Physics 403 Problem Set 3

Model Solution

February 6, 2021

Problem 1. Photon Beams

Consider a blackbody of diameter 1 m at a temperature T contained in a spherical shell with a surface a distance 10 m away from the center of the blackbody. Suppose we colimate the light emitted by this body through a circular hole of radius 1 cm in the shell and that we apply a filter of wavelength $\lambda_0 = 5890$ angstroms with wavelength bandwidth $\Delta\lambda = 0.01$ angstroms.

(a) What temperature must the blackbody be to have a power of 10^{-4} W in the beam?

Solution: The blackbody radiates energy as photons, and the spectral energy density u_s of these photons is given by the Planck distribution:

$$u_s(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/k_B T} - 1} \tag{1.1}$$

A region of area dB on the blackbody radiates a spectral power dp given by:

$$dp(\nu) = \frac{cu_s(\nu)}{4}dB \tag{1.2}$$

The blackbody is a convex object, and therefore radiates outwards. Furthermore, it is spherically symmetric, so the radiation for each region dB will be identical. Thus, the total power P_b radiated by the blackbody will be the differential power per unit area multiplied by the surface area of the blackbody, that is:

$$P_b = 4\pi r_b^2 \frac{dp}{dB} \tag{1.3}$$

where r_b is the radius of the blackbody. Next, we know by the conservation of energy, along with the fact that the blackbody radiates outwards, that the total power emitted by the blackbody will be the total power observed at the outer shell. Furthermore, the spherical symmetry of the blackbody and the outer shell tells us that the power observed will be spherically symmetric over the surface area of the shell as well. Thus, the spectral power dP_s observed at a region dA on the boundary of the shell will be:

$$dP_s = \frac{P_b}{4\pi r_s^2}dA \tag{1.4}$$

where r_s is the radius the shell. This does not depend on dA, so we can trivially integrate over the area of the hole, and obtain the result:

$$P_s = \frac{P_b r_h^2}{r_s^2} \tag{1.5}$$

where r_h is the radius of the hole. Using our previous expressions to write this in terms of the frequency, we find:

$$P_s(\nu) = \frac{8\pi^2 h\nu^3 r_h^2 r_b^2}{c^2 r_s^2 (e^{h\nu/k_B T} - 1)} \tag{1.6}$$

It follows that the amount of power radiated in the infinitesimal region $(\nu, \nu + d\nu)$ is given by:

$$P_s(\nu)d\nu = \frac{8\pi^2 h\nu^3 r_h^2 r_b^2}{c^2 r_s^2 (e^{h\nu/k_B T} - 1)}d\nu \tag{1.7}$$

We shall now convert this into an expression in terms of wavelengths. We have the transformation rule $\nu \mapsto c/\lambda$, which we can differentiate to obtain the differential transformation $d\nu \mapsto c\,d\lambda/\lambda^2$. This tells us that:

$$P_s(\lambda)d\lambda = \frac{8\pi^2 hc^2 r_h^2 r_b^2}{r_s^2 \lambda^5 (e^{hc/k_B T\lambda} - 1)}d\lambda \tag{1.8}$$

Next, the filter blocks all of the light with wavelengths that are not in the regime $(\lambda_0 - \Delta\lambda, \lambda_0 + \Delta\lambda)$. Then, the total power P_r radiated through the hole is given by integrating our expression on this interval:

$$P_r = \frac{8\pi^2 hc^2 r_h^2 r_b^2}{r_s^2} \int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} \frac{d\lambda}{\lambda^5 (e^{hc/k_B T\lambda} - 1)} \tag{1.9}$$

We can now Taylor expanding the integrand to see thaat:

$$\int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} \frac{d\lambda}{\lambda^5 (e^{\beta hc/\lambda} - 1)} = \sum_{k=0}^{\infty} \left(\frac{d^k}{d\lambda^k} \frac{1}{\lambda^5 (e^{hc/k_B T\lambda} - 1)} \right)_{\lambda = \lambda_0} \int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} d\lambda \, \frac{(\lambda - \lambda_0)^k}{k!} \tag{1.10}$$

where we have used the fact that the term pulled out of the integral is evaluated at $\lambda = \lambda_0$, and therefore does not depend on the variable of integration. Evaluating the integral on the right hand side, we have:

$$\int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} \frac{d\lambda}{\lambda^5 (e^{\beta hc/\lambda} - 1)} = \sum_{k=0}^{\infty} \left(\frac{d^k}{d\lambda^k} \frac{1}{\lambda^5 (e^{hc/k_B T\lambda} - 1)} \right)_{\lambda = \lambda_0} \frac{1 + (-1)^k}{(k+1)!} \Delta\lambda^{k+1} \tag{1.11}$$

