can understand (20) graphically; the velocity

V of the external body is just the slope of the line connecting the origin to the appropriate point of the dispersion curve $\varepsilon(k)$ shown above.

It is then clear there is a minimum value for all the allowed quasiparticles which occurs when the straight line defined by ε / k just touches the dispersion curve. When V is equal to this critical velocity V_c, the only quasiparticles that can be created are those at the point where the line touches the curve. Thus, if we try to move an object through He II, it will accelerate up to this velocity, and then emit a roton, of energy Δ and momentum k_0 (see figure above).

One can do this experiment with neutrons, and also by moving heavy ions through the superfluid with electric fields. If we apply the electric field, the ions will simply accelerate up to the critical velocity as though they were in free space, and then begin to emit rotons – they end up moving at a terminal velocity roughly equal to V_c, where the energy loss from roton emission is balanced by the energy gain from the electric field. Note that this will happen for arbitrarily small electric field – if the field is small, it will simply take longer for the particle to reach the critical velocity.

In He II the critical velocity depends on the pressure; one has $V_c = \frac{\varepsilon_{\vec{k}}}{\hbar k} = \frac{\Delta}{\hbar k_0} \approx 5 \times 10^3 cm \cdot s^{-1}$.

On the other hand if we are in the normal He I phase, the dispersion curve for the fluid is quite different. We can guess that the response of the normal fluid will be a quadratic function of k given roughly by $\varepsilon = \frac{(\hbar k)^2}{2m}$ where m is roughly the mass of a He atom. We guess this because in the normal phase, where the system behaves like an ordinary fluid, it is easy to excite the motion of individual atoms of ^{4}He, and this what their dispersion relation looks like in the gas phase – the interactions between the atoms should not radically change this.

In that case there is no critical velocity since the function $\frac{\varepsilon_{\vec{k}}}{\hbar k} = \frac{\hbar^2 k^2}{2m\hbar k} = \frac{\hbar k}{2m}$ has no lower limit. For any velocity there exist quasiparticles that can be excited. Thus the response of the normal fluid to the motion of an external object through it will be frictional – there will be a drag force on the object and under an electric field it will reach a critical velocity which depends on the field strength (and goes to zero when the field goes to zero). This the behavior is completely different from that in the superfluid phase.

One could say a lot more about superfluid 4He, but let us now move to another example.

(ii) **<u>Superfluids in Neutron Stars</u>:** Stars, at the end of their lives, typically end up as white dwarfs, which then very slowly cool to black dwarfs – I will describe these in more details later. However if they are sufficiently massive, they can evolve into neutron stars – either via a supernova explosion or by other less violent processes. Both the supernova explosion and the resulting neutron star are amongst the most spectacular and bizarre phenomena in the universe. Although neutron stars were not discovered until 1967, they had been predicted theoretically at the beginning of the 1930s, and so it did not take long to figure out many of their properties after they were discovered.

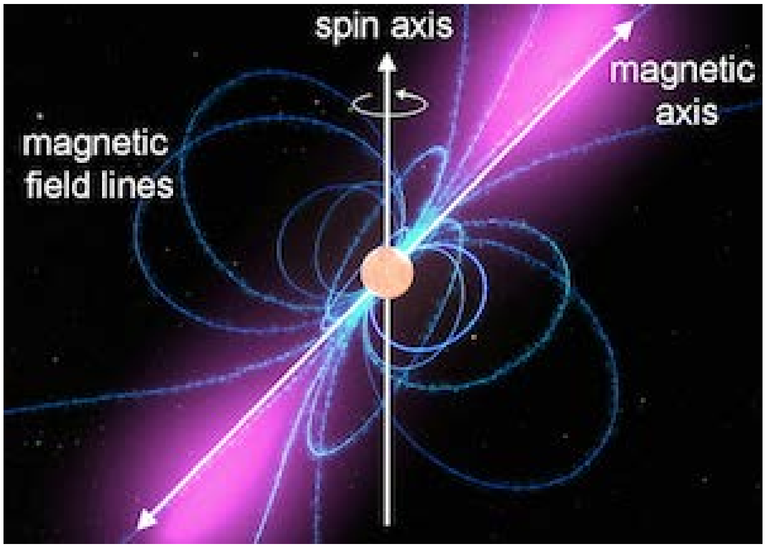
We can begin with their birth, in a supernova explosion. When a massive star exhausts its nuclear fuel, it undergoes a dramatic and sudden collapse from a diameter of 10-100 million km down to perhaps 20 km – this process may take only 30-200 secs, at which point the density has reached more than 10^{15} g/cm^3, exceeding even nuclear densities. At this point extremely powerful repulsive forces between the nucleons cause an explosive rebound, with a release of fantastic quantities of energy – we get a supernova explosion, which may shine with a luminosity of over 10 billion suns.

The "Crab Nebula" in Taurus, remnant of a supernova seen in 1054 AD.

