Radiation transport model for ablation hollows on snowfields

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[1] The ablation hollows or “suncups” that form on the surface of snowfields in summer are a wonderful example of pattern formation in nature. Suncups reduce the albedo of the snow and set a characteristic length for interaction of wind with the snowpack. They also contain information about the properties of the snow and its ablation rate, which could be extracted if we had a more quantitative understanding of how suncups form. A mathematical model is proposed that explains the shape, size, and dynamical behavior of suncups in terms of the interaction of solar radiation with the snowpack. Using a perturbation method, we derive a nonlinear partial differential equation for the time-dependent shape of the snow surface from an approximate physical model for the interaction of solar radiation with snow. The resulting equation, which is similar to the Kuramoto-Sivashinsky equation in fluid mechanics, has solutions with characteristic length and amplitude. We find expressions for the characteristic size of suncups in terms of the spectrally averaged diffusion length of solar radiation in snow. The model correctly describes the shape of suncups, with their spatially ordered patterns of parabolic valleys and V-shaped ridges. It is also in remarkably good agreement with the observed length scales and growth rates. Depending on the relative values of the coefficients of the nonlinear terms in the differential equation, the suncup patterns can be either stationary in time or chaotic.


1. Introduction

[2] Ablation hollows or “suncups” are characteristic features that appear in summer on the surface of snowfields after an extended period of ablation by exposure to solar radiation [Post and LaChapelle, 2000]. Suncups are quasi-periodic, two-dimensional patterns that form during the summer melting season with a typical periodicity of 20–80 cm [Post and LaChapelle, 2000; Herzfeld et al., 2003]. They are typically 2–50 cm deep [Rhodes et al., 1987], smoothly rounded on the bottom with sharp ridges between neighboring suncups, sometimes forming a hexagonal pattern. A photograph of suncups on a snowfield in southern British Columbia is shown in Figure 1. The surface morphology of snow is important because it affects the albedo of the snow and hence the rate at which it melts when exposed to solar radiation. Melting rates are needed in hydrological flow models in order to predict water flow from snowmelt, and are of interest for global climate studies [Herzfeld et al., 2003; Corripio, 2003]. It is conceivable that a better understanding of the mechanism of suncup formation could open the door to synthetic methods to control their formation, and thereby modify the rate of snowmelt.

The shape of the snow surface also affects the interaction of the snow with the atmosphere. Many people who have traveled in the mountains in the summer have wondered about the origin of these striking patterns on the snow surface.

[3] There have been a number of experimental and theoretical studies directed at understanding the physical mechanisms that lead to the formation of suncups [see Post and LaChapelle, 2000; Betterton, 2002; Rhodes et al., 1987, and references therein]. Ablation hollows can appear on both clean and dirty snow [Rhodes et al., 1987]. Various ideas have been proposed for the formation of ablation hollows in the case of clean snow including heat transfer associated with atmospheric turbulence [Betterton, 2002]. Solar radiation is an obvious factor to consider since it is the dominant source of heat in the ablation of snow [Betterton, 2002; Corripio, 2003]. Also, it has been reported that during periods of overcast weather, suncups decay in amplitude [Rhodes et al., 1987]. For clean snow, the mechanism of suncup formation most frequently discussed in the literature is the effect of surface topography on the focusing of solar radiation.

[4] In the solar radiation model, suncups are caused by the fact that hollows in the snow trap incident sunlight more efficiently than peaks. This means that the hollows melt faster than the peaks leading to an instability where hollows grow spontaneously from noise and then become progressively deeper. Although this qualitative picture explains the origin of the instability it says nothing about the character-
istic length, the shape of suncups, or what prevents them from growing without limit. In this paper, we present a mathematical model for suncups driven by solar radiation. This model is derived from the effect of the snow surface shape on the absorption and diffusion of light in snow.

Snow is a granular medium that strongly scatters incident light. Some of the light incident on the snow surface is scattered back into the air, some of it is absorbed inside the snow and in the case of a rough surface, the radiation can scatter from one part of the surface to another. The fraction of light scattered back into the atmosphere is known as the albedo and the fraction that is absorbed is the co-albedo. The scattering properties of snow vary widely depending on the wavelength of the light, and the grain size and density of the snow. In the blue region of the spectrum the diffusion length of light can be a fraction of a meter, while in the infrared the diffusion length is very short, on the order of a millimeter or less, owing to strong absorption. In the analysis below we will assume that the surface topography is weak so that $|\nabla h| < 1$, where $\nabla$ is the two-dimensional gradient operator and $h(x, y, t)$ is the surface height relative to a horizontal reference plane, as a function of position $(x, y)$ and time $t$.

It is important to emphasize that the process of snow ablation by solar radiation is exceedingly complex. For example, the scattering properties of snow depend on the grain size and packing density of the ice particles, which are functions of the time/temperature history of the snow. Light scattering from rough surfaces and granular media is difficult to describe quantitatively. Additionally, the mechanics of snow collapse during melting is not well understood; thermal radiation from the snow surface can be significant; the direction and intensity of the Sun and the temperature of the air are constantly changing; the snow can ablate by both sublimation and melting; and the role of thermal conductivity is unclear. Given the complexity of the processes involved, a model of suncup formation must focus on the most important factors and will of necessity be approximate. It is in this spirit that we present the model in this paper.

Although there are differences in suncup patterns on mountain slopes facing toward or away from the Sun, for example on sloping surfaces suncups tend to tilt toward the Sun, nevertheless the basic phenomenon is not strongly sensitive to the average Sun angle. Therefore, for simplicity, we assume that the Sun is directly overhead at all times.

2. Interaction of Light With Snow

The distribution of scattered light below the surface of the snow is shown schematically in Figure 2 for sloping and curved surfaces when the incident light is in the form of a
collimated beam. Figure 2a illustrates how light at off-normal incidence to the snow surface is more likely to be scattered out of the snow than light at normal incidence. This scattering out of the snow should depend quadratically on the slope because the term that is linear in the slope cancels by symmetry. Intuitively, the increased loss of light on the downslope side of the point of entry into the snow will be balanced by a corresponding reduction in light escaping through scattering out of the upslope side, to first order in the surface slope. We therefore expect a reduction in the absorbed light intensity by a factor of \[ \frac{1}{C_0 a(q^2)} \] to lowest order in the surface slope \[ r_h \], where \( a \) is a dimensionless constant of order unity.

