PHYS 403: FINAL EXAM 2021 – MARKING SCHEME (SECTION A)

Each question nominally counts for 10 points. In some cases there will be bonus marks so it is possible for somebody to get more than 10 for a question, and it is possible for their total mark to exceed 40, for the 4 questions they have to answer. Please be flexible in marking and give a bonus where deserved.

No need whatsoever to provide any details or explanations of the marks you award - but you should indicate what marks you give for each part of the questions you mark, and also, wherever there is any ambiguity or subtlety about how you award a mark, make some sort of note that will allow you, at some later date, to recall your reasoning for awarding the mark that you did give.

Please (i) compile an excel page that shown which mark you gave to each student for each question, along with a final total, for section A, for each student; and then (ii) add this mark to the main Excel page which has all the marks for the different homework assignments.

SECTION A: SHORT QUESTIONS (ANSWER 4 of THESE)

QUESTION A.1: QUANTUM GASES

(i): Why does the diameter of a white dwarf decrease when its mass increases?

In a non-relativistic analysis, this is because the negative gravitational energy increases faster with mass (going as M^2) than the positive degeneracy energy increases (as $M^{5/3}$; hence incresing the mass decreases the radius at which the energy is minimized. Roughly speaking, the gravitational attraction becomes ever more important as M increases. The same is even more true in a relativistic analysis.

(ii) Why does the chemical potential of a gas (Bose, Fermi, or classical) never increase (and almost always decreases) as one raises the temperature?

Roughly speaking this is because the free energy $F = U - TS$ is governed by the behaviour of the entropy when T is large (and also when the density of the system is low); in both these cases the occupation numbers of the states are low. The entropy is increased by allowing the system to spread over more states. Since the chemical potential $\mu = (\partial F/\partial N)$, and adding a particle will cause an increase in entropy, we get $\mu < 0$.

One can see this purely mathematically as well, by examining the usual equation relating N to μ for a Fermi or Bose gas; at high T this can only be satisfied by allowing μ to decrease relative to the $T = 0$ value.

Part (i) gets **FOUR (4) Marks**: This is standard stuff - done in the lectures and in the notes. The explanation should clearly show how the result follows from the form for the sum of the gravitational and degeneracy pressures (ie., from minimizing the energy).

Part (ii) gets **SIX (6) marks**: This is a more subtle question which can be answered in various ways. I have given 2 ways. An answer in either of the ways way give above, if done properly, will get 6 marks. If someone gives both ways, and shows how they complement each other, then they get up to 2 bonus marks.

If any other way is used to to answer this question - ask me.

QUESTION A.2: SPECIFIC HEAT

(i): Consider a set of N non-interacting 2-level systems. What is the difference $\Delta S = [S(T = \infty) - S(T = 0)]$ for this system?

Answer is $\Delta S = N k_B \ln 2$. This can be found really simply by using $S = k_B \ln W$, or by working it out explicitly for this model.

(ii) Suppose we can approximate the specific heat $C_V(T)$ of this system by the simple formula

$$C_V(T) \;=\; C_o \left[1 - 4 \left(\frac{T - T_o}{T_o} \right)^2 \right]$$

for $T_o/2 < T < 3T_o/2$, and zero otherwise. Using the relation between the specific heat and the entropy, and the result you found for ΔS in (i) above, find the value of C_o.

We use $C_V(T) = T(dS/dT)_V$. It then follows that $\Delta S = \int_0^\infty dT\,(C_V(T)/T)$.
Doing the integral, we find that

$$\Delta S \;=\; Nk_B \ln 2 \;=\; \int_0^\infty dT\, \frac{C_V(T)}{T} \;\rightarrow\; C_o(4 - 3\ln 3)$$

so that we get finally that

$$C_o \;=\; Nk_B\, \frac{\ln 2}{4 - 3\ln 3}$$

Part (i) gets **THREE (3)marks**: Full 3 marks for correct answer, no matter how it is derived

Part (ii) gets **SEVEN (7) marks**: This is straightforward - there is really only one way to get the answer.

QUESTION A.3: 2-LEVEL SYSTEMS

(i): Consider a set of N non-interacting 2-level systems (TLS), with level energies E_1 and E_2 for each of the TLS. At temperature T, what is the average energy $U(T)$ for the total system? Derive also the specific heat $C_V(T)$.