Supernovae have been seen since the beginning of recovered history – one of the best known was seen in 1054 AD, and recorded in great detail by Chinese astronomers at that time. Although rare, they have caused considerable excitement, in popular culture, because if they occur near enough to us, they shine extremely brightly in the sky. In the figure above we show the "Crab Nebula", the remnant of the 1054 explosion, which requires a telescope to see at all – it is 6,300 light years away from us. However when the explosion occurred, it was seen on earth as a star which outshone every other night-time object except for the full moon, and which remained visible to the eye for nearly a year afterwards. The Crab nebula comprises those parts of the star that were blown out by the explosion – it now extends over roughly 13 light years (its expansion rate having been slowed considerably by collision of the expanding stellar remnants with the interstellar medium).

After the supernova explosion, those parts of the star that are not ejected then recollapse, and if the total mass of the remaining object is not too large, it settles down as a neutron star with mean density ~ 4.5 x 10^{14} g/cm^3. This object is typically spinning very fast (up to 1000 revolutions per second at the moment of formation). Even though it is very small – roughly the size of Vancouver – its mass will exceed that of the sun – the mass usually ranges from about 1.3 to 2.8 solar masses.

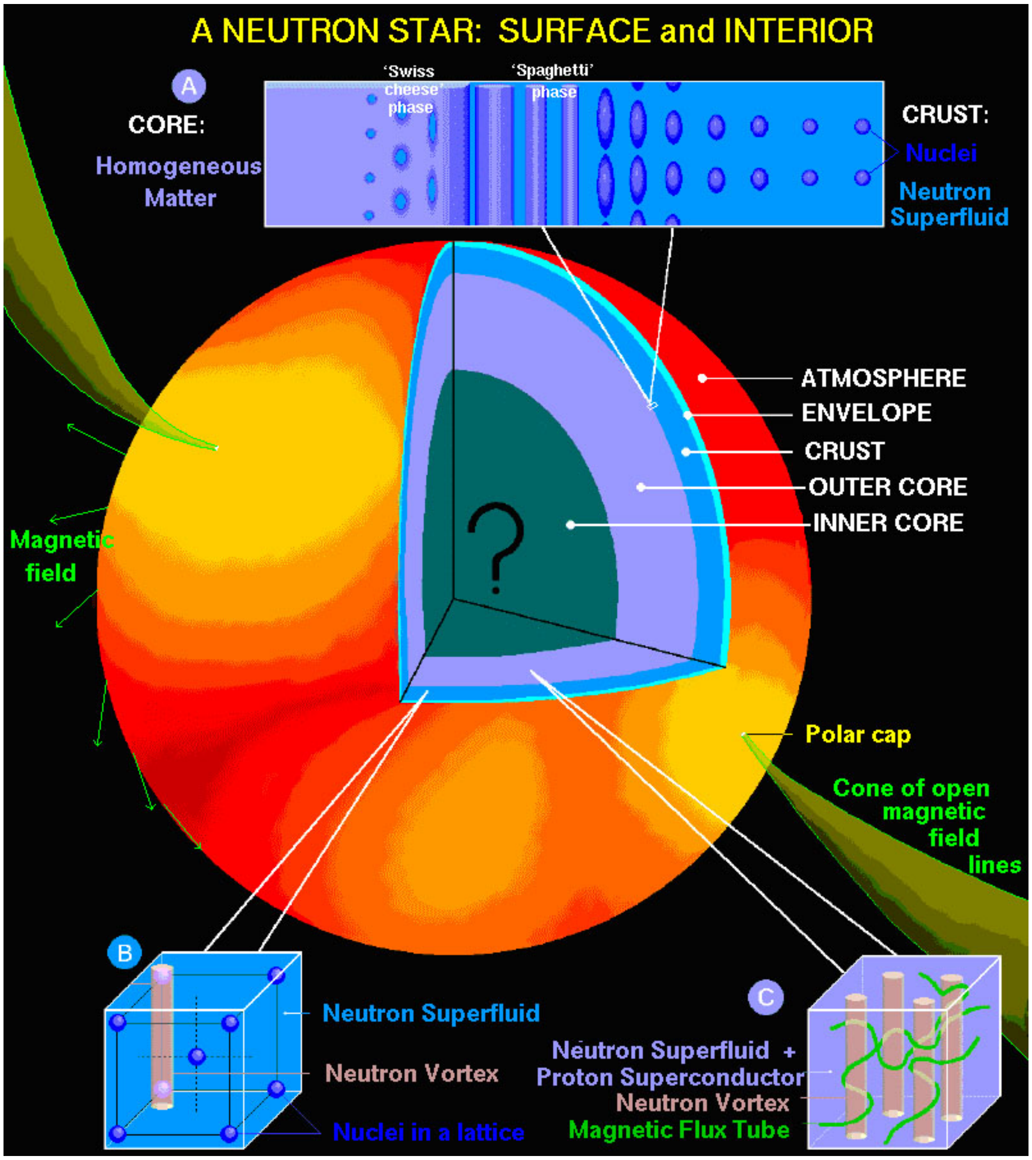
If the mass exceeds about 2.8 solar masses, the gravitational field it generates will overcome any conceivable repulsive force, and the system collapses into a black hole.