This conclusion is supported by the measurements and quantitative light scattering model presented by Wiscombe and Warren [1981]. In particular the data and model for the albedo \( a(0) \) presented in Figure 6 of Wiscombe and Warren [1981], can be fit by the following simple function [see also Pfeffer and Bretherton, 1987],

\[
1 - a(0) = [1 - a(0)][1 - 0.5(1 - \cos \theta)]
\]  

(1)

where \( \theta \) is the solar zenith angle. This expression fits the angle dependence of the albedo rather well for the five different values of the normal incidence co-albedo that are presented by Wiscombe and Warren [1981]. If the Sun is directly overhead, we can make the replacement \( \cos \theta \approx 1 - (\nabla h)^2/2 \) in the small angle limit \( \theta \approx \nabla h \ll 1 \). In this case, equation (1) reduces to

\[
1 - a(0) = [1 - a(0)][1 - 0.25(\nabla h)^2].
\]  

(2)

This reproduces the functional form derived on the basis of the symmetry argument described above and provides a value for the dimensionless constant, \( \alpha = 0.25 \). Although the small angle approximation is mathematically correct only in the limit that the surface slope approaches zero, it is accurate to 2% for slopes up to 0.5, and 30% for slopes up to one.

Following similar reasoning light incident on a curved snow surface will be scattered out of the snow with increased probability for a convex snow surface and reduced probability for a concave surface. (The bottom of a depression in the snow is defined as concave.) An intuitive explanation illustrated in Figure 2b is that the light scattered out of the incident beam must travel farther through the snow to escape if the surface is concave, and is therefore more likely to be absorbed. The magnitude of this effect will depend on the curvature of the surface relative to the curvature of the scattering volume \( 1/l_s \), where \( l_s \) is the photon diffusion length in the snow at wavelength \( \lambda \).
curvature-related effect will therefore increase the co-albedo by a factor of order $\left[1 + \beta \lambda, \nabla^2 h \right]$ where $\beta$ is a dimensionless numerical constant of order unity. This expression is expected to be valid in the limit of low surface curvature where $|\nabla^2 h| \ll 1/l_c$.

[11] In addition to modifying the light trapping characteristics of the snow itself, surface topography also causes multiple reflections from one area of the surface to another. In particular, concave regions of the surface will experience additional radiation intensity due to scattering from neighboring parts of the surface as shown in Figure 2c. For a sinusoidal surface, diffuse reflections will cause the light intensity to be concentrated in the valleys, and reduced on the ridge tops. In this respect, multiple scattering has a similar effect to the curvature-dependent in-snow scattering process discussed above and illustrated in Figure 2b; that is, light trapping within the snow increases with surface curvature. In a concave region of the surface the intensity of the solar radiation at a particular point will be increased by an amount proportional to the solid angle of snow visible from this point and the albedo of the surrounding snow [see also Betterton, 2002]. For example, at the bottom of a circularly symmetric parabolic region of diameter $l_c$, the fraction of the exposed $2\pi$ solid angle that faces snow elsewhere on the surface is

$$\frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\theta_{\text{max}}} \cos \theta \cos \phi \, d\theta \, d\phi \approx \frac{l_c \nabla^2 h}{8}.$$  \hspace{1cm} (3)

where $\theta_{\text{max}} = l_c \nabla^2 h / 8$. The comparable fraction of the exposed $2\pi$ solid angle that faces snow for a point on the surface just below the rim of the paraboloid is equal to

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_{0}^{\theta_{\text{max}}} \cos \theta \cos \phi \, d\theta \, d\phi \approx \frac{l_c \nabla^2 h}{4\pi}.$$  \hspace{1cm} (4)

Note that the solid angle at the edge differs from the value in the valley bottom only by the factor $2/\pi$.

[12] The surface topography of the ablation features on snow that we wish to describe, is observed to have a characteristic length $l_c$. In this case, following the argument above, we can approximate the effect of multiple scattering from the surface with a curvature dependent factor in the absorbed light intensity equal to $\left[1 + \alpha \lambda, l_c, \nabla^2 h / 8 \right]$, where $\alpha = l_c$ is the wavelength dependent normal incidence albedo of the snow. We have replaced the true diameter of the suncup with the diameter of an average suncup for purposes of estimating the contribution of scattering from nearby parts of the surface. The characteristic length $l_c$ of the snow surface topography will be determined self-consistently as discussed below. Clearly this expression is only valid for concave surfaces where $\nabla^2 h > 0$, as reflections from neighboring regions cannot be reduced below zero when $\nabla^2 h < 0$; however, snow ablation topography is commonly observed to be concave almost everywhere once the suncups are fully developed. Prior to the development of the characteristic suncup topography this approximation will exaggerate the concentration effect by reducing the intensity in convex regions. In addition, this approximation obviously does not take into account multiple scattering between neighboring suncups.

[13] This approach to the multiple scattering from the snow surface can be compared with the results of Pfeffer and Bretherton [1987], who calculated the albedo of a V-shaped trench in snow. For shallow trenches they find that the albedo is relatively constant across the trench and decreases linearly with opening angle (the opening angle is the angle at the bottom of the V). In our notation the analog of the opening angle is $\pi - 2\theta_{\text{max}}$. For the particular snow model, which they use, the spectrally averaged co-albedo of the trench is found to increase by the factor $(1 + 0.57 \theta_{\text{max}})$ as $\theta_{\text{max}} \to 0$. Our analysis will be applicable at the top edge of the trench. In this case, following equation (4) above, we find that the co-albedo increases by the factor $(1 + 2\alpha / \theta_{\text{max}}/\pi)$, which has the same linear dependence on surface angle as the Pfeffer and Bretherton [1987] result. If we use 0.5 for the spectrally averaged albedo (see below), in our treatment the co-albedo should increase by a factor of $(1 + 0.32 \theta_{\text{max}})$. This is consistent with the results of Pfeffer and Bretherton [1987].