Notice that the bandwith $\Delta\lambda$ is much smaller than the central wavelength λ_0, so the quantity $\Delta\lambda/\lambda_0$ is small. Furthermore, every derivative will add a factor of λ^{-1}. Furthermore, the odd terms vanish, so each higher order terms will add a factor of $(\Delta\lambda/\lambda_0)^2$. We can subsequently neglect all terms other than the lowest order one, and we find that:

$$\int_{\lambda_0 - \Delta\lambda}^{\lambda_0 + \Delta\lambda} \frac{d\lambda}{\lambda^5 (e^{hc/k_B T\lambda} - 1)} \approx \frac{2\Delta\lambda}{\lambda_0^5 (e^{hc/k_B T\lambda_0} - 1)} \tag{1.12}$$

Substituting this into our expression for the total power radiated through the hole, we find:

$$P_r = \frac{16\pi^2 hc^2 r_h^2 r_b^2 \Delta\lambda}{r_s^2 \lambda_0^5 (e^{hc/k_B T\lambda_0} - 1)} \tag{1.13}$$

Solving for T yields:

$$T = \frac{hc}{k_B \lambda_0} \left[\log\left(1 + \frac{16\pi^2 hc^2 r_h^2 r_b^2 \Delta\lambda}{P_r r_s^2 \lambda_0^5} \right) \right]^{-1} \tag{1.14}$$

2

Recall that the parameters in the previous expression have the following values:

$$h = 6.636 \times 10^{-34} \text{ J} \cdot \text{m} \qquad k_B = 1.38 \times 10^{-23} \text{ J/m} \qquad c = 3.0 \times 10^8 \text{ m/s}$$
$$\lambda_0 = 5.890 \times 10^{-7} \text{ m} \qquad \Delta\lambda = 1 \times 10^{-11} \text{ m} \qquad r_h = 0.01 \text{ m}$$
$$r_b = 0.5 \text{ m} \qquad r_s = 10 \text{ m}$$

Substituting in these values and evaluating the result, we find:

$$\boxed{T = 6923 \text{ K}} \tag{1.15}$$

(b) *What would the difference be between this light and light from a laser with the same frequency and bandwith? How could you tell the difference in an experiment?*

Solution: Laser light is coherent; all of the photons are emitted together and therefore have the same phase. In a blackbody the emitted light is incoherent; each particle making up the blackbody emits photons independent of the others, and the resulting photons (even after collimation) do not have the same phase.

There are several experiments that can be used to distinguish between coherent and incoherent light source. One example is through the use of a Mach Zehnder interferometer, shown below:
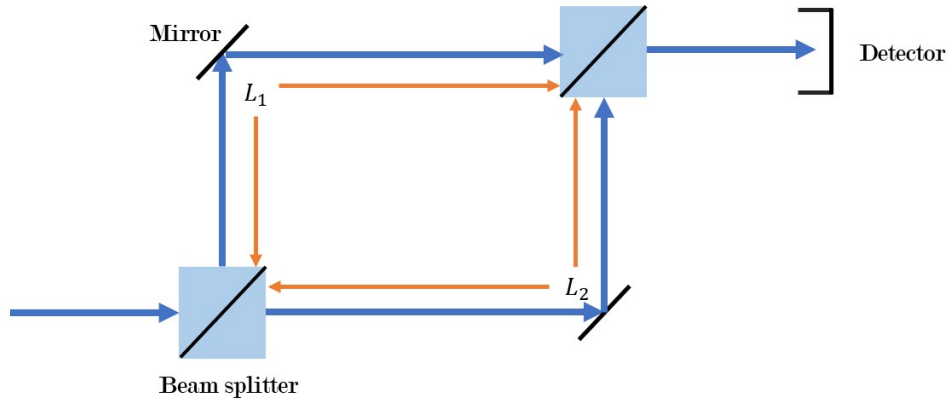
We shine the output light through a beam-splitter, reflect each resultant beam off a mirror, then recombine the beams with a second beam splitter. The resultant amplitude of the wave is measured with a photodetector. Suppose two beam paths L_1 and L_2 have a difference in length ΔL.