If we could sit close to a neutron star (which would be pretty close – it is only 15-20 km in diameter), what we would see is shown schematically in the figure above. Most of the star is a mixture of protons, neutrons and electrons – however these are partly separated into different regions, and because the system is spinning extremely fast, very powerful magnetic fields are generated – in some cases up to 10^{10}-10^{11} T (units of Tesla). To get some idea of what this means, note that the earth's field is ~ 2 x 10^{-4} to 4 x 10^{-4} T at the earth's surface, that the highest static lab fields on earth are ~ 30-40 T, that the pressure needed to confine a field of 50 T causes *Cu* metal to flow like toothpaste, and that the energy density of a static field increases like the ***square*** of the magnetic field strength. This means that matter behaves in very strange ways in a neutron star.

Size of a neutron star compared to Vancouver.

The structure of a neutron star is extremely complex. This is not only because of the magnetic fields – it is also because as one goes deeper, the enormous weight of the outer layers compresses what is below, so that the density varies between roughly 10^4 g/cm^3 at the surface up to ~ 8 x 10^{14} g/cm^3 in the core. A piece of neutron star core material the size of a sugar lump would then have a mass of roughly 4 x 10^9 tons, not much less than the mass of Grouse Mountain in Nth Vancouver.



Because of the large change in density, the nature of the matter varies drastically as we go deeper into the star. Let's divide them as follows:

(i) **Surface Crust**: at the very surface the atoms are distorted into extended spindle shapes, oriented along the magnetic field, because the field energy is so much larger than the Coulomb energy binding the electrons to the nuclei. Classically we can imagine the electron spiraling along cylinders oriented parallel to the field, in cyclotron orbits, only very weakly confined by the Coulomb interaction. These atomic spindles form a crystalline array – the crust is actually solid.

As one descends through the crust, the pressure rapidly increases the density to a level where the atoms are crushed and we then have a neutron superfluid moving through a lattice of neutrons. The neutron superfluid forms through the Cooper pairing mechanism – neutrons are fermions but a Cooper pair is a boson, and these bosons Bose condense. The Bose condensation energy is 2-3 MeV (ie., 2-3 \times 10^{10} K), which is far higher than the actual thermal energy in the neutron star (whose temperature ranges down from maybe 10^7 K at the time of creation, to much lower temperatures if it is old and has cooled). Thus as far as superfluidity is concerned, we are in the extreme low T limit. Because the neutron star is rotating extremely fast, the superfluid is permeated by a dense array of vortex lines.

Note that neutrons cannot decay into protons here because the electron density is so high – the Fermi energy of the electrons is much higher than the Fermi energy of the neutrons, and so there are no available states for the protons to go to.

(ii) **Outer Core**: Here it gets harder for theorists to predict what we will see, because the calculations are exploring densities that are higher than those in nuclei, and the results depend very much on the 'equation of state' that one calculates, using what we currently understand about inter-nuclear forces. It is however believed protons start to appear as the density gets still higher – as a result the neutron superfluid mixes with a charged proton superconductor, which of course generates a very large magnetic field because the neutron star is spinning rapidly. This type II superconductor is run through with superconducting flux lines.

Finally one gets to the inner core about which very little is known. The main ways in which theory and experiment are tied together for neutron stars is by observing the very complicated electromagnetic phenomena associated with them and by looking at how the rotation period changes with time (since this is telling us about how vortex lines and flux lines are redistributing themselves).

7(d) The Photon Gas

We finally come to the case of massless photons. These behave very differently from a gas of massive bosons, because their number is not conserved – indeed one can freely add or subtract photons from a system (and the same is true of other massless bosons like phonons) . As a consequence of this the chemical potential must be zero, since m is just measuring the change in energy or in free energy if we do add or subtract a particle (compare eqtns. (1.50) - (1.52)).