[14] As discussed above, the sloping sidewalls and the bottom of the paraboloid receive approximately the same solid angle of radiation scattered from elsewhere on the surface. However, the sloping sidewalls have more surface area per unit horizontal distance according to the factor $\sqrt{1 + (\nabla h)^2} \approx 1 + (\nabla h)^2/2$. This produces a slope dependent increase in incident light flux analogous to equation (2), but with the opposite sign. The incremental increase in incident light intensity on a sloping surface can be approximated by $\alpha \lambda, \Delta \omega \left(\nabla h \right)^2/4\pi$ where $\alpha = l_c$ is the albedo of a flat, horizontal snow surface at wavelength $\lambda$ and $\Delta \omega < 2\pi$ is the solid angle subtended at the point of interest by other parts of the snow surface. According to equations (3) and (4), the solid angle $\Delta \omega$ is weakly dependent on position for a paraboloid. Putting all the slope and curvature effects discussed above together, the co-albedo of a curved snow surface as a function of surface slope and curvature, for small slopes and curvatures, is

$$(1 - \tilde{\alpha}) = (1 - \alpha) \left[1 + \left(\beta \lambda, \frac{\alpha l_c}{8} \right) \nabla^2 h \right]$$

$$+ \left(\alpha \frac{\Delta \omega}{4\pi} - 0.25 \right) \left(\nabla h \right)^2.$$  \hspace{1cm} (5)

[15] The final light absorption effect we will consider is the tendency for light scattered in the snow to migrate in the “upslope” direction from the point at which it first strikes the surface of the snowpack. In the case of light incident vertically on a sloping surface as illustrated in Figure 2a, more of the incident light that scatters in the downslope direction tends to escape from the snowpack than the light that scatters in the upslope direction. This causes a net shift in the center of gravity of the absorbed radiation distribution in the upslope direction relative to the entry point at the snow surface. In addition, as illustrated in Figure 2d, a sloping surface will tend to capture light scattered from other parts of the surface that is propagating in a near-horizontal direction. This light will also be absorbed at a position that is horizontally displaced in the upslope direction from the point of entry into the snowpack. These two
processes will generate a wavelength dependent lateral light intensity flow $J_{S,\lambda}$, that is proportional to the product of the surface slope and the diffusion length of the light in the snow, as follows:

$$J_{S,\lambda} = \frac{L_\lambda}{\tau_\lambda} \nabla h,$$  \hspace{1cm} (6)

where $L_\lambda$ is the light energy density in the snow per unit of horizontal area per unit wavelength, depth-integrated, $\tau_\lambda$ is the lifetime of a photon of wavelength $\lambda$ in the snow, before it is absorbed, and $\delta$ is a dimensionless constant of order unity. $J_{S,\lambda}$ is the lateral flux of light energy in the horizontal plane driven by the surface slope.

[16] In addition, there is a lateral flux of light energy associated with diffusion,

$$J_{D,\lambda} = D_\lambda \nabla L_\lambda,$$  \hspace{1cm} (7)

where the two-dimensional diffusion coefficient is $D_\lambda \equiv \frac{L_\lambda^2}{(2\tau_\lambda)}$. This flux is caused by lateral variations in the intensity of the light in the snow arising from the topography dependent variations in the co-albedo discussed above. For simplicity, we assume that the diffusion of light in snow is isotropic so that the lateral diffusion length is the same as the vertical diffusion length: $D_\lambda = D_\lambda \equiv \frac{L_\lambda^2}{(2\tau_\lambda)}$.  

3. Derivation of Surface Topography Equation

[17] Combining the phenomena discussed in the previous section with the extinction rate of photons in the snow $l$, we can write a continuity equation for the light intensity $L_\lambda$ in the snow at wavelength $\lambda$ as follows:

$$\frac{\partial L_\lambda}{\partial t} + \nabla \cdot (J_{S,\lambda} + J_{D,\lambda}) = F_\lambda (1 - \tilde{a}_\lambda) - \frac{L_\lambda}{\tau_\lambda}. \hspace{1cm} (8)$$

The subscript $\lambda$ indicates that the various quantities are wavelength dependent. The wavelength-integrated values will be presented later without the $\lambda$ subscript. $L_\lambda$ is the depth-integrated light energy in the snow per unit wavelength per unit area. $F_\lambda$ is the incident intensity of the Sun per unit wavelength, and $(1 - \tilde{a}_\lambda)$ is the co-albedo of a curved/sloping surface at wavelength $\lambda$, from equation (5). Substituting equations (5), (6), and (7) into equation (8), we obtain the following equation that describes the flow of light in the horizontal plane:

$$\frac{\partial L_\lambda}{\partial t} = f_\lambda \left[ 1 + \frac{c_\lambda}{l} \nabla^2 h - \frac{s_\lambda (\nabla h)^2}{2} \right] - \frac{L_\lambda}{\tau_\lambda} + \frac{L_\lambda^2}{\tau_\lambda} \nabla^2 L_\lambda - \frac{L_\lambda}{\tau_\lambda} \nabla \cdot (L_\lambda \nabla h), \hspace{1cm} (9)$$

where $f_\lambda = F_\lambda (1 - \tilde{a}_\lambda)$ is the flux of solar radiation at wavelength $\lambda$ that would be absorbed at normal incidence on a flat surface. The coefficient of the $\nabla^2$ term in equation (9) is

$$c_\lambda = \frac{(3l + a_\lambda l_c)}{8}$$  \hspace{1cm} (10)

according to the analysis presented above and

$$s_\lambda = \left( 0.25 - \frac{a_\lambda \Delta \omega}{\pi} \right). \hspace{1cm} (11)$$