For the coherent light, the phase difference $\Delta\phi$ will be very close to $2\pi\Delta L/\lambda_0$. The amplitude of the output scales with $\cos^2(\Delta\phi)$, so we can tune this phase difference to maximize the amplitude (at $\Delta\phi = 0$) or minimize the amplitude (at $\Delta\phi = \pi$). For incoherent light, the the phase difference between two photons going through the interferometer will be $2\pi\Delta L/\lambda_0 + \phi$, where ϕ is the initial phase difference between the photons. However, the initial phase difference will be random for every pair of photons, so there will not be a consistent output amplitude. The reading from the detector will be the time average of the readings from sequential pairs of photons hitting it, so our measurement will be some constant independent of ΔL. □

Problem 2. Photons

(a) Explain under what circumstances and why the chemical potential of photons vanishes. Where do we expect to see photons with a non-vanishing chemical potential? Why? What is the density of states and internal energy for an n-dimensional photon gas?

Solution: Consider a system of particles with vanishing chemical potential, which is true if and only if the contribution to the differential free energy $\mu \, dN$ is zero. This is equivalent to saying that changing the number of particles and leaving all else equal leaves the free energy unchanged. Thus, for a system of bosons to have zero chemical potential, there must be some mechanism by which the number of particles can change independent of the other parameters of the system without changing the energy.

Let us now consider the specific case of photons. Photons are massless particles, which tells us that the energy of two photons with momenta $\hbar k_1$ and $\hbar k_2$ is the same as the energy of one photon with momentum $\hbar(k_1 + k_2)$. Then, as matter at thermal equilibrium can absorb photons of two frequencies and emit a single photon of a higher energy without having any of its properties change, photons will have zero chemical potential as long as they can freely interact with matter at thermal equilibrium.

Photons will have nonzero chemical potential when they cannot be created or annihilated without leaving the other parameters of the system unchanged, for example the number of other particles in the system. This occurs when the photons are produced by a photochemical reaction. In this case, changing the photon number and leaving all of the other parameters fixed would change the free energy of the system. Thus, $\mu \, dN$ and therefore μ would be nonzero. A well known example is the case of a semiconducting diode, where we can treat the photons as being produced by the chemical reaction of electrons in the conduction band and holes in the valence band.

Now consider the n-dimensional photon gas. In this case, the chemical potential vanishes, and the Bose distribution f_γ becomes:

$$f_\gamma(E) = \frac{1}{e^{\beta E} - 1} \tag{2.16}$$

We shall begin by determining the density of states g_γ for this gas in n-dimensions. We consider a gas constrained to a n-dimensional box with side length L. We have the condition that $\vec{k}$ must vanish at the boundary of the box. Then, each component k_j of the wave-vector must satisfy:

$$k_j = \frac{m_j \pi}{L} \tag{2.17}$$

for positive integers m_j. This tells us that each allowed wave-vector component must be separated by a minimum of π/L, so each allowed wave-vector occupies an n-dimensional volume $(\pi/L)^n$ in k-space. The number of states G with energy less than E is given by:

$$G(E) = \frac{1}{2^n} \int_0^{k(E)} d^n \vec{k} \tag{2.18}$$

where we divide by 2^n to account for the fact that the wave-vector modes are equivalent under sign changes in the components, so we only need to consider the modes where all components are positive. Writing this

in n-spherical coordinates, we have:

$$G(E) = \frac{1}{2^n} \int_0^{k(E)} dk \, d\Omega_{n-1} \, k^{n-1} \tag{2.19}$$

where $d\Omega_{n-1}$ is the solid angle of the $n-1$-sphere. The integrand is spherically symmetric, so the integral over the solid angle yields the volume of the unit $n-1$ sphere. Then, we can write:

$$G(E) = \frac{2\pi^{n/2}}{2^n \Gamma(n/2)} \int_0^{k(E)} dk \, k^{n-1} \tag{2.20}$$

> **Note:** The unit $n-1$ sphere is an $n-1$-dimensional surface, so its volume is the surface area of the unit $n-1$ ball. Checking the prefactor above for $n = 2$ and $n = 3$ yields the familiar results of 2π (the 1-dimensional "volume" of the circumference of the unit circle) and 4π (the 2-dimensional "volume" of the surface area of a sphere).