The Planck Distribution: To treat the photon gas, let's put it in a cubic box of side L, so that no photons can pass through the walls (this does not stop their number changing – this can be done on interaction with the walls). Physically, this would mean that the walls are conducting. We therefore work in the grand canonical ensemble, with a heat bath outside the box at temperature *T*. We assume that the only thing in the box is photons. Since photons can't get through the walls, we must fix an appropriate boundary condition for the photon wave eqtn. Consider, eg., the electric field component, which obeys the wave eqn.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0 \qquad (21)$$

Then as we approach any wall boundary, the component of $\vec{E}$ which is tangential to the boundary must vanish at that boundary. In addition, since there is no charge in the cavity, only photons, we have

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0\,. \qquad (22)$$

The modes are of then of the form:

$$\begin{aligned} E_x &= E_{x0} \sin \omega t \cos(k_x x) \sin(k_y y) \sin(k_z z) \\ E_y &= E_{y0} \sin \omega t \sin(k_x x) \cos(k_y y) \sin(k_z z) \\ E_z &= E_{z0} \sin \omega t \sin(k_x x) \sin(k_y y) \cos(k_z z) \end{aligned} \qquad (23)$$

where each mode is characterized by a wave vector $\vec{k} = \left(\frac{m\pi}{L}, \frac{n\pi}{L}, \frac{l\pi}{L}\right)$ (24)

where n, l *and* m are positive non-zero integers. (Note negative integers correspond to the same mode but with a phase differing by 180^o). In order to satisfy the wave equation in (21) we must then have

$$\left(\frac{\omega}{c}\right)^2 = \left(\frac{\pi}{L}\right)^2 \left(m^2 + l^2 + n^2\right) = k^2 \qquad (25)$$

implying that the photon frequencies satisfy $\nu = \frac{ck}{2\pi}$ (26)

where c is the velocity of light, and $k = |\vec{k}\,|$ is the magnitude of the wave vector (NB: one often sees this relation written as w = ck, without the factor of 2p. This is because the frequency is sometimes defined in terms of radians per second, and sometimes in terms of wave numbers per second – we have done the latter here, in eqtns. (25) and (26)).

The additional requirement that $\nabla \cdot \vec{E} = 0$ implies that $\vec{E}_0 \cdot \vec{k} = 0$. Thus the electric field is always perpendicular to the wave vector, and consequently there are just two independent polarization directions for each wave vector. This is not quite the same as one finds for massless acoustic phonons, where we label the normal modes for vibration according to whether they are transverse or longitudinal – that leads to 3 independent polarization directions for each phonon wave vector, with 2 transverse and one longitudinal. However a longitudinal photon mode is physically meaningless – it corresponds to an oscillation along the photon direction, but there is no rest frame for the photon in which one could have this oscillation.

Let us now count the photon modes so as to find the density of states. We easily see that the number of modes up to a frequency ν is just

$$G(\nu) = 2\frac{1}{8}\frac{4\pi k^3/3}{(\pi/L)^3}\frac{1}{L^3} = \frac{1}{3\pi^2}\left(\frac{2\pi\nu}{c}\right)^3 \qquad (27)$$

(here we use *G(ν)*, as opposed to *N(k)* for the number of modes of to a given wave-vector as in chapter 4). Differentiating this with respect to ν gives the density of states in energy space as

$$g(\nu) = \frac{dG}{d\nu} = \frac{8\pi\nu^2}{c^3} \qquad (28)$$

As in the QM oscillator the energy in each cavity mode is quantized in units of $h\nu$ i.e, one can have energies $0, h\nu, 2h\nu, 3h\nu, \ldots nh\nu$, etc., this last corresponding to a number *n* photons in this particular mode. Note that this was the key assumption made by Planck in 1900 that lead to his famous radiation law. Historically, it had profound implications it opened the way to quantum mechanics (although another 25 yrs were needed).; indeed this crucial step marked the beginning of quantum mechanics.

As e have already seen in discussing the SHO, the partition function for a single mode with frequency ν must then be

$$Z = \sum_{n=0}^{\infty}\exp[-n\beta h\nu] = \frac{1}{1-\exp[-\beta h\nu]} \qquad (29)$$

And likewise the mean energy in the mode with frequency ν is simply

$$\langle E\rangle = \frac{1}{Z}\sum_{n=0} nh\nu\exp[-n\beta h\nu] = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{h\nu}{Z}Z^2\exp[-\beta h\nu] = \frac{h\nu}{\exp[\beta h\nu]-1} \qquad (30)$$

And since this is simply $\langle E\rangle = \langle n\rangle h\nu$ we have

$$\langle n\rangle = \frac{1}{\exp[\beta h\nu]-1} \qquad (31)$$

Which is Planck's famous distribution function, ie., the Bose-Einstein distribution with $\mu = 0$.