[18] In the case of snow ablation the surface morphology changes over a period of many hours; therefore on the timescale of the morphology changes of interest the time derivative of the light intensity can be neglected [see, e.g., Strogatz, 1994]. This quasi-steady state approximation eliminates the time derivative of the light intensity as a variable. Accordingly, we can set $\frac{\partial L_\lambda}{\partial t} = 0$ in equation (9) and solve the resulting homogeneous equation for $L_\lambda$. This equation can be solved for the light intensity to lowest order in the surface gradients by applying the following inverse operator,

$$\frac{g}{1 + f + c} \approx \frac{g}{1 + f} - \frac{1}{1 + f} \tilde{O} \left( \frac{g}{1 + f} \right). \hspace{1cm} (12)$$

In our case, the operator $\tilde{O} \equiv \delta h \nabla \cdot \nabla - \frac{L_\lambda^2}{\tau_\lambda}$, and the function $f \equiv \delta h \nabla^2 h$. This approximation is valid in the limit of a smoothly varying surface with low slopes. A derivation of this expression is presented in Appendix A for one dimension. Briefly, the derivative operators can be inverted by Fourier transforming the continuity equation (equation (9)), solving the transformed equation to lowest order in the Fourier variable $k^2$, then transforming back to real space. Solving equation (9) for $L_\lambda$ in the quasi-steady state limit using the expansion in equation (12) we obtain an expression for $L_\lambda$ up to fourth order in the spatial derivatives and second order in the surface height,

$$L_\lambda \approx f_\lambda \tau_\lambda \left[ 1 + \left( c_\lambda - \delta h \right) \nabla^2 h + \frac{(c_\lambda - \delta h) L_\lambda^2}{2} \nabla^4 h - s_\lambda (\nabla h)^2 \right. \left. - \frac{s_\lambda L_\lambda^2 + (c_\lambda - \delta h) L_\lambda^2}{2} \nabla^4 h \right]. \hspace{1cm} (13)$$

This equation describes the dependence of the light intensity in the snow on the surface shape. The next step is to relate the local light intensity to the ablation rate of the snow.

[19] We assume that the snow surface collapses vertically downward under the influence of gravity as the snowmelt or sublimates underneath. In this case, the rate of change of surface height at a given position in the horizontal plane is proportional to the rate at which solar radiation is absorbed at that location:

$$\frac{dh(x)}{dt} = -\frac{1}{Q} \int \frac{L_\lambda(x)}{\tau_\lambda} d\lambda, \hspace{1cm} (14)$$

where $Q$ is the latent heat per unit volume of the snow, and $x = (x, y)$ is a position in the horizontal plane. The latent heat will be different depending on whether the snow is sublimating or melting. By substituting equation (13) into
equation (14) we obtain a dynamical equation for the shape of the snow surface.

\[
\frac{dh}{dt} = \frac{f}{Q} \left[ -1 - \langle c_s - \delta h \rangle \nabla^2 h - \frac{\langle c_s h^2 - \delta h^2 \rangle}{2} \right] \nabla^4 h + \langle s_i \rangle (\nabla h)^2 + \left( \frac{s_i h^2 + c_s \delta h^2 - \delta h^2}{2} \right) \nabla^2 (\nabla h)^2 \right] \tag{15}
\]

In this equation, \( f \) is the total solar radiation flux integrated over all wavelengths that would be absorbed on a flat surface and \( \langle \ldots \rangle \) means average over the entire solar spectrum. In the spectral averages of the form \( \langle c_s \rangle \) involving the parameter \( c_s = 3\delta_h + a_s \frac{\lambda}{8} \), we set \( \langle a_s \rangle = 0.5 \) following Betterton [2002], who uses 0.5 for the spectrally averaged albedo of old snow.

[20] The nonlinear partial differential equation in equation (15) is similar to the Kuramoto-Sivashinsky (KS) equation, differing only by the presence of the additional nonlinear term \( \nabla^4 (\nabla h)^2 \). The KS equation has been used to describe a variety of nonlinear phenomena in fluid mechanics and surface morphology [see, e.g., Facsko et al., 2004]. The modified KS equation in equation (15) has been studied by Raible et al. [2000, 2001] and Castro and Cuerno [2005]. For low-amplitude surface morphology, the surface topography is controlled by the linear terms and the nonlinear terms can be neglected. The two linear terms are competing in the sense that the second order linear term \( -\nabla^2 \delta h \) is unstable, causing surface oscillations to grow, while the fourth order linear term \( -\nabla^4 h \) is stable, causing surface topography to decay. As these two linear terms are dominant at low and high spatial frequencies, respectively, the competition between them leads to a characteristic spatial frequency, which shows the most rapid exponential growth. This leads to the characteristic length \( l_c \) discussed above.

[21] In the linear low-amplitude regime, the surface will show a smooth, almost sinusoidal surface topography with up-down symmetry. Consider the linear equation with constant coefficients,

\[
\frac{dh}{dt} = -c_1 \nabla^2 h - c_2 \nabla^4 h. \tag{16}
\]

For a sinusoidal surface topography with spatial frequency \( q \), the surface amplitude grows at the rate \( c_1 q^2 - c_2 q^4 \), which is a maximum when the characteristic length of the topography is \( l_c = 2\pi \sqrt{c_2/c_1} \). We use this expression to define the characteristic length in equation (15), and obtain

\[
l_c = \frac{2\pi}{\sqrt{\frac{\langle c_s h^2 - \delta h^2 \rangle}{\langle c_s - \delta_h \rangle}}} = \frac{16(3-\delta)\langle l_i \rangle}{16(3-\delta)\langle h_i \rangle + l_i}. \tag{17}
\]

Since the right-hand side of equation (17) is a function of \( l_i \), we can use this equation to obtain a self-consistent expression for the characteristic length in terms of the spectral averages of the photon diffusion length. This produces a cubic equation for \( l_i \); therefore an analytical solution is available.