Next, we know that the energy E of a photon with wave-vector $\vec{k}$ is given by the relativistic equation:

$$E = \hbar c k \tag{2.21}$$

Then, we can rewrite our integral expression for $G(E)$ as:

$$G(E) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^E dE' \, \frac{(E')^{n-1}}{(2\hbar c)^n} \tag{2.22}$$

The degeneracy of states $g_\gamma(E)$ is defined to be the number of states in the system between E and $E+dE$. We recognize that this is the infinitesimal increase dG over an increase dE, or:

$$g_\gamma(E) = \frac{dG}{dE} \tag{2.23}$$

Using our expression for G, we see that:

$$\boxed{g_\gamma(E) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{E^{n-1}}{(2\hbar c)^n}} \tag{2.24}$$

The energy density U of the system is given by integrating the product of the energy and the number of particles at that energy level over all energies. The number of particles at a given energy level is the product of the degeneracy af states g_γ with the distribution f_γ. Then:

$$U = \int_0^\infty dE \, E g_\gamma(E) f_\gamma(E) \tag{2.25}$$

Using our expressions for f_γ and g_γ, we find:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n} \int_0^\infty dE \, \frac{E^n}{e^{\beta E} - 1} \tag{2.26}$$

We can shhall now evaluate this integral. Using the geometric series formula, we can write the denominator in the following way:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n} \int_0^\infty dE \sum_{k=1}^\infty E^n e^{-\beta k E} \tag{2.27}$$

Notice that the terms of the series are all positive and we know that it converges. Thus, it is absolutely convergent, and we can interchange the order of the series and the integral. This yields:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n} \sum_{k=1}^\infty \int_0^\infty dE \ E^n e^{-\beta k E} \tag{2.28}$$

We now use the substitution $u = \beta k E$, which yields:

$$U = \frac{2\pi^{n/2}}{\Gamma(n/2)(2\hbar c)^n \beta^{n+1}} \sum_{k=1}^\infty \frac{1}{k^{n+1}} \int du \ u^n e^u \tag{2.29}$$

The integral is the definition of the Γ function and the series is the definition of the ζ function. Writing the internal energy in terms of these quantities, we find:

$$U = \frac{2\pi^{n/2}\Gamma(n+1)\zeta(n+1)}{\Gamma(n/2)(2\hbar c)^n \beta^{n+1}} \tag{2.30}$$

Rewriting this expression in terms of the temperature T, we find:

$$\boxed{U = \frac{2\pi^{n/2}\Gamma(n+1)\zeta(n+1)(k_B T)^{n+1}}{\Gamma(n/2)(2\hbar c)^n}} \tag{2.31}$$

(b) *Consider now a two dimensional system of area A with density of states per unit area $g(\epsilon) = g_0\epsilon$. Find expressions for the grand canonical partition function per unit area, the total energy, and the pressure. What is the relationship between the total energy, the pressure, and the area?*

Solution: The photon gas has no chemical potential, so the partition function Ξ is given by:

$$\Xi = \sum_{\text{states } s} e^{-\beta E_s} \tag{2.32}$$

Because our volume is finite, there are countably many single photon states, which we can enumerate $1, 2, ...$. Let E_i denote the energy of state i, and let n_i denote the number of photons in the state i. Then, we can associate each state to the list $(n_1, n_2, ...)$ for the n_i being non-negative integers. The energies for the state $(n_1, n_2, ...)$ will be $\sum_i n_i E_i$, so the partition function is:

$$\Xi = \sum_{(n_1, n_2, ...)} \exp\left(-\beta \sum_{i=1}^\infty n_i E_i\right) \tag{2.33}$$

We can rewrite the exponential in product form in the following way:

$$\Xi = \sum_{(n_1, n_2, ...)} \prod_{i=1}^\infty e^{-\beta n_i E_i} \tag{2.34}$$

6

From this equation, we see that the partition function is a sum of the products of $e^{-n_i \beta E_i}$ for all possible n_i. We can subsequently factor it in the following way:

$$\Xi = \prod_{i=1}^{\infty} \sum_{n_i=0}^{\infty} e^{-\beta n_i E_i} \tag{2.35}$$

Using the geometric series formula, we can write this as:

$$\Xi = \prod_{i=1}^{\infty} \frac{1}{1 - e^{-\beta E_i}} \tag{2.36}$$

Taking the logarithm of both sides and using the properties of logarithms yields:

$$\log \Xi = -\sum_{i=1}^{\infty} \log\left(1 - e^{-\beta E_i}\right) \tag{2.37}$$