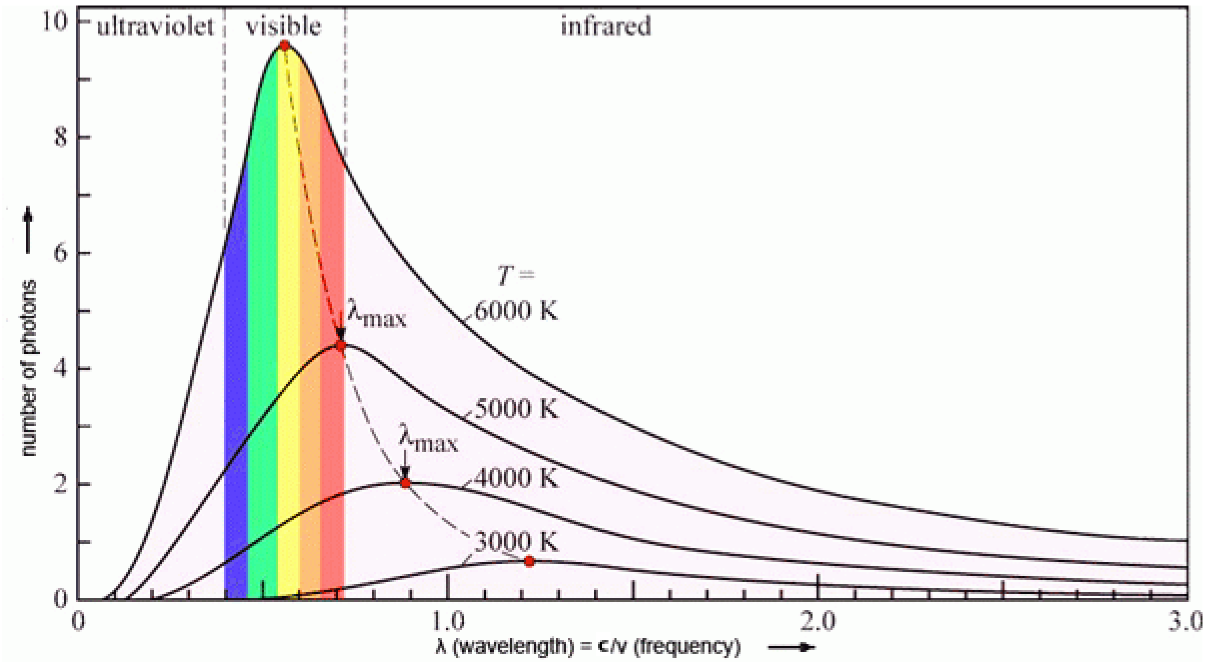
From here we can easily find the total energy per unit volume for the photons. We first define the ***spectral energy density*** as the energy per unit frequency interval per unit volume [or equivalently the energy per photon times the number of photons per unit volume per unit frequency (energy)]. In any case we have

$$u_s(\nu,T) = h\nu g(\nu)\langle n\rangle = h\nu\frac{8\pi\nu^2}{c^3}\frac{1}{\exp[\beta h\nu]-1} = \frac{8\pi(k_BT)^3}{h^2c^3}\frac{x^3}{\exp x-1} \qquad (32)$$

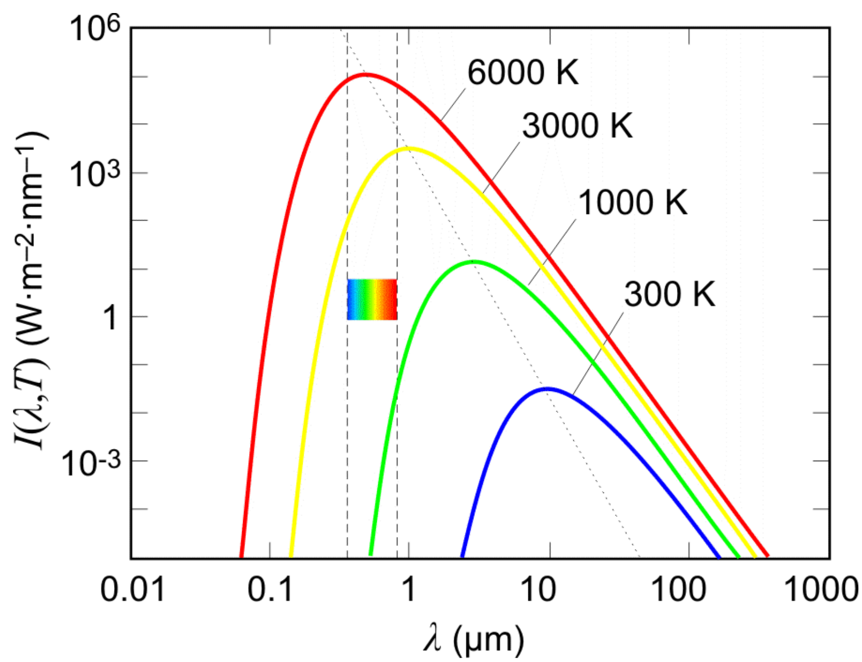
where $x = \beta h\nu$. This is the famous ***Planck distribution***; the total energy density is then found by integrating over all frequencies, using $d\nu = \frac{k_BT}{h}dx$ to get

$$u(T)=\int_0^\infty u(\nu,T)d\nu=\frac{8\pi(k_BT)^4}{h^3c^3}\int_0^\infty\frac{x^3}{\exp x-1}dx=\frac{8\pi(k_BT)^4}{h^3c^3}\frac{\pi^4}{15}=\frac{8\pi^5k_B^4}{15h^3c^3}T^4 \tag{33}$$