[22] The diffusion length of light in snow \( l_i \) is a strong function of wavelength varying by more than two orders of magnitude across the solar spectrum. In this case, the spectral averages have the property \( \sqrt{\langle l_i^2 \rangle} > \sqrt{\langle l_j^2 \rangle} > \langle l_i \rangle \). We set the dimensionless constants \( \beta \) and \( \delta \) equal to 1.2 and 0.8, respectively. These values give results that are consistent with observations as discussed below. Equation (17) will be solved numerically, however an approximate solution is

\[
l_c \approx 6.3\langle l_i \rangle^{1/4}, \tag{18}
\]

provided that \( \beta > \delta \). We can substitute these values for \( \beta, \delta \) into equation (15) and obtain the following surface evolution equation:

\[
\frac{dh}{dt} = \frac{f}{Q} \left[ -1 - (0.4\langle h_i \rangle + 0.063l_c) \nabla^2 h - \left( 0.2\langle l_i \rangle + 0.031\langle l_i \rangle l_c \right) \nabla^4 h + \langle s_i \rangle (\nabla h)^2 + \left( \frac{s_i l_i^2}{2} + 0.16\langle l_i \rangle + 0.025\langle l_i \rangle l_c \right) \nabla^2 (\nabla h)^2 \right] \tag{19}
\]

In this analysis, multiple surface reflections make the largest contribution to the unstable Laplacian in equation (19). This is the unstable term that drives the formation of the characteristic surface features. We will assume that \( \langle s_i \rangle \langle h_i \rangle \approx \langle s_i \rangle \langle l_i \rangle \).

[23] The coefficients of three of the terms in equation (19) depend on the characteristic length of the surface structure \( l_i \) to varying degrees. Strictly speaking these are not constant parameters, since the characteristic length of the surface depends on the solution of the equation. In the case of a sinusoidal surface topography \( h(x) \sim \cos(qx) \), the characteristic length \( l_i \) equals \( 2\pi/q \). This means that in Fourier space, the unstable linear term in equation (19) (i.e., \( -0.063l_i \nabla^2 \delta h \)) should have a linear \( q \) dependence. Betterton [2002] also found a linear \( q \) dependence from an analysis of the surface reflections. Since sincups have a characteristic length, we will retain the correct magnitude of the unstable term if we replace the linear \( q \) term by the \( q^2 \) term in equation (19) multiplied by \( l_c \). Although this term does not have the right \( q \) dependence, it does have the correct magnitude at the characteristic length. A linear \( q \) dependence for the unstable term will shift the most unstable spatial frequency to smaller \( q \) values relative to the quadratic \( q \) dependence in equation (19). (If the growth rate in the linear regime has the form \( c_1 q^2 - c_2 q^4 \), then the peak growth rate would be at a characteristic length \( l_c = 2\pi (4c_2/c_1)^{1/3} \). (The constant \( c_1 \) replaces \( c_1 \) when the first term in equation (16) has a linear \( q \) dependence.) Solving the last expression self-consistently for \( l_c \) using the parameters in equation (19), we find a characteristic length that differs from the result in equation (18) only by a factor of \( 2^{1/3} = 1.26 \). This argument suggests that the approximation will have only a small effect on the characteristic length of the solution.

[24] When the amplitude of the surface topography becomes sufficiently large that the nonlinear terms dominate in equation (19), the nature of the surface shape changes qualitatively, namely the up-down symmetry of the surface is lost. In the nonlinear regime, the amplitude of the surface topography computed with equation (19) may achieve a constant limiting value at long times, depending on the value of the coefficient of the \( (\nabla h)^2 \) term. If this coefficient...
is negative, the surface amplitude saturates at long times. On the other hand, if the coefficient is positive \((s_h) > 0\) in equation (19), then there is a special spatial frequency for which the \((\nabla h)^2\) and \(\nabla^2 (\nabla h)^2\) terms cancel. If this spatial frequency is within the band of frequencies that are unstable in the linear regime, then the surface amplitude at this spatial frequency will grow without limit \([\text{Castro and Cuerno, 2005}]\). If the coefficient of the \((\nabla h)^2\) term is zero \((s_h) = 0\), the solution also grows without limit, but with a slow linear time dependence, forming parabolic surface profiles that grow in both width and height \([\text{Raible et al., 2000}]\). In practice, the instability in equation (19) when \((s_h) \geq 0\) may be limited by higher order nonlinearities that are not included in this analysis.

The signs of the nonlinear terms determine whether the surface has sharp ridges and rounded valleys, or the opposite morphology with V-shaped valleys and rounded ridges. The characteristic sharp ridges and rounded valleys of suncups are favored by a positive \(\nabla^2(\nabla h)^2\) term and a negative \((\nabla h)^2\) term. If the two nonlinear terms have the same sign the ridge/valley shapes depend on the relative magnitudes of the two competing terms. The coefficient \((s_h)\) of the \((\nabla h)^2\) term in equation (19) equals 0.25 \((1 - \Delta \omega/2\pi)\) for a spectrally averaged normal incidence albedo equal to 0.5, according to our model. For typical suncup heights we would expect the solid angle \(\Delta \omega < \pi\) so that \(0.1 < (s_h) < 0.25\). As discussed above a positive value for this coefficient favors V-shaped valleys and rounded ridge tops, contrary to the observed shapes. In addition, in the case of a positive coefficient for \((\nabla h)^2\), the solution of equation (19) may be unstable and grow without limit. Neither of these outcomes is consistent with observations. The problem is that according to the light-scattering model, the sloping parts of the surface should ablate more slowly than the horizontal parts, whereas observations of the surface shapes suggest the opposite. Therefore we must assume that some important physical process has been overlooked. Perhaps our estimate for the slope effect in the multiple scattering is too low, or different phenomena may be important. Water may drain out of the sloping parts more efficiently leading to reduced thermal conductivity and heat capacity. Alternatively, surface tension of water between snow grains may mean that the snow does not collapse vertically as we have assumed. Nevertheless we will proceed assuming that the coefficient \((s_h)\) has a similar magnitude to the model and the opposite sign, so that the surface shapes are consistent with observations.