We shall now rewrite this in terms of the allowed photon frequencies. Let $\epsilon_1, \epsilon_2, ...$ be the allowed energies. For each allowed state energy E_i, there must be some allowed energy $\epsilon_j = E_i$. However, this association is not distinct – there are multiple allowed states for each frequency, the multiplicity of which is given by the density of states per unit area. Then, for each ϵ_j, there will be $g(\epsilon_j)$ allowed states, so each ν_j will show up $g(\epsilon_j)$ times in our series expression for $\log \Xi$. We can therefore rewrite our previous expression for $\log \Xi$ as:

$$\log \Xi = -\sum_{j=1}^{\infty} g(\epsilon_j) \log\left(1 - e^{-\beta \epsilon_j}\right) \tag{2.38}$$

We assume that our area is very large, which means the spacing between the frequencies will be small. We can then take this series into the continuum limit, which yields:

$$\log \Xi = -g_0 \int_0^{\infty} d\epsilon \; \epsilon \log\left(1 - e^{-\beta \epsilon}\right) \tag{2.39}$$

where we have substituted in our expression for g. We shall now evaluate this integral. Writing the logarithm function as a Taylor series, we find:

$$\log \Xi = g_0 \int_0^{\infty} d\epsilon \; \epsilon \sum_{k=1}^{\infty} \frac{e^{-k \beta \epsilon}}{k} \tag{2.40}$$

This series converges absolutely to the logarithm function, so we can exchange the order of the series and the integral. This yields:

$$\log \Xi = g_0 \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{\infty} d\epsilon \; \epsilon e^{-k \beta \epsilon} \tag{2.41}$$

Integrating by parts, gives us the following equation:

$$\log \Xi = g_0 \sum_{k=1}^{\infty} \frac{1}{\beta k^2} \int_0^{\infty} d\epsilon \; e^{-k \beta \epsilon} \tag{2.42}$$

7

Evaluating the integral gives us the following expression:

$$\log \Xi = \frac{g_0}{\beta^2} \sum_{k=1}^{\infty} \frac{1}{k^3} \tag{2.43}$$

This series is the definition of the zeta function ζ evaluated at 3, so we conclude that:

$$\boxed{\Xi(T) = \exp\left(g_0 \zeta(3)(k_B T)^2\right)} \tag{2.44}$$

Because the chemical potential for the photon gas is zero, the internal energy density u is given by:

$$u = -\frac{1}{\Xi}\frac{\partial \Xi}{\partial \beta} \tag{2.45}$$

Substituting in our expression for Ξ, we find:

$$u = -\exp\left(-g_0 \zeta(3)(k_B T)^2\right) \left(\frac{\partial \beta}{\partial T}\right)^{-1} \frac{\partial}{\partial T} \exp\left(g_0 \zeta(3)(k_B T)^2\right) \tag{2.46}$$

Which evaluates to the following expression:

$$u = 2g_0 \zeta(3)(k_B T)^3 \tag{2.47}$$

The total internal energy U is just the internal energy density multiplied by the volume of the system. In this case, we have a two-dimensional system, so the volume of the system is an area A. Thus, we conclude that:

$$\boxed{U = 2A g_0 \zeta(3)(k_B T)^3} \tag{2.48}$$

Next, we shall determine the pressure of the system. The first law of thermodynamics tells us that when the chemical potential vanishes, the differential internal energy is given by:

$$dU = T\,dS - P\,dA \tag{2.49}$$

where we have used the fact that the 2-dimensional volume in this system is the area A. Taking the derivative with respect to A while keeping S constant, we find:

$$P = -\frac{\partial U(S, A)}{\partial A} \tag{2.50}$$

Notice that our expression for U is a function of the volume and the temperature, not the volume and the entropy. We must now transform U into a function of the entropy. Once more, we can use the first law of thermodynamics to tell us that:

$$T = \frac{\partial U(S, A)}{\partial S} \tag{2.51}$$

We can now rearrange our expression for the internal energy to solve for the temperature, which yields:

$$T = \frac{1}{k_B}\left(\frac{U}{2A g_0 \zeta(3)}\right)^{1/3} \tag{2.52}$$

8

Substituting this into our previous equation, we obtain the differential equation:

$$\frac{\partial U}{\partial S} = \frac{1}{k_B} \left(\frac{U}{2Ag_0\zeta(3)} \right)^{1/3} \tag{2.53}$$