The Planck distribution is one of the most famous curves in physics We show it here in 2 different ways; first, as a linear-linear plot, with the number of photons plotted against the wavelength, ie., against the inverse frequency:



Next, we plot a log-log graph, of the same variables – this allows us to see how things vary over a much larger range of temperatures and photon numbers:



Essentially the Planck distribution is a plot of how radiation would be emitted from a "black body", ie, one which shows no preference for any particular wavelength. Since stars are very nearly blackbodies, and so are many other celestial sources, this is very useful. The peak frequency in the Planck distribution occurs at an energy $h\nu \sim 3k_B T / h$; at 300K the peak is in the mid infrared. Organisms like us that rely on the sun will be sensitive preferentially to wavelengths near the peak in the solar distribution – the solar surface temperature is roughly 6,000K, and so our eye sensitivity is maximized near the peak in the spectral density from the sun corresponding to a wavelength $\lambda = 750nm$.

At low frequencies, where $\beta h\nu << 1$ the spectral density has the limiting ***"Rayleigh-Jeans"*** form:

$$u_s(\nu,T) = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp[\beta h\nu]-1} \approx \frac{8\pi h}{c^3}\frac{\nu^3}{1+\beta h\nu - 1} = \frac{8\pi\nu^2}{c^3}k_B T \qquad \text{(for } h\nu << k_B T\text{)} \qquad (34)$$

which does not depend on Planck's constant. This is reasonable since under these conditions the equipartition theorem applies, and QM is irrelevant. Note that if we applied this form for the distribution at all frequencies (as Rayleigh and Jeans did, before QM), the whole thing blows up at high frequency and the total energy density diverges – this is the famous "UV catastrophe". However energy quantization effectively cuts off contributions to $u_s(\nu,T)$ from high frequency modes where $h\nu >> k_B T$; these are *frozen out*. On the other hand it has little effect on the low frequency modes $h\nu << k_B T$ which are in the classical limit.

More on Black Bodies: Let us look in more detail at what was said above for black bodies. Suppose we take our cubic box again, and then cut a small hole in the box with area *A*. We then measure the energy being emitted from the hole.

Let the energy emitted per unit area be J_u. Consider a semispherical shell a distance ct from the hole of thickness cdt. The only photons that can reach the hole in a time interval $(t, t+dt)$ must be in this shell. For each element of volume dV in the shell the energy passing through the hole in the time interval $(t, t+dt)$ is:

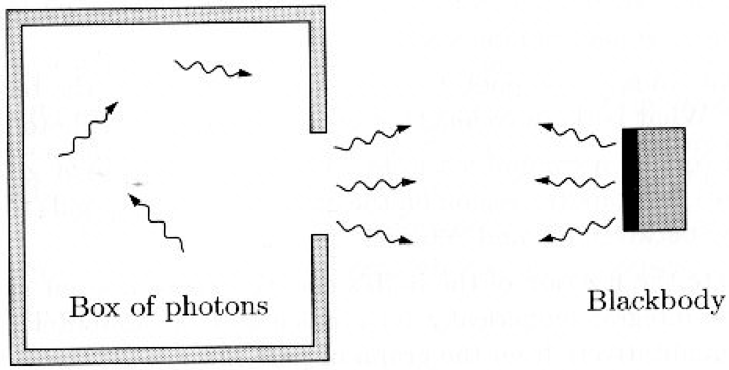
$$dE = u(T)dV\frac{\cos\theta A}{4\pi(ct)^2} = u(T)\sin\theta d\theta d\varphi (ct)^2 cdt\frac{\cos\theta A}{4\pi(ct)^2} = u(T)\cos\theta\sin\theta d\theta d\varphi cdt\,\frac{A}{4\pi} \qquad (35)$$

The total energy passing through the hole per unit time per unit area is then obtained by integrating over θ and ϕ, to get

$$J_u = \frac{dE}{Adt} = \frac{cu(T)}{4\pi}\int_0^{\pi/2}\cos\theta\sin\theta d\theta\int_0^{2\pi}d\phi = \frac{cu(T)}{4\pi}\left[\frac{\cos^2\theta}{2}\right]_{\pi/2}^{0}2\pi = \frac{cu(T)}{4} = \frac{2\pi^5 k_B^4}{15h^3c^2}T^4 = \sigma_B T^4 \qquad (36)$$

This is called the Stefan-Boltzmann law; we see that the escaping energy is propeortional to the energy density calculated from Planck's distribution (cf. eqn. (33) above). Stefan's constant is then found to be $\sigma_B = 5.67\times 10^{-8} WK^{-4}m^{-2}$; it was first measured experimentally well before QM, but its explanation required Planck's radiation law.