4. Comparison With Observations

The surface shapes produced by equation (19) can be illustrated by numerical solutions. The equation is most conveniently solved with the space and time dimensions scaled to a standard form. By choosing the units of length in the horizontal and vertical directions and the units of time appropriately, one can set the coefficients of the two linear terms in equation (19) equal to unity \((c_1 = c_2 = 1)\) and enforce the constraint \(c_3 + c_4 = 1\) on the two remaining coefficients of the nonlinear terms. This leaves a single parameter \(c_3/c_4\), which controls the shape of the solution. Using this standard form, lateral length is measured in units of \(\sqrt{c_2/c_1}\), time is in units of \(c_2/c_1^2\) and height is in units of \([c_3/c_1^2 + (c_4/c_2)^2]^{-1/2}\). In these units, the characteristic length of the surface features in the linear regime will be \(2\pi \sqrt{2} \approx 8.9\).

Figure 3 shows 3D renderings of surface profiles obtained by solving equation (19) numerically as a function of time with Gaussian-smoothed, white noise as the initial condition. Cross sections of the surface profiles are shown in Figure 4. The parameters \(c_3/c_4 = -0.32\) and \(c_3/c_4 = -1.38\) are used, which bracket the magnitude of \([c_3/c_1^2 + (c_4/c_2)^2]^{-1/2}\). In these units, the characteristic length of the surface features suggest the opposite. Therefore we must assume that some important physical process has been overlooked. Perhaps our estimate for the slope effect in the multiple scattering is too low, or different phenomena may be important. Water may drain out of the sloping parts more efficiently leading to reduced thermal conductivity and heat capacity. Alternatively, surface tension of water between snow grains may mean that the snow does not collapse vertically as we have assumed. Nevertheless we will proceed assuming that the coefficient \((s_h)\) has a similar magnitude to the model and the opposite sign, so that the surface shapes are consistent with observations.
Figure 4. Two-dimensional cross sections through the surfaces shown in Figure 3 with coefficients \(c_1, c_2, c_3, c_4\) equal to (a) \((1, 1, -0.31, 0.95)\) and (b) \((1, 1, -0.81, 0.59)\). Note that the surface pattern is stable at long times in Figure 4a and constantly changing in Figure 5b even though the RMS amplitude is relatively constant. The times indicated on the figure are in units of \(c_2/c_1\). The lateral surface position is in units of \(\sqrt{c_2/c_1}\) and the height is in units of \(\left[\left(c_3/c_1^2\right) + \left(c_4/c_2^2\right)^2\right]^{-1/2}\).

forming an approximately hexagonal pattern. The surface patterns produced in the numerical solutions of equation (19) are in good qualitative agreement with field observations of suncups, as illustrated in Figure 1, for example. Also noteworthy is that the solution with the small negative value of \(c_3/c_4\) produces a quasi-stationary pattern in the nonlinear regime, whereas the larger negative value of \(c_3/c_4\) produces a less ordered pattern, exhibiting temporally chaotic behavior. The RMS amplitude remains relatively constant in both cases.

In order to make a quantitative comparison of the size of the features shown in Figure 3 with field observations, we need to evaluate the spectral averages of the one-dimensional diffusion lengths of the form \(l_n^k\) where \(n = 1, 2, 3\). We equate the one-dimensional diffusion length of the light with the extinction depth for light at normal incidence on a horizontal snow surface. The extinction depth for light in snow can be computed as a function of wavelength using the theoretical model and parameters of Bohren and Barkstrom [1974] and Wiscombe and Warren [1981]. According to Wiscombe and Warren [1981], \(0.1–0.3\) mm grain size is typical of fine grained older snow and \(1.0–1.5\) mm is typical of old snow near the melting point. As a representative example, we consider snow with a grain size of \(0.5\) mm and a relative density of \(0.5\). In the visible part of the spectrum, where the wavelength is small compared to the grain size, the light is weakly absorbing and will scatter many times from different snow grains, and penetrate far into the snowpack before being absorbed. In the infrared, the snow is strongly absorbing and the light will only diffuse a short distance before being absorbed. The diffusion length is a strong function of wavelength, but its average value can be readily obtained by numerical integration over the solar spectrum. The following values for the spectrally averaged diffusion lengths are obtained: \(\langle l_n^k \rangle^{1/2}\), \(\langle l_n^3 \rangle^{1/2}\), \(\langle h_s \rangle = 7.9, 5.2\) and \(2.1\) cm, respectively. Substituting these numbers into equation (17), we find a characteristic periodicity \(l_c = 57\) cm for the surface topography. This length is consistent with the observed size of suncups, for which \(20–80\) cm is typical (observations by the authors in the British Columbia Coast Range and by Post and LaChapelle [2000]). We note that our value for the spectrally averaged diffusion length \(\langle h_s \rangle = 2.1\) cm is consistent with an estimate by Betterton [2002], who concludes from various literature sources that the extinction length in old snow is “of order \(1\) cm.” With these parameters, the first linear term in equation (19) is always unstable independent of the choice of the parameters \(\beta\) and \(\delta\) in the range \([0, 1]\).

[29] We can also estimate the saturation amplitude of the surface topography. Qualitatively, when the surface amplitude reaches the point that the nonlinear terms on the right hand side of equation (19) are equal to the linear terms, the surface amplitude saturates. Physically, the saturation happens because the nonlinear terms accelerate the erosion of the steep sidewalls, which balances the unstable linear term. From the numerical solutions in Figure 3 the saturation amplitude is \(h_s \approx 3\left[(c_3/c_1)^2 + (c_4/c_2)^2\right]^{-1/2}\). Using the parameters \(\beta = 1.2, \delta = 0.8\) and \(\langle h_s \rangle = -0.16\), and the values for the light diffusion lengths discussed above, we find the peak surface height to be \(h_s \approx 59\) cm. This is close to the right range, with observed values reported to be \(2–50\) cm by Rhodes et al. [1987].