The average energy for the system is easy, since there are only 2 levels; we have

$$U(T) \;=\; N\frac{E_1 e^{-\beta E_1} + E_2 e^{-\beta E_2}}{e^{-\beta E_1} + e^{-\beta E_2}} \;=\; N\frac{E_1 + E_2 e^{-\beta \Delta_o}}{1 + e^{-\beta\Delta_o}}$$

where $\Delta_o = |E_1 - E_2|$. The specific heat is easily found from this to be

$$C_V(T) \;=\; N\left(\frac{\Delta_o}{kT}\right)^2 \frac{e^{-\Delta_o/kT}}{(1 + e^{-\Delta_o/kT})^2}$$

(ii) Find expressions for $U(T)$ and $C_V(T)$ when $kT \gg |E_1 - E_2|$. You should find the $T = \infty$ result, and also the first correction to this result, for finite (but very large) T.

The high-T limiting result for the energy is obtained by expanding the exponent for small β; the result is

$$U(T) \;=\; N\left[\tfrac{1}{2}(E_1 + E_2) - \frac{\Delta_o^2}{4kT}\right] \qquad\qquad (T \rightarrow \infty)$$

and this just gives the specific heat in this limit as

$$C_V(T) \;=\; \frac{N}{4}\left(\frac{\Delta_o}{kT}\right)^2 \qquad\qquad (T \rightarrow \infty)$$

Part (i) gets **SIX (6) marks**: give 4 marks for the first part and 2 for the 2nd.

Part (ii) gets **FOUR (4) marks**:

QUESTION A.4: FERMI DISTRIBUTION

(i): The grand canonical partition function for a single fermion state of energy ϵ is $z(\epsilon) = \sum_n \exp[n\beta(\mu - \epsilon)] = 1 + \exp[\beta(\mu - \epsilon)]$. Show that the mean occupation number for this state is just the Fermi function, ie., that $\langle n \rangle \rightarrow f(\epsilon - \mu) \equiv \{1 + \exp[\beta(\epsilon - \mu)]\}^{-1}$, which we also write as $f(x) = [1 + e^{\beta x}]^{-1}$, where $x = (\epsilon - \mu)$.

This is a standard exercise, done in the course notes. We write for the probability $P(n) = z^{-1} e^{\beta n(\mu - \epsilon)} \;=\; z^{-1} e^{n\beta x}$, and then the expectation value $\langle n \rangle \;=\; z^{-1} \sum_n nP(n)$, which since $n = 0$ or $n = 1$ is just

$$\langle n \rangle \;=\; \frac{1}{[1 + \exp[\beta(\epsilon - \mu)]}$$

(ii) Then show that the probability of finding n particles in this state is

$$p(n) \; = \; \frac{[1 - f(-x)]^n}{[f(-x)]^{n-1}}$$

This is a little finicky because of the "$-$" signs. We have $P(n) \; = \; z^{-1} e^{\beta n(\mu - \epsilon)} \equiv e^{-n\beta x}/[1 + e^{-\beta x}]$. Now compare with the expression given above

$$P(n) \; = \; \frac{(1 - f(-x))^n}{f(-x)^{n-1}} \;\; \rightarrow \;\; \frac{e^{-n\beta x}}{1 + e^{-\beta x}} \qquad\qquad QED.$$

Part (i) gets **FOUR (4) marks**: This is a derivation done in the notes and lectures.

Part (ii) gets **SIX (6) marks**: This question caused huge problems because students were confused by the notation - I had to change the wording of the question several times. If any questions arise about how to mark this, ask me. Students who really understand things should get bonus marks.

QUESTION A.5: INTERATOMIC POTENTIAL

(i): Consider the 1-dimensional potential

$$V(x) \; = \; V_o \left[\left(\frac{a_o}{x} \right)^{12} - 2 \left(\frac{a_o}{x} \right)^6 \right]$$

Find the value of x for which $V(x)$ is a minimum, and find the "curvature" d^2V/dx^2 at this point. What is the frequency of small oscillations of a particle of mass M about the minimum in this potential?

The minimum of the potential, given by the value of x for which $V(x)$ is minimized, is at $x = a_o$. The 2nd derivative is

$$\frac{d^2V}{dx^2} \; = \; \frac{12V_o}{x^2} \left[13 \left(\frac{a_o}{x} \right)^{12} - 7 \left(\frac{a_o}{x} \right)^6 \right] \Bigg|_{x=a_o} \;\; \rightarrow \;\; 72 \frac{V_o}{a_o^2}$$

This SHO potential then has a small oscillation frequency ω_o, for a mass M, given by $\omega_o^2 = 72(V_o/a_o^2 M)$.