This equation can be solved using separation of variables. Going through this process, we find:

$$U(S, A) = \left(\frac{2S}{3k_B} \right)^{3/2} \frac{1}{\sqrt{2Ag_0\zeta(3)}} \tag{2.54}$$

Now, we can differentiate this with respect to A to find:

$$\frac{\partial U}{\partial A} = -\sqrt{\frac{1}{g_0\zeta(3)}} \left(\frac{S}{3k_BA} \right)^{3/2} \tag{2.55}$$

We can now rearrange our expression for $U(S, A)$ to solve for the entropy, which yields:

$$S = 3k_B \left(\frac{Ag_0\zeta(3)U^2}{4} \right)^{1/3} \tag{2.56}$$

Substituting this into our expression for $\partial U/\partial A$, we find:

$$\frac{\partial U}{\partial A} = -\frac{U}{2A} \tag{2.57}$$

In terms of the temperature, this becomes:

$$\frac{\partial U(S, A)}{\partial A} = -g_0\zeta(3)(k_BT)^3 \tag{2.58}$$

Using our expression for the pressure, we recognize that:

$$\boxed{P = g_0\zeta(3)(k_BT)^3} \tag{2.59}$$

Writing this in terms of the internal energy U, we see that:

$$\boxed{P = \frac{U}{2A}} \tag{2.60}$$

Problem 3. The Influence of the Density of States

(a) *Consider a set of N bosons, where N is conserved, and Suppose that the density of states is given by:*

$$g(\epsilon) = g_0\sqrt{\epsilon} \tag{2.61}$$

where g_0 is some constant. Write down an expression for the number density ρ of particles for this system, and explain why it will show Bose condensation. Find the temperature T_c at which Bose condensation occurs, and find the temperature dependence of the condensate density ρ_s as a function of the temperature

for $T < T_c$.

Solution: The particle density ρ in the system is given by integrating the denisty of particles n with an energy ϵ over all energies added to the density of particles ρ_0 in the ground state. That is:

$$\rho = \rho_0 + \int_0^\infty d\epsilon \; n(\epsilon) \tag{2.62}$$

The ground state is the zero energy state with zero momentum, which we can take to have degeneracy g_g. Thus, ρ_0 is just the value of the Bose distribution at $E = 0$ multiplied by g_g. The density of particles $n(\epsilon)$ with an energy ϵ is given by the product of the density of states and the Bose distribution. Then, our expression for ρ is:

$$\rho = \frac{g_g}{e^{-\beta\mu} - 1} + \int_0^\infty d\epsilon \; \frac{g_0\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)} - 1} \tag{2.63}$$

Using the geometric series formula, we can write the denominator of the integrand as a series in the following way:

$$\rho = \frac{g_g}{e^{-\beta\mu} - 1} + g_0 \sum_{k=1}^\infty \int_0^\infty d\epsilon \; \sqrt{\epsilon}e^{k\beta(\mu-\epsilon)} \tag{2.64}$$

We can now use the substitution $u = k\beta E$ to rewrite this integral as:

$$\rho = \frac{g_g}{e^{-\beta\mu} - 1} + \frac{g_0}{\beta^{3/2}} \sum_{k=1}^\infty \frac{e^{k\beta\mu}}{(k\beta)^{3/2}} \int_0^\infty du\sqrt{u}e^{-u} \tag{2.65}$$

We recognize that the integral is a definition for the Γ-function evaluated at $3/2$, which is equal to $\sqrt{\pi}/2$. We also recognize that the series is the definition of the polylog function $\mathrm{Li}_{\frac{3}{2}}$. Then, we can rewrite this expression as:

$$\boxed{\rho = \frac{g_g}{e^{-\beta\mu} - 1} + \frac{g_0\sqrt{\pi}}{2\beta^{3/2}}\mathrm{Li}_{\frac{3}{2}}\left(e^{\beta\mu}\right)} \tag{2.66}$$

We know that the chemical potential μ cannot be positive, so $e^{\beta\mu}$ is restricted to the interval $(0, 1]$. The polylogarithim function is bounded on this interval, so only finitely many particles can be added to the excited states of the system at any finite temperatures. If the particle density N/V is above this bound, all additional particles must be added to the ground states. Thus, a Bose Einstein condensate will form.