One can continue this line of argument as follows. Suppose we have a black disk (ie., one which completely absorbs all incident radiation), which has the same diameter as the hole, and that the box and disk are at the same temperature. The hole and disk are set up to face one another, as shown in the figure below. For simplicity we assume all the outside surfaces of the box and the back side surface of the disk are perfectly reflective. Thus the only radiation coming from the box exits through the hole and the only radiation leaving the disk comes off the black surface. It is then clear that the total amount of radiation emitted by the disc must be exactly equal to total emitted by the hole. This follows from the fact that they have the same area and the two objects are in thermal equilibrium - there can be no net energy flow between two objects at the same temperature, by the 1st law of thermodynamics



Now suppose we place a filter between the two objects which only allows radiation to pass at a given frequency. Again the amount of radiation emitted by the hole at that frequency must be exactly equal to the radiation emitted by the disc at that frequency in order for the two objects to be in thermal equilibrium. The implication is that any black body (i.e. totally absorbing) will emit the same spectrum of radiation as the hole in our box assuming they have the same area.

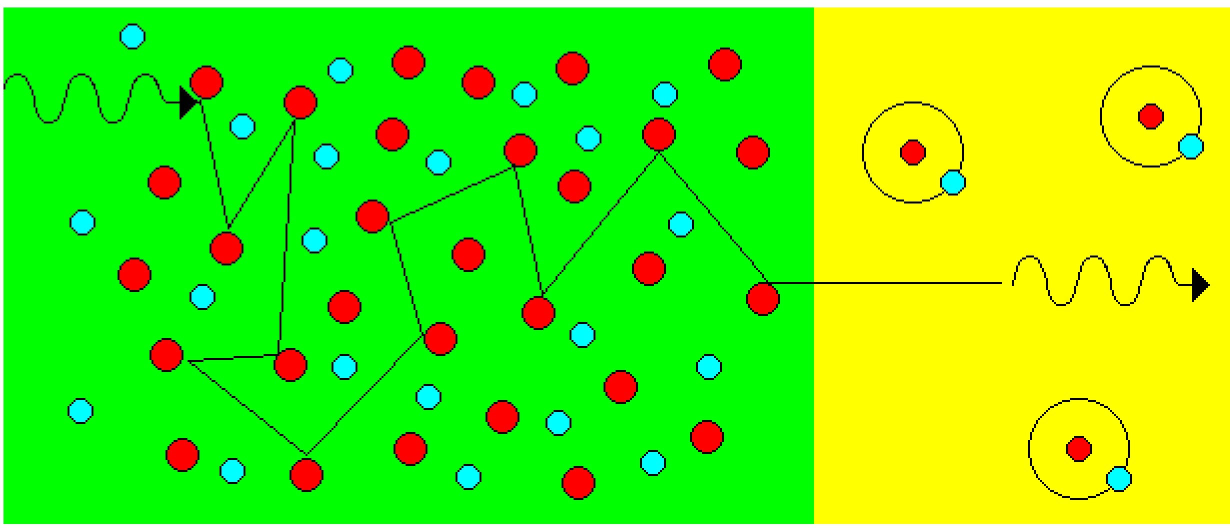
Now suppose the disk covers the hole (black side facing in). Assume also that the inside surface of the disc absorbs a fraction $a < 1$ of the incident radiation, where a is called the absorptivity, and reflects a fraction $r = 1 - a$ (the reflectivity). In order to remain in thermal equilibrium with the box the disk must emit the same fraction $e = a$ relative to a truly black body, so that equilibrium is maintained, where e is the emissivity. Note that $e=a$ is just Kirchoff's law. Basically it says the same amount of radiation emitted by the partially absorbing disk is exactly equal to that which is absorbed (not reflected). This is a necessary condition for equilibrium.

Thought experiments like these were crucial in the formulation of thermodynamics of radiation in the 19th century, and the link with statistical mechanics.

Example – Microwave background: The most well studied black body spectrum in Nature is the cosmic microwave background (CMB) radiation. It represents a profound confirmation of the big bang theory of the universe. The CMB was first predicted in 1948 by Gamow and Alpher. In 1965, Arno Penzias and Robert W. Wilson from of Bell Labs accidentally discovered the CMB using a radiometer intended to use for radio astronomy and satellite communication experiments. Their instrument had an excess radiation level corresponding to 3.5 K.

The early universe was a far more symmetric and homogeneous place to live than it is today, albeit much hotter. At $10^{-5}s$ after the big bang it had a temperature of roughly $10^{12}K$ and was therefore a relativistic soup of fundamental particles including quarks, gluons W's Z's photons, leptons and still unknown constituents called dark matter.