(ii) Draw a picture of the potential $V(x)$, and explain briefly how it can be used to model interatomic interactions. For such interaction, what do you think are typical values for V_o and a_o?

The key here is to point out that the short range repulsion is from the Pauli principle, and the long-range attraction is van der Waals; such forces exist between all atomic clouds. Typically $V_o \sim 2 - 10 \times 10^{-4}$ eV, ie., $V_o \sim 2 - 10$ K; and a_o is a few Angstroms.

Part (i) gets **SIX (6) marks**: Give 2 marks for the 1st part and 4 marks for the 2nd.

Part (ii) gets **FOUR (4) marks**: The correct numbers should get 2 marks, and an explanation 2 marks. If they correctly identify/understand the role of the van der Waals and Pauli principle, then one bonus mark.

QUESTION A.6: ARGON in ATMOSPHERE

(i): Roughly 1 percent of the volume of the earth's atmosphere is composed of ^{40}Ar. Suppose you are in a bedroom with a volume of 60 m^3. Roughly how many ^{40}Ar atoms are in the room, and what is their total mass?

I did say that they should estimate this "roughly". So if somebody says something like the following:

"10^3 kg of water, with molecular weight 18, occupies 1 m^3. Atmospheric pressure supports a column of water roughly 10m high (this is easy to show - atmospheric pressure is $\sim 10^5 Nm^-2$ and $g \sim 10 \; ms^{-2}$, so atmospheric pressure is equivalent to a weight $\sim 10^4 \; kgm^{-2}$). The scale height of the atmoshere is ~ 5 km, so the atmospheric pressure is equivalent to that coming from a 10 km column of air at atmospheric density. Thus 1 kg of air occupies roughly 1 m^3. The atomic weights of N_2 and O_2 molecules are 28 and 32 respectively, so the mass fraction of air coming from ^{40}Ar is roughly $4/3 \times 0.01 \sim 0.013$, with total mass $\sim$ 13g, ie., roughly 1/3 mole of ^{40}Ar. Since Avogadro's number is 6×10^{23}, this implies we have roughly 2×10^{23} atoms of ^{40}Ar per cubic metre. Multiplying by 60, we then get a number $N \sim 1.2 \times 10^{25}$ atoms in the bedroom, with mass of roughly 0.78 kg."

There will be many ways to get this rough answer, so I won't try to guess them all.

(ii) In MKS units, roughly what is the total thermal energy associated with the ^{40}Ar atomic motion?

The energy per atom is $3k_BT/2$. Let's assume room temperature (ie., $T \sim$ 300K). Then, using $k_B = 1.38 \times 10^{-23} m^2 kg s^{-2} K^{-1}$, we have a total thermal energy in the room given by $E = 3Nk_BT/2 \sim 7.45 \times 10^4$J, at room temperature.

Part (i) gets **SEVEN (7) marks**: There are so many ways to get the right answer here - you should be flexible in how you mark it. Note that the question ius asking for an ESTIMATE. My derivation is perhaps idiosyncratic. Contact me if you have questions.

Part (ii) should get **THREE (3) marks**: Note that a mistake in getting N in the first question should not penalize them in the 2nd part.

QUESTION A.7: NEGATIVE TEMPERATURE

(i): A set of non-interacting or very weakly interacting spin-1/2 spins has an entropy which looks roughly like $S(U) = S_o - \alpha U^2$, for $U^2 < \alpha$, as a function of the total energy U, and is zero for $U^2 > \alpha$. From the definition of temperature T in terms of S and U for a system in equilibrium, find U in terms of T, and sketch a graph of it.

The definition of temperature T tells us that $T = (dU/dS)_V \rightarrow -(1/2\alpha U)$. It then follows that $U(T) = -1/2\alpha T$. Here $\alpha > 0$, and the graph is just a set of hyperbolae, which diverge to $U \rightarrow \pm\infty$ as $T \rightarrow -\infty, +\infty$ respectively. This is clearly unphysical, and is cutoff at $U^2 \sim \alpha$, ie., for $|T| = T_o \sim 1/2\alpha^{3/2}$, at which point we expect to see plateaux in the energy at values $U_o = \pm\alpha^{1/2}$. Thus in the range $T_o > T > -T_o$, the curve for $U(T)$ flattens out.

(ii) What is the specific heat of this system, in the temperature range $-\infty < T < \infty$? How do you interpret this result for $T < 0$?

