Holographic Entanglement and Renyi Entropies

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Based on

[XD 1409.????]
[XD 1310.5713]
[Barrella, XD, Hartnoll & Martin 1306.4682]

Quantum Information in Quantum Gravity, University of British Columbia
Holographic Entanglement Entropy

A remarkably simple prescription in QFTs dual to Einstein gravity:

\[ S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \]

[Ryu & Takayanagi '06]
A remarkably simple prescription in QFTs dual to Einstein gravity:

\[ S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \] [Ryu & Takayanagi '06]

- Satisfies strong subadditivity. [Headrick & Takayanagi '07]
- Reproduces exact results for one interval in 1+1D CFTs. [Holzhey, Larsen & Wilczek '94; Calabrese & Cardy '04]
- First derived for spherical entangling surfaces. [Casini, Huerta & Myers '11]
- Proven for 2D CFTs with large $c$. [Hartman 1303.6955; Faulkner 1303.7221]
- Derived generally for Einstein gravity. [Lewkowycz & Maldacena 1304.4926]
- Bulk one-loop corrections: [Barrella, XD, Hartnoll & Martin 1306.4682]
  [Faulkner, Lewkowycz, & Maldacena 1307.2892; Engelhardt & Wall 1408.3203]
- Higher spin gravity: [Ammon, Castro & Iqbal 1306.4338; de Boer & Jottar 1306.4347]
- Bulk EOMs from EE first law: [Lashkari et al. 1308.3716; Faulkner et al. 1312.7856]
1. Holographic Replica Trick

2. Entanglement Entropy for Higher Derivative Gravity

3. Universal Terms in Holographic Renyi Entropy

4. Conclusion and Open Questions
Replica Trick

Introduce Rényi entropy:

\[ S_n = -\frac{1}{n-1} \log \text{Tr} \rho^n \]

\[ \Rightarrow S_{EE} = \lim_{n \to 1} S_n = -\text{Tr} \rho \log \rho \]

At integer \( n \), Rényi entropy can be written in terms of partition functions:

\[ S_n = -\frac{1}{n-1} (\log Z[M_n] - n \log Z[M_1]) \]

- \( M_1 \): original (Euclidean) spacetime manifold.
- \( M_n \): \( n \)-fold cover = \( n \) copies of \( M_1 \) glued together along \( A \) in cyclic order.
- \( \tau \): angle around \( \partial A \), range extended to \( 2\pi n \).
- \( n \)-fold cover does not make much sense for non-integer \( n \).
- Holographic dual side provides much “better” analytic continuation. [Lewkowycz & Maldacena]

E.g. 1+1D QFT
Holographic Dual of the $n$-Fold Cover

Build a bulk solution $B_n$ whose boundary is $M_n$:

$$Z[M_n] = e^{-S[B_n]} + \ldots$$

**Basic idea**

1. Use gauge-gravity duality to calculate $S[B_n]$.
2. Analytically continue it to non-integer $n$.
3. Expand to $O(n - 1)$ to extract EE.

Very complicated in general, can be explicitly worked out only in special cases e.g. AdS$_3$/CFT$_2$. [Faulkner 1303.7221; Barrella, XD, Hartnoll & Martin 1306.4682]

**But...**

- We do not need $B_n$ explicitly.
- For EE, only need $S[B_n]$ near $n \approx 1$:
  $$S_n = -\frac{1}{n - 1} (S[B_n] - nS[B_1])$$
- If we can find a family of bulk configurations interpolating between integer $n$, then we can expand in $n - 1$!
The $n$-fold cover has $Z_n$ symmetry: $\tau \to \tau + 2\pi$.

Assume: $Z_n$ symmetry extends to the bulk $B_n$.

Agrees with e.g. [Faulkner 1303.7221]

Then consider the orbifold: $\hat{B}_n = B_n/Z_n$

- Regular except at fixed points.
- Fixed points form codimension 2 surface $C_n$.
- $C_n$: conical defect with opening angle $2\pi/n$, anchored at $\partial A$: $ds^2 = \rho^{-2(1-\frac{1}{n})}(d\rho^2 + \rho^2 d\tau^2) + \cdots$

How does this help us calculate EE?

By construction: $S[B_n] = nS[\hat{B}_n]$ at integer $n$

$$\Rightarrow \quad S_n = \frac{n}{n-1} \left( S[\hat{B}_n] - S[\hat{B}_1] \right)$$

Note: $S[\hat{B}_n]$ does not include contributions from $C_n$. Now plausible that we can analytically continue $\hat{B}_n$. 
There are two equivalent methods.

