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PHYS 403: MID-TERM TEST

(10.00 hrs, Feb 17th, 2023)

This is a MID-TERM TEST. It will last 45 mins. The only material you will be allowed to use will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators. You should try to do **all four questions**.

QUESTION (1) PROBABILITY: A drunk is taking random steps, one step every 5 seconds, of length 1 metre each step, along a road. At each step, with probability 2/3 he takes a step forward, with probability 1/3 a step backwards.

(i) After 15 seconds, what is the probability that the drunk will have walked 2 or more metres forward?

(ii) After N steps, what is the probability distribution $P_n(N)$ that the drunk will have walked n steps forward (here n can be positive or negative)?

QUESTION (2) TWO-LEVEL SYSTEMS: Suppose we have a set of N non-interacting two-level systems, each of which can have energy $\epsilon_j = \pm\Delta_o$.

(i) The canonical partition function for a system with states of allowed energy E_j is $\mathcal{Z}(\beta) = \sum_j e^{-\beta E_j}$, where $\beta = 1/k_B T$ is the inverse temperature. What is the canonical partition function for each of the individual spins? And what is the canonical partition function for the total system of N spins??

(ii) The mean energy of the system is given by $U = \sum_j p_j E_j$, where p_j is the probability that the state with energy E_j is occupied. Show that the mean energy at temperature T is given by $U = -N\Delta_o \tanh \beta\Delta_o$.

(iii) Suppose I have two sets of spin-1/2 systems, each containing N spins in an applied magnetic field which splits each spin level by an amount $2\Delta_o$. We assume that they are in every way identical except that one of the N-spin systems is at temperature $T = 0$, while the other is at temperature $T = \infty$.
Now, I combine the 2 sets of spins. What is the final energy of the combined system? And if we allowed them to reach thermal equilibrium at this energy, what would be the final temperature?

QUESTION (3) HARMONIC OSCILLATOR: Suppose we have a 1-dimensional oscillator with energy levels $\epsilon_n = (n + \frac{1}{2})\hbar\omega_o$, where ω_o is the frequency of the oscillator.

(i) Show the canonical partition function is $\mathcal{Z}(\beta) = \frac{1}{2} cosech\, (\hbar\beta\omega_o/2)$, where $cosech\, x \equiv 2/(e^x - e^{-x})$.

(ii) As already noted in 2(ii) above, the mean energy $U = \sum_j p_j E_j$ for any thermodynamic system. Show that this implies that $U = \langle E \rangle = -\mathcal{Z}^{-1}(\partial\mathcal{Z}/\partial\beta)$. Given this, find an expression for the mean energy U of the oscillator. Finally, draw a graph of this energy U as a function of the dimensionless variable $x = k_B T/\hbar\omega_o$.

QUESTION (4) THERMODYNAMICS: An elastic string has equilibrium length L and mass ρ per unit length when in equilibrium. Suppose it is stretched from its equilibrium length by an infinitesimal length dl. It is found that the work required to do this is $dW = kldl$ where l is the extension (ie., the total length is now $L + l$).
Find expressions for the total energy U and the free energy F of the string; and for the change dF in the energy under a change dl in the length.

END of MID-TERM TEST