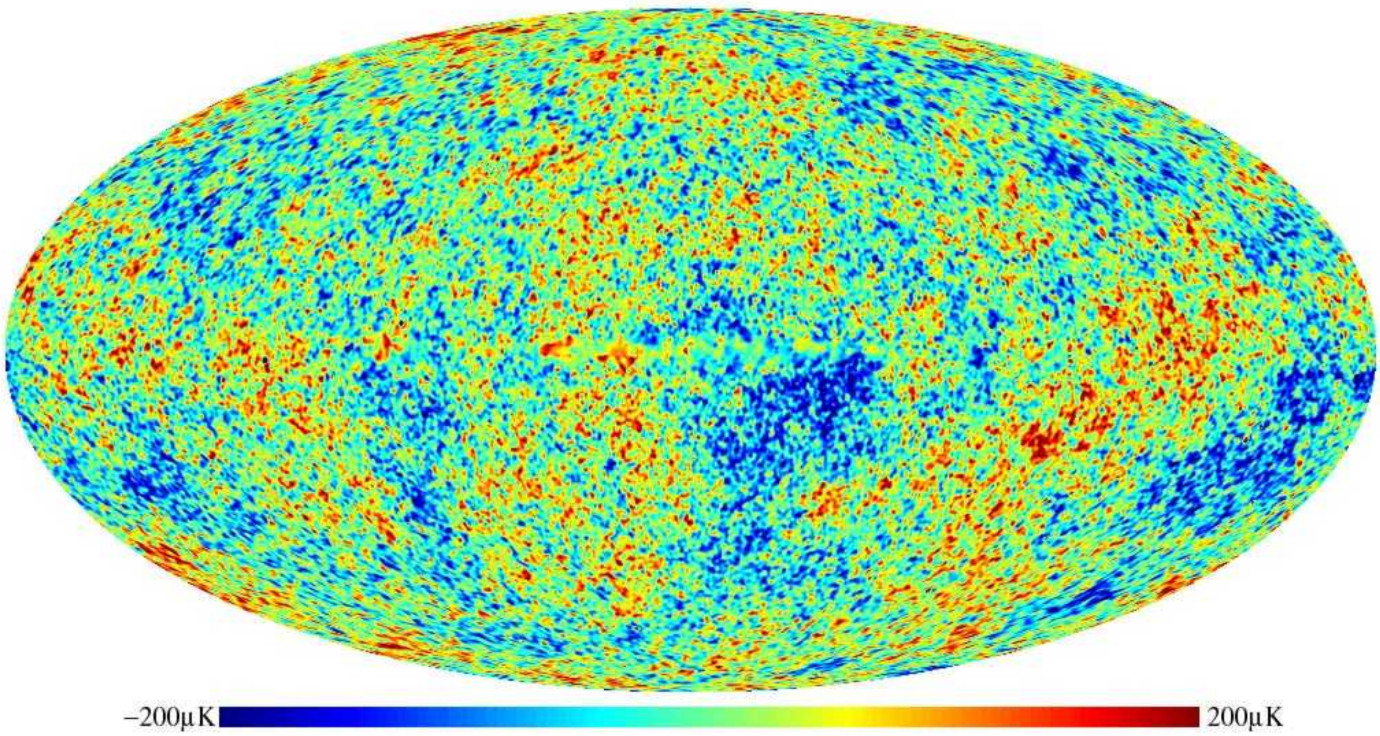
After 400,000 years the universe had become a plasma with temperature ~ 3000 K containing electrons, protons, and photons,along with a lot of dark matter and dark energy. One can imagine the universe at this time as an expanding box, filled with protons, electrons and photons in thermal equilibrium with each other. However, further expansion transformed this completely. Below 3000K the plasma of protons, electrons and photons condensed into a gas of neutral H atoms and photons plus other things that interact weakly with the photons. After this the photons in the box were effectively *decoupled* from the particles, since photons interact very weakly with neutral atoms (but very strongly with charged particles). Thus the photon mean free path went from being very short before the transition, to being essentially infinite after it. This situation is illustrated schematically here:



The photon gas in our 'box' universe then evolved in a very simple way: isentropic expansion (constant entropy) took place, whereby the number of photons in each mode stayed the same but the frequency of each mode decreased. One can imagine our box expanding slowly in such a way that the photon occupation numbers remained unchanged, but the fundamental frequency of each mode $\nu = c/2L$ decreased inversely with the dimension of the box. Thus the whole frequency spectrum was compressed by same factor as the box expanded. At the present time the temperature of the photon gas is 2.73K compared to about 3000K when decoupling first occurred.

What is remarkable about this is that by looking a the microwave background we are essentially looking at a very strongly red-shifted "snapshot" of the universe roughly 400,000 yrs after it was formed. Hence the great interest in looking at the microwave background in greater detail. It is incredibly uniform around the sky; but the small fractional variations in intensity (at the level of 10^{-5}) reveal the onset of density fluctuations in the universe at that time.

The most accurate measurements so far of these fluctuations have been done using the WMAP probe – the results are shown below:



The fluctuations of the intensity appear here as fluctuations in the temperature around the mean value of 2.73K. Analysis of these gives support to the “inflationary picture of the beginning of the universe – which is another story….