The specific heat $C_V(T) \sim dU/dT$, which here gives, for $|T| > T_o$, the result $C_V(T) \sim 1/2\alpha T^2$. In the range $T_o > T > -T_o$, where the curve for $U(T)$ flattens out, we expect $C_V(T)$ to fall rapidly to zero for $|T| \ll T_o$.

Part (i) gets **FIVE (5) marks**: The algebra is straightforward - what is more subtle is the realization that the hyperbolae have to be cut off as one approaches small $|T|$. For this latter point give 2 out of the 5 marks.

Part (ii) gets **FIVE (5) marks**: Same again - two marks for understanding the plateau.

There is scope for bonus marks in this question, since the whole plateau business is subtle. They may well compare with the exact answer, known from stat mechanics - refer to me if you have questions here.

QUESTION A.8: RADIATION PRESSURE

The radiation pressure p from photons is equal to $p = 4J/3c$, where J is the radiation flux. A star like the sun emits black-body radiation with flux $J = \sigma T^4$ per unit area of its surface, where temperature T is measured in Kelvin units; here $\sigma = 5.67 \times 10^{-8} W m^{-2} K^{-4}$, and the sun's surface temperature is $6,000\ K$. The radius of the sun is $R_S \sim 0.7 \times 10^6$ km.

(i): Consider the forces on an electron at the sun's surface. If the cross-section for photon-electron scattering is $\sim 6.6 \times 10^{-29}\ m^2$, and the electron mass is $\sim 9 \times 10^{-31}$ kg, then how do the gravitational and radiation forces on the electron at the sun's surface compare (assume here that all the photon energy is taken up by the electron)? You can assume that the solar mass is 2×10^{30} kg, and that the gravitational constant $G = 6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2}$.

Here is what I find. The radiation flux at the sun's surface is $J(R_S) = \sigma T^4$, which I get to be $J(R_S) = 4.35 \times 10^7\ W m^{-2}$. This gives a pressure $p(R_S) = 4J(R_S)/3c = 0.327 N m^{-2}$, using a value $c = 2.998 \times 10^8\ m s^{-1}$. On an electron, with cross-section $6.6 \times 10^{-29}\ m^2$, this gives a force $f_p(R_S) \sim 2.157 \times 10^{-29}\ N$. The gravitational force is just $f_g(R_S) = -GM_S m_e/R_S^2$, which using the numbers given (and a mass $m_e = 9.11 \times 10^{-31}$kg for the electron) the result $f_g(R_S) = -2.45 \times 10^{-28}\ N$. Thus in this calculation the total force $f(R_S) = f_p(R_S) + f_g(R_S)$ is almost entirely gravitational (in reality, charged particles are accelerated away from the sun by shock waves and coronal mass ejections).

(ii) How do the radiation force and gravitational force on the electron behave as a function of the distance r from the sun (for $r > R_o$)?. What then is the equation of motion for $r(t)$, and what is its solution as a function of time, if the electron starts at a distance $r_o = r(t = 0)$ from the sun?

Both the radiation force and the gravitational force fall off like $1/r^2$. The radial equation of motion of the particle is then just $m\ddot{r} = -f_g(r)$, neglecting the radiation force, so that $\ddot{r}(t) = -GM_S/r^2$. Although this is not part of the course, the easiest way to solve this problem is to use the conservation of total energy for the electron. I did not specify the initial velocity, but suppose that at $r = r_o$, we have $\dot{r} = 0$. The we have the result, by conservation of energy, that

$$\left(\frac{dr}{dt}\right)^2 = 2GM_S\left(\frac{1}{r(t)} - \frac{1}{r_o}\right) \qquad \Longrightarrow \qquad t = \frac{1}{(2GM_S)^{1/2}}\int_{r_o}^{r(t)} dr\left[\frac{1}{r} - r_o^{-1}\right]^{-1/2}$$

and I don't expect anyone to do more than this.

Part (i) gets **FIVE (5) marks**: Give 3 for the radiation force and 2 for the gravitational force.

Part (ii) gets **FIVE (5) marks**: Give 1 mark for the equation of motion and 1 mark for the recognition that both forces fall of as $1/r^2$. Then give 3 marks for a basic understanding of how to solve the equation of motion. There is scope for bonus marks here, depending on ho far they get with solving the equation of motion. If they get as far as I did above, they should get 1 bonus mark. They may go even further - refer to me if you have questions about this.