The condensate will form when the density of particles in the system $\rho = N/V$ exceeds the upper bound on the excited state occupation. We shall now determine this bound. Notice that each term in the polylog function scales as $e^{k\beta\mu}$, so it will increase with the chemical potential μ. Then, the excited state occupation is maximized at $\mu = 0$. In this limit, the polylog function becomes $\zeta(3/2)$, so the excited state occupation ρ_e has the bound:

$$\rho_e \leq \frac{g_0\zeta(3/2)\sqrt{\pi}}{2\beta^{3/2}} \tag{2.67}$$

The Bose condensate will then form when the particle density increases above this limit, that is when:

$$\rho \geq \frac{g_0\zeta(3/2)\sqrt{\pi}}{2\beta^{3/2}} \tag{2.68}$$

10

Writing this in terms of the temperature then solving for T, we find:

$$T \leq \frac{1}{k_B \pi^{1/3}} \left(\frac{2\rho}{g_0 \zeta(3/2)} \right)^{2/3} \tag{2.69}$$

We conclude thate the critical temperature is given by:

$$\boxed{T_c = \frac{1}{k_B \pi^{1/3}} \left(\frac{2\rho}{g_0 \zeta(3/2)} \right)^{2/3}} \tag{2.70}$$

For $T < T_c$, the excited state occupation will be maximized, and the condensate density ρ_s will be the particle density that is not in the excited states. Using our expression for the maximum bound of the excited state density, this becomes:

$$\boxed{\rho_s = \rho - \frac{g_0 \zeta(3/2)\sqrt{\pi}(k_B T)^{3/2}}{2}} \tag{2.71}$$

(b) *Consider a system of non-interacting bosons with a density of states is given by:*

$$g(\epsilon) = \frac{g_0}{E_1 - E_2} \Big(\Theta(\epsilon - E_1) - \Theta(\epsilon - E_2) \Big) \tag{2.72}$$

where Θ is the Heaviside step function, and where E_1 and E_2 are positive. Write down the expression for the particle density ρ. Suppose instead that we are dealing with Fermions where $E_1 < \mu < E_2$. What would be the relationship between the particle density ρ, the chemical potential μ, and g_0? Suppose that μ is given by:

$$\mu = \frac{E_2 + E_1}{2} \tag{2.73}$$

What is the zero temperature value of the particle density in terms of g_0? What is the total energy U of the system up to $\mathcal{O}(T^2)$, assuming that $k_B T$ is much less than $\mu - E_1$.

Following the same procedure as in the previous part, we can write the particle density as:

$$\rho = \frac{g_0}{E_1 - E_2} \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} \Big(\Theta(\epsilon - E_1) - \Theta(\epsilon - E_2) \Big) \tag{2.74}$$

where we have dropped the ground state density because the minimum energy for which we have allowed states is $\min\{E_1, E_2\}$. The heaviside functions restrict the bounds of integration to the region (E_1, E_2), so this expression becomes:

$$\rho = \frac{g_0}{E_1 - E_2} \int_{E_1}^{E_2} \frac{d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} \tag{2.75}$$

We shall now evaluate the integral. We see that:

$$\rho = \frac{g_0}{E_1 - E_2} \int_{E_1}^{E_2} d\epsilon \left(\frac{e^{\beta(\epsilon - \mu)}}{e^{\beta(\epsilon - \mu)} - 1} - 1 \right) \tag{2.76}$$

11

Next, we can use the substitution $u = e^{\beta(\epsilon - \mu)}$, which yields:

$$\rho = \frac{g_0}{E_1 - E_2}\left[\frac{1}{\beta}\left(\int_{e^{\beta(E_1-\mu)}}^{e^{\beta(E_2-\mu)}} \frac{du}{u-1}\right) - \left(\int_{E_1}^{E_2} d\epsilon\right)\right] \tag{2.77}$$

Evaluating each and simplifying yields:

$$\boxed{\rho = g_0\left[\frac{1}{\beta(E_1 - E_2)}\log\left(\frac{e^{\beta(E_2-\mu)} - 1}{e^{\beta(E_1-\mu)} - 1}\right) + 1\right]} \tag{2.78}$$

Next, we consider the case where we have a system of fermions. In this case, the distribution function becomes the Fermi-Dirac distribution. Then, our expression for the particle density becomes:

$$\rho = \frac{g_0}{E_1 - E_2}\int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}\Big(\Theta(\epsilon - E_1) - \Theta(\epsilon - E_2)\Big) \tag{2.79}$$

