PHYS 403: MID-TERM TEST

(March 4th, 2022)

This is a MID-TERM TEST. It will last 45 mins. The only material you will be allowed to use will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators. You should try to do **all four questions**.

QUESTION (1) PROBABILITY:

(i) Suppose we throw 2 dice, each of which may end up in a state with equal probability which shows one of the the numbers from 1 to 6. What is the probability that the sum of the scores for the 2 dice is 10?

(ii) Consider a system in which we have 6 spin-1/2 spins, which are not interacting with each other, and each with possible value $s_j^z = \pm 1/2$, where $j = 1, 2, ...6$. What are the possible values for the total polarization $S_z = \sum_j s_j^z$? If the applied field is zero, what is the probability that $S_z = 1$?

QUESTION (2) TWO-LEVEL SYSTEMS: Suppose we have a set of N non-interacting two-level systems, each of which can have energy $\epsilon_j = \pm\Delta_o$.

(i) The canonical partition function for a system with states of allowed energy E_j is $\mathcal{Z}(\beta) = \sum_j e^{-\beta E_j}$, where $\beta = 1/k_B T$ is the inverse temperature. What is the canonical partition function for each of the individual spins? What are the allowed energies E_j for the total system of N spins? ? And what is the canonical partition function for the total system of N spins??

(ii) The mean energy of the system is given by $U = \sum_j p_j E_j$, where p_j is the probability that the state with energy E_j is occupied. Show that the mean energy at temperature T is given by $U = -N\Delta_o \tanh \beta\Delta_o$.

QUESTION (3) HARMONIC OSCILLATOR: Suppose we have a 1-dimensional oscillator with energy levels $\epsilon_n = (n + \frac{1}{2})\hbar\omega_o$, where ω_o is the frequency of the oscillator.

(i) Show the canonical partition function is $\mathcal{Z}(\beta) = \frac{1}{2} cosech\,(\hbar\beta\omega_o/2)$, where $cosech\, x \equiv 2/(e^x - e^{-x})$.

(ii) As already noted in 2(ii) above, the mean energy $U = \sum_j p_j E_j$ for any thermodynamic system. Show that this implies that $U = \langle E \rangle = -\mathcal{Z}^{-1}(\partial\mathcal{Z}/\partial\beta)$. Given this, find an expression for the mean energy U of the oscillator.

QUESTION (4) HELIUM ATOMS: Consider a set of Helium atoms which can exist in one of 4 states: an atomic state with 2 bound electrons and energy $-E_o$, two singly-ionized state with one bound electron and energy $-E_1$, and a doubly-ionized state, in which it has lost both electrons, with energy $-E_2 = 0$. Note that there are two possible singly-ionized states because the electron that is lost can be either spin up or spin down.

(i) List the possible states for the Helium atom, with values of n and E_j for each, and then write down an expression for the grand-partition function $\Xi(\mu, \beta) = \sum_{j,n} \exp[\beta(\mu n - E_j(n))]$, where the E_j are the energies of the states and n the number of bound electrons.

(ii) Suppose that $E_o = 2\Delta_o$ and $E_1 = \Delta_o$. Then find a simple expression for $\Xi(\beta, \mu)$. What are the limiting values of this expression when $kT \to \infty$, and when $kT \to 0$?

END of MID-TERM TEST