[30] It is also of interest to estimate the time required to form suncups. According to equations (16), (17) and (19), the characteristic time for the exponential growth of suncups in the linear regime is given by

\[
\tau_e \approx 0.8l_c Q/f. \tag{20}
\]

Figure 5. RMS surface amplitude as a function of time for the same parameters as in Figure 3 showing the exponential growth in the linear regime and the saturation of the surface height in the nonlinear regime. The timescale is in units of \(c_2/c_1\) and the surface height is in units of \(\left[\left(c_3/c_1^2\right) + \left(c_4/c_2^2\right)^2\right]^{-1/2}\).
The constant $f/Q$ is the average ablation rate of the snow surface. Ignoring the numerical constant, equation (20) predicts that the growth rate of suncups is equal to the ablation rate of the snow divided by the diameter of the suncups. Betterton [2002] has estimated the snow ablation rate in full sun for melting and sublimation. We translate her estimates into 14 cm/day and 2 cm/day for melting and sublimation respectively assuming 4 hours of full sun per day. Corripio [2003] found a snow surface-lowering rate of 5 cm/day in the Andes in clear summer weather near the equator. In northern temperate latitudes the ablation rate may be less. If we take 5 cm/day as a representative value of $f/Q$ in equation (20), we find the characteristic time in the exponential growth region to be $\tau_c = 9$ days. The time required for the surface amplitude to increase by a factor of 100 from 2 mm to 200 mm will then be 41 days of full sun. This is also in the right range. From experience in the British Columbia Coast Range, suncups typically develop in mid to late summer following a few months of melting.

[31] There is considerable variability in the size of suncups in nature, varying by at least a factor of 3 in size, depending on local conditions so that an exact match with the data is not expected. Nevertheless, the consistency between our model and the observed size and growth rate of suncups is remarkable given the complexity of the processes involved and the simplifying assumptions that have been made. The equation itself is derived in the form of a perturbation expansion in derivatives and powers of the surface slope. Owing to incomplete knowledge of the underlying physical phenomena we do not know whether neglecting the higher-order terms is a good approximation for the surface structures typical of real suncups. It is possible that the surface shapes are not very sensitive to the exact form of the nonlinearities in the growth equation. The good agreement with observations is partly due to the fact that we have selected values for the dimensionless parameters $\beta = 1.2, \delta = 0.8$ that match observations. On the other hand, the characteristic length and growth rate of the suncups are not strongly sensitive to the value of these parameters, given the range of acceptable values. The amplitude and chaotic dynamical behavior of the suncup patterns are relatively sensitive to the value of these parameters.

[32] It would be interesting to test the model with field measurements. Some important things to measure are: (1) the suncup diameter in relation to the extinction length of solar radiation in the snow; (2) how the suncup growth rate compares with the ablation rate of the snow; and (3) the changes in the suncup patterns as a function of time in the saturation regime.

5. Conclusion

[33] In conclusion, we have presented a radiation transport model for the striking ablation features commonly known as suncups, that form on snowfields in mid to late summer at high elevations and temperate latitudes. The model describes the surface shape as a function of time in terms of a nonlinear differential equation, which generates a quasiperiodic surface pattern from a random initial condition. We obtain remarkably accurate values for the size and rate of formation of suncups using optical properties of snow reported in the literature. In order to reproduce the correct shape for the suncups (V-shaped ridges, rounded valleys), we must assume that the sloping parts of the suncups ablate faster than the horizontal parts even though the optical model suggests the opposite. An unexpected result is that the mathematical model predicts that the pattern of ablation hollows can be stationary with time or a constantly evolving, chaotic pattern, with fixed amplitude depending on the relative values of the parameters in the nonlinear equation. In addition, depending on the relative signs of the nonlinear terms the nonlinearity can cancel causing the surface to be unstable. This behavior is suggestive of penitents, an extreme form of suncups which occur when there is strong sublimation.

[34] The model indicates that the size of suncups is proportional to a spectral average of the extinction length of light in snow. If this connection can be substantiated by field measurements, then the suncup diameter could be used as a way to estimate the extinction length. This quantity depends on the grain size and density of the snow. Therefore the suncups may provide useful information on the properties of the snow. Similarly, the model indicates that the growth rate of suncups is approximately equal to the snow ablation rate divided by the suncup diameter. This suggests that before the amplitude of the suncups saturates, their amplitude could be used to determine how much snow has ablated. Similarly, in the saturation regime if the suncup patterns turn out to be chaotic in time, the rate at which the patterns change may provide information on how fast the snow is ablat ing.

Appendix A

[35] We prove the operator expansion in equation (12) for the special case of one dimension for the second derivative operator $O = d^2/dx^2$. Consider the equation

$$\begin{align*}
(1 + f(x))L(x) + \frac{d^2 L(x)}{dx^2} &= g(x).
\end{align*}$$

(A1)

where $f(x)$ and $g(x)$ are two functions of $x$. Divide both sides by $(1 + f(x))$ and take the Fourier transform. This operation is possible as long as the divisor is nonzero which will always be true if $|f(x)| < 1$.

$$\begin{align*}
\hat{L}(k) &= FT\left(\frac{g(x)}{1 + f(x)}\right) + FT\left(\frac{1}{1 + f(x)}\right) \ast k^2 \hat{L}(k).
\end{align*}$$

(A2)

In this expression, $FT$ means Fourier transform, $\ast$ means convolution and $L(k) \equiv FT(L(x))$. This can be solved iteratively if $k^2 \hat{L}(k)$ is small (i.e., if $L(x)$ is smooth). Only the two lowest order approximations are shown,

$$\begin{align*}
\hat{L}_0(k) &= FT\left(\frac{g(x)}{1 + f(x)}\right) \\
\hat{L}_1(k) &= FT\left(\frac{g(x)}{1 + f(x)}\right) + FT\left(\frac{1}{1 + f(x)}\right) \ast k^2 FT\left(\frac{g(x)}{1 + f(x)}\right).
\end{align*}$$

(A3)

(A4)
Now we take the inverse Fourier transform of equation (A4) to obtain the solution to first order in the operator,

$$L(x) = \frac{g(x)}{1 + f(x)} - \frac{1}{1 + f(x)} \frac{d^2}{dx^2} \left( \frac{g(x)}{1 + f(x)} \right).$$

This reproduces equation (12) in one dimension.