1. "Boundary condition" method

Solve all EOMs and demand the metric near $C_n$ as

$$ds^2 = \rho^{-2\epsilon} (d\rho^2 + \rho^2 d\tau^2) + (g_{ij} + 2K_{aij}x^a)dy^i dy^j + \cdots$$

- An unconventional "IR" boundary condition.
- Justified by considering integer $n$ and impose $Z_n$ symmetry.
- In general has conical defect with deficit $2\pi \epsilon = 2\pi \left(1 - \frac{1}{n}\right)$.

2. "Cosmic brane" method

Replace $C_n$ by a codimension 2 brane! Solve all EOMs resulting from

$$S_{\text{total}} = S_{\text{EH}} + S_B = -\frac{1}{8\pi G_N} \int d^D x \sqrt{G} R + \frac{\epsilon}{4G_N} \int d^d y \sqrt{g}$$

Cosmic branes are "straight" allowing us to glue $\hat{B}_n$ back to $B_n$ for $n \in \mathbb{Z}$. 
1. Holographic Replica Trick

2. Entanglement Entropy for Higher Derivative Gravity

3. Universal Terms in Holographic Renyi Entropy

4. Conclusion and Open Questions
After all, string theory produces $\alpha'$ corrections.

\[
S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \Rightarrow S_A = \frac{(???)_\text{min}}{4G_N}
\]
Holographic Entanglement for Higher Derivative Gravity

\[ S_A = \frac{(\text{Area})_{\text{min}}}{4G_N} \quad \Rightarrow \quad S_A = \frac{(???)_{\text{min}}}{4G_N} \]

After all, string theory produces \( \alpha' \) corrections.

- Analogous to: Bekenstein-Hawking Entropy \( \Rightarrow \) Wald Entropy for BHs:

\[
S_{\text{Wald}} = -2\pi \int d^d y \sqrt{g} \frac{\partial L}{\partial R_{\mu \rho \nu \sigma}} \varepsilon_{\mu \rho} \varepsilon_{\nu \sigma}
\]

[Hald '93]

- In general, \( S_{\text{Wald}} \) cannot be \( S_{EE} \). [Hung, Myers & Smolkin '11]
- Even before Wald, there existed a different formula \( S_{JM} \) for BH entropy in Lovelock gravity. [Jacobson & Myers '93]
- They differ only by extrinsic curvature terms (\( = 0 \) for Killing horizons).
- For Gauss-Bonnet, \( S_{JM} \) passes consistency checks as \( S_{EE} \). [Hung, Myers & Smolkin '11]
Entropy Formula for Higher Derivative Gravity

General entropy formula for $L(R_{\mu\rho\nu\sigma})$:

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \frac{\partial L}{\partial R_{zz\bar{z}\bar{z}}} + \sum_\alpha \left( \frac{\partial^2 L}{\partial R_{izj} \partial R_{\bar{z}k\bar{z}l}} \right) \frac{8K_{zij} K_{\bar{z}kl}}{q_\alpha + 1} \right\}$$

- Wald’s formula
- “Anomaly” from extrinsic curvature

Encompasses previous results of special cases (e.g. giving $S_{JM}$ for Gauss-Bonnet): [Fursaev, Patrushev, & Solodukhin 1306.4000; Chen & Zhang 1305.6767; Bhattacharyya, Sharma, & Sinha 1305.6694, 1308.5748; \ldots]

Can show minimization prescription for at least 3 classes of examples: f(R), Lovelock, general 4-derivative gravity.

Covariant version exists.

Although derived for entanglement entropy, this formula also applies for BH entropy, generalizing Wald’s formula to non-stationary BHs.
\[ S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \frac{\partial L}{\partial R_{zz\bar{z}\bar{z}}} + \sum_{\alpha} \left( \frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{l}}} \right)_{\alpha} \frac{8K_{zij} K_{\bar{z}kl}}{q_{\alpha} + 1} \right\} \]

Outline for derivation

- Calculate \( S_{on}(\text{bulk with conical deficit}) \)
- Take small \( n-1 \) limit, conical deficit \( \epsilon \approx n-1 \).
- First-order variation of \( S_{on} \) localizes at defect: from either \( \delta \)-function or potential logarithmic divergences:
  \[ R_{zizj} \sim \frac{\epsilon}{\rho} K_{zij} + \cdots \Rightarrow \delta S_{on} \propto \int \rho d\rho \frac{\epsilon^2}{\rho^2} \rho \# \epsilon \sim \frac{\epsilon}{\#} \]
1. Holographic Replica Trick

2. Entanglement Entropy for Higher Derivative Gravity

3. Universal Terms in Holographic Renyi Entropy

4. Conclusion and Open Questions
In even-dimensional CFTs, certain logarithmically divergent terms are universal, i.e. they do not depend on much of the theory besides a few numbers such as anomaly coefficients.