Once more, the step functions restrict the bounds of integration, which allows us to write the equation in the following way:

$$\rho = \frac{g_0}{E_1 - E_2}\int_{E_1}^{E_2} \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} + 1} \tag{2.80}$$

We now use the same procedure as in the previous part to evaluate the integral. We can rewrite the integral as:

$$\rho = \frac{g_0}{E_1 - E_2}\int_{E_1}^{E_2} d\epsilon\left(1 - \frac{e^{\beta(\epsilon-\mu)}}{1 + e^{\beta(\epsilon-\mu)}}\right) \tag{2.81}$$

Evaluating the integral, we find:

$$\boxed{\rho = g_0\left[\frac{1}{\beta(E_1 - E_2)}\log\left(\frac{1 + e^{\beta(E_1-\mu)}}{1 + e^{\beta(E_2-\mu)}}\right) - 1\right]} \tag{2.82}$$

Now, suppose that the chemical potential μ satisfies the relation:

$$\mu = \frac{E_1 + E_2}{2} \tag{2.83}$$

Substituting this into our expression for the particle density, we find:

$$\rho = g_0\left[\frac{1}{\beta(E_1 - E_2)}\log\left(\frac{1 + e^{\beta(E_1-E_2)/2}}{1 + e^{\beta(E_2-E_1)/2}}\right) - 1\right] \tag{2.84}$$

Factoring the quantity in the logarithm and simplifying, we find:

$$\rho = g_0\left[\frac{1}{\beta(E_1 - E_2)}\log\left(e^{\beta(E_1-E_2)/2}\right) - 1\right] \tag{2.85}$$

Which evaluates to the following expression:

$$\boxed{\rho = -\frac{g_0}{2}} \tag{2.86}$$

Next, we shall determine the internal energy U. We know that the internal energy is given by equation:

$$U = \int_0^\infty d\epsilon \ \epsilon g(\epsilon) f(\epsilon) \tag{2.87}$$

Substituting in our expression for the density of states for and using the Fermi-Dirac distribution, we find:

$$U = \frac{g_0}{E_1 - E_2} \int_{E_1}^{E_2} d\epsilon \frac{\epsilon}{e^{\beta(\epsilon-\mu)} + 1} \tag{2.88}$$

The Sommerfeld expansion tells us that up to second order in the temperature, we can write this equation as:

$$U = \frac{g_0}{E_1 - E_2} \left(\int_{E_1}^{\mu} d\epsilon \ \epsilon \right) + \frac{\pi^2 g_0 (k_B T)^2}{6(E_1 - E_2)} \left(\frac{d\epsilon}{d\epsilon} \right)_\mu + \mathcal{O}(T^3) \tag{2.89}$$

Dropping the higher order terms and evaluating the integral and the derivative, we find:

$$U = \frac{g_0(\mu^2 - E_1^2)}{E_1 - E_2} + \frac{\pi^2 g_0 (k_B T)^2}{6(E_1 - E_2)} \tag{2.90}$$

Substituting in our value for the chemical potential, we conclude that:

$$\boxed{U = \frac{\pi^2 g_0 (k_B T)^2}{6(E_1 - E_2)} - \frac{g_0(3E_1 + E_2)}{4}} \tag{2.91}$$

(c) Consider a system of non-interacting bosons with a density of states $g(\epsilon) = g_0$. Find an expression for ρ and show that the system will not show Bose condensation.

Using the same argument as in the previous parts, we know that the particle density ρ is:

$$\rho = \frac{g_g}{e^{-\beta\mu} - 1} + g_0 \int_0^\infty \frac{d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} \tag{2.92}$$

We can evaluate this integral by once again using the geometric series formula to convert the denominator into a geometric series, anuse the substituion $u = \beta k\epsilon$. This yields:

$$\rho = \frac{g_g}{e^{-\beta\mu} - 1} + g_0 \sum_{k=1}^\infty \frac{e^{k\beta\mu}}{k} \int_0^\infty du \ e^{-u} \tag{2.93}$$

The integral evaluates to 1, and we recognize that the series is the Taylor expansion of the logarithm function. Then, we see that:

$$\boxed{\rho = \frac{g_g}{e^{-\beta\mu} - 1} + g_0 \log\left(1 - e^{-\beta\mu}\right)} \tag{2.94}$$

We recognize that the excited state density diverges as μ goes to zero. Thus, a Bose condensate does not form. $\square$