**Appendix B**

[36] We use a pseudospectral method to obtain numerical solutions to equation (19): Spatial derivatives are calculated in Fourier space while nonlinear multiplication is carried out in real space. MATLAB’s built in ODE15s adaptive time step stiff integrator is used to advance the solution in time [Shampine and Reichlet, 1997]. The integrator is provided with the Fourier spectrum of the initial surface and a right-hand-side function coded to calculate the time rate of change for each Fourier coefficient. Upon completion of the calculation, each output time increment of the solution is inverse Fourier transformed to obtain the surface profile.

[37] On a periodic, square domain with sides of length L, equation (19) can be discretized on an N by N-point grid and discrete Fourier transformed to obtain

$$\frac{\partial h_{m,n}}{\partial t} = (c_1 - c_2(q_m^2 + q_n^2))h_{m,n} + (c_3 + c_4(q_m^2 + q_n^2))$$

$$\cdot \left( \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right) + \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right) \right)$$

$$+ \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right)$$

$$+ \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right)$$

$$q_m = \frac{2\pi (m - 1)}{L} x_j = \frac{(j - 1)}{N} \frac{2\pi}{L},$$

where m, n, j and k are integers ranging from 1 to N and * denotes 2D cyclic convolution. Equation (B1) is solved numerically with N = 64 and L = 5(2π) to calculate the data shown in Figures 3, 4, and 5.

[38] In a Fourier basis, the linear differential operators are greatly simplified to multiplication by spatial frequency and are of order (L/N)^3 compared to typical finite difference methods of order (L/N)^2 or (L/N)^4 [Boyd, 2000]. Conversely, the nonlinear multiplication becomes a computationally intensive convolution. It is thus significantly more efficient to make use of the (Inverse) Fast Fourier Transform ((I)FFT) and the identity

$$\left( \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right) + \frac{1}{N^2} \left( q_w h_{w,m'} \right)^* \left( q_w h_{w,m'} \right) \right)$$

$$= FFT \left( \frac{1}{N^2} \left( q_w h_{w,m'} \right)^2 + FFT \left( \frac{1}{N^2} \left( q_w h_{w,m'} \right)^2 \right) \right).$$

This reduces the computational complexity from $O(N^3)$ to $O(N\log N)$ [Press et al., 1992].

[39] An undesirable side effect of spatial discretization is that the convolution in equations (B1) and (B2) is cyclic in Fourier space. In order to eliminate aliasing resulting from wrap around effects, zero padding is added at high spatial frequencies before the IFFT is performed. In this way, the real space function is interpolated to a 3N/2 by 3N/2 grid according to the Orszag 2/3 rule [Orszag, 1971]. The additional coefficients are truncated after the result is brought back to Fourier space by the FFT so that the number of grid points remains constant.

[40] Finally, as h is a real-valued function, we take advantage of the Hermitian symmetry of its discrete Fourier transform. Fourier coefficients greater than the Nyquist frequency in one spatial direction are not explicitly calculated as they contain redundant information.

**Notation**

- $a_\lambda$: albedo of snow for normal incidence on a flat surface at wavelength $\lambda$, dimensionless.
- $\tilde{a}_\lambda$: albedo for light of wavelength $\lambda$ incident on a curved/sloping surface, dimensionless.
- $a(\theta)$: albedo averaged over the solar spectrum for light incident on a horizontal surface at zenith angle $\theta$, dimensionless.
- $c_\lambda$: coefficient relating surface curvature to albedo, m.
- $c_1$: coefficient of unstable $\nabla^2 h$ term in surface shape equation, m^2/s.
- $c_{11}$: analogous to $c_1$ when the unstable term has a linear q dependence, m/s.
- $c_2$: coefficient of $\nabla^4 h$ term in surface shape equation, m^4/s.
- $c_3$: coefficient of $(\nabla h)^2$ term in surface shape equation, m/s.
- $c_4$: coefficient of $\nabla^2(\nabla h)^2$ term in surface shape equation, m^6/s.
- $D_\lambda$: wavelength dependent diffusion constant for light in snow, m^2/s.
- $F, F_\lambda$: flux of solar radiation on a horizontal plane, W/m^2, and flux at wavelength $\lambda$, W/m/nm.
- $f, f_\lambda$: absorbed flux of solar radiation at normal incidence on snow, W/m^2, and flux at wavelength $\lambda$, W/m/nm.
- $h(x, t)$: height of snow surface above a horizontal reference plane at position x and time t, m.
- $h_c$: characteristic peak height of suncups in the nonlinear regime, m.
- $J_{D\lambda}, J_{D,\lambda}$: vertically integrated light intensity flux in horizontal plane due to diffusion in the snow, W/m, and flux at wavelength $\lambda$, W/m/nm.
- $J_{S\lambda}, J_{S,\lambda}$: vertically integrated light flux in horizontal plane associated with a sloping surface, W/m, and vertically integrated light flux at wavelength $\lambda$, W/m/nm.
- $L, L_\lambda$: light energy density in the snow, vertically integrated, J/m^2, and light energy at wavelength $\lambda$, J/m^2/nm.
- $l_c$: characteristic length of surface topography, m.
- $l_\lambda$: diffusion length of light in snow at wavelength $\lambda$, m.
- $q$: spatial frequency of surface topography, m^{-1}.
- $Q$: latent heat of melting (or sublimation) for snow, J/m^3.
\( s_\lambda \) dimensionless parameter describing how the absorbed light intensity depends on surface slope.

\( x \) position vector in horizontal plane, m.

\( \alpha \) dimensionless constant relating surface slope to co-albedo.

\( \beta \) dimensionless constant relating surface curvature to co-albedo.

\( \delta \) dimensionless constant relating surface slope to lateral light intensity flow.

\( \tau_\lambda \) average length of time required for light (of wavelength \( \lambda \)) to be absorbed after it enters snow.

\( \tau_c \) characteristic time for exponential growth of suncups in the linear regime.

\( \langle \ldots \rangle \) average over solar spectrum.

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