PHYS 403 : MID-TERM
Feb 17th, 2023

ANSWERS

Q1 It was explained during the exam that the drunk took 3 steps, and ended up having advanced 2^+ metres forward. In spite of this, and of the clear wording of the question, quite a few people solved the different (and easier) problem in which 2 of the 3 steps taken were forward steps. So somewhat fewer marks were given for this answer, and for the same interpretation when n and N were substituted in place of 2 and 3.

(i) CORRECT INTERPRETATION : After 3 steps, advance 2^+ steps forward; this implies only one possibility, viz., that the drunk advanced 3 steps forward. The probability is

$$P_3(3) = (2/3)^3 = 8/27 \qquad 7$$

INCORRECT INTERPRETATION : 2^+ forward steps taken; then we have a total probability

$$P = P_3(2) + P_3(3)$$
$$= C^3_2 \left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right) + C^3_1\left(\tfrac{2}{3}\right)^3 = \left[\left(3 \times \tfrac{4}{27}\right) + \left(1 \times \tfrac{8}{27}\right)\right] = \frac{20}{27} \qquad 8$$

(ii) CORRECT INTERPRETATION : Advances n steps forward from origin (n can be $-$ve): Then we $(N-n)$ forward steps, $N-2n$ backward steps, and the probability is

$$P_N(N-n) = C^N_{\frac{N+n}{2}} \left(\frac{2}{3}\right)^{\frac{N+n}{2}} \left(\frac{1}{3}\right)^{\frac{N-n}{2}} \qquad 10+6$$

INCORRECT INTERPRETATION : One takes n forward steps, and $N-n$ backward steps, so that one advances $2n-N$ steps forward. The probability is then

$$\mathcal{P}_N(n) = C^N_n \left(\tfrac{2}{3}\right)^n \left(\tfrac{1}{3}\right)^{N-n} \qquad (8)$$

TOTAL : $\boxed{17+6}$ CORRECT INTERP.

$\boxed{14}$ INCORRECT INTERP.

2

(i) For each spin, $Z_1 = \left(e^{\beta\Delta_0} + e^{-\beta\Delta_0}\right)$ (2)

For N spins: $Z_N = \left(e^{\beta\Delta_0} + e^{-\beta\Delta_0}\right)^N$. (1)

(3)

(ii) The spins are independent, so their total energy is $U = \sum_j U_j = NU_1$, where U_1 is the energy of each spin.

For a single spin

$$U_1 = \sum_j p_j E_j = \frac{1}{Z_1}\left(\Delta_0 e^{-\beta\Delta_0} - \Delta_0 e^{\beta\Delta_0}\right)$$

$$= -\Delta_0 \frac{e^{-\beta\Delta_0} - e^{-\beta\Delta_0}}{e^{\beta\Delta_0} + e^{-\beta\Delta_0}} = -\Delta_0 \tanh \beta\Delta_0$$

So for N spins $U = -N\Delta_0 \tanh \beta\Delta_0$ (8)

(iii) For the $T=0$ system, all spins have ground state energy $-\Delta_0$, so $U = -N\Delta_0$ (1)

For the $T=\infty$ system, $P_\uparrow = P_\downarrow$ for each spin, so the net energy is $U=0$ (2)

Thus when the spins are combined the net energy is $-N\Delta_0$, with mean energy $-\frac{\Delta_0}{2}$ per spin. (4)

TOTAL: $\boxed{24}$

The final temperature will satisfy $-\frac{\Delta_0}{2} = -\Delta_0 \tanh\beta\Delta_0$, ie $\tanh\beta\Delta_0 = \frac{1}{2}$

Hence $\frac{\Delta_0}{kT} = \tanh^{-1}(\frac{1}{2})$ or $kT = \Delta_0 / \tanh^{-1}(\frac{1}{2})$ (6) (13)

Q3 (i) For SHO, $E_n = (n+\frac{1}{2})\hbar\omega_0$

so

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\frac{1}{2}\beta\hbar\omega_0} \sum_{n=0}^{\infty} e^{-\beta\hbar n\omega_0}$$

$$= \frac{e^{-\frac{1}{2}\beta\hbar\omega_0}}{1 - e^{-\beta\hbar\omega_0}} = \frac{1}{e^{\beta\hbar\omega_0/2} - e^{-\beta\hbar\omega_0/2}}$$

$$= \tfrac{1}{2}\operatorname{cosech}\frac{\beta\hbar\omega_0}{2} \qquad (6)$$

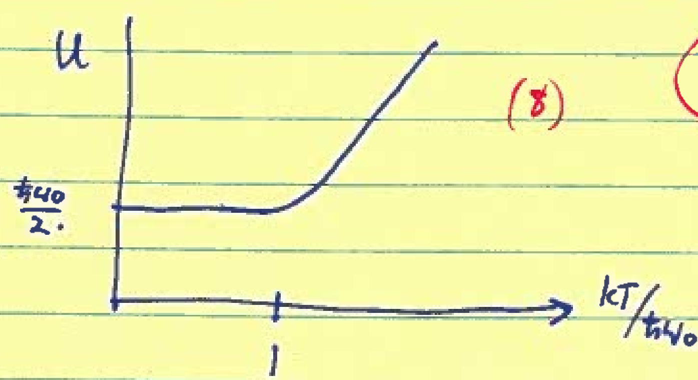
(ii) $$U = \sum_j p_j E_j = \frac{1}{Z}\sum_j E_j e^{-\beta E_j}$$

$$= -\frac{1}{Z}\frac{\partial}{\partial\beta}\sum_j e^{-\beta E_j} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} \qquad (2)$$

So $$U = \frac{1}{e^{\beta\hbar\omega_0/2} - e^{-\beta\hbar\omega_0/2}} \frac{\partial}{\partial\beta}\left(e^{\beta\hbar\omega_0/2} - e^{-\beta\hbar\omega_0/2}\right)$$

$$= \frac{\hbar\omega_0}{2}\,\frac{e^{\beta\hbar\omega_0/2} + e^{-\beta\hbar\omega_0/2}}{e^{\beta\hbar\omega_0/2} - e^{-\beta\hbar\omega_0/2}} = \frac{\hbar\omega_0}{2}\coth\left(\frac{\hbar\beta\omega_0}{2}\right) \qquad (8)$$

Finally: the graph is a function of $kT/\hbar\omega_0$, NOT $\frac{\hbar\omega_0}{kT}$; we get



(8) (18)

TOTAL : $\boxed{24}$

Q4

For an elastic string it is clear that if $dW = kl\,dl$, then we have

$$U = \int dU = \tfrac{1}{2} k l^2 \qquad (4)$$

and

$$F = U - TS \qquad (2)$$

Then we have
$$\left.\begin{aligned} dF &= -S\,dT + dU \\ &= -S\,dT + kl\,dl \end{aligned}\right\} (4)$$

TOTAL : $\boxed{10}$

TOTAL MARKS WERE $\boxed{75}$