- **Partition function:**
  \[
  \log Z = \text{(power divergences)} + \log \epsilon \int d^d x \sqrt{g} \mathcal{A} + \text{(finite)}
  \]
  \[
  \mathcal{A}(d = 2) = \frac{c}{24\pi} R, \quad \mathcal{A}(d = 4) = \frac{a}{16\pi^2} E(4) - \frac{c}{16\pi^2} l(4)
  \]

- **Entanglement entropy across a codimension-2 surface \( \Sigma \):**
  \[
  S_{EE}(d = 2) \sim -\frac{c}{6} \text{Volume}(\Sigma) \log \epsilon
  \]
  \[
  S_{EE}(d = 4) \sim \log \epsilon \left[ \frac{a}{2\pi} \int_\Sigma R_\Sigma + \frac{c}{2\pi} \int_\Sigma \left( \text{Tr} K^2 - \frac{1}{2} (\text{Tr} K)^2 - C^{ab}_{\ ab} \right) \right]
  \]

Can derive these by PBH (Penrose–Brown–Henneaux) transformations.
Universal Terms in Renyi Entropies

Renyi entropies $S_n$

- Contain richer information about $\rho$ than $S_{EE}$.
- Are interesting at special $n$: $n = 1/2$ (negativity), $n = 0$, $n \to \infty$.
- Have nice holographic interpretation in terms of cosmic branes.

They also have universal logarithmic terms in even dimensions.

$d = 2$

$$S_n \sim -\frac{c}{12} \left(1 + \frac{1}{n}\right) \text{Volume } (\Sigma) \log \epsilon$$

$d = 4$

$$S_n \sim \log \epsilon \left[ \frac{f_a(n)}{2\pi} \int_\Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int_\Sigma \left( \text{Tr}K^2 - \frac{1}{2}(\text{Tr}K)^2 \right) - \frac{f_c(n)}{2\pi} \int_\Sigma C^{ab}_{\Sigma} \right]$$

[Fursaev '12]
Universal Terms in Renyi Entropies for 4D CFTs

\[ S_n \sim \log \epsilon \left[ \frac{f_a(n)}{2\pi} \int_{\Sigma} R_{\Sigma} + \frac{f_b(n)}{2\pi} \int_{\Sigma} \left( \text{Tr} K^2 - \frac{1}{2} (\text{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_{\Sigma} C_{ab}^{ab} \right] \]

\( f_a(n) \) is computed by considering a spherical \( \Sigma \) in flat space:

- The \( n \)-fold cover may be conformally mapped to a hyperboloid \( H^3 \times S^1 \), with the size of \( S^1 \) being \( \beta = 2\pi n \). [Casini, Huerta & Myers '11]

\[ f_a(n) \] is completely determined by \( \log Z[H^3 \times S^1] \propto \text{Volume} (H^3) \).

This can be computed holographically as the dual geometry is a hyperbolic black hole.
\[ S_n \sim \log \varepsilon \left[ \frac{f_a(n)}{2\pi} \int_{\Sigma} R_{\Sigma} + \frac{f_b(n)}{2\pi} \int_{\Sigma} \left( \text{Tr} K^2 - \frac{1}{2} (\text{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_{\Sigma} C^{ab}_{\alpha\beta} \right] \]

What about \( f_b(n) \) and \( f_c(n) \)?

- Not much was known about them until [Lewkowycz & Perlmutter 1407.8171] proposed that \( f_c(n) \) may be derived from \( f_a(n) \):
  \[ f_c(n) = \frac{n}{n-1} [a - f_a(n) - (n-1)f'_a(n)] . \]

- It has also been conjectured that \( f_b(n) = f_c(n) \). [Lee, McGough & Safdi 1403.1580]

- I will propose a holographic derivation of these relations.
Deformed Hyperboloid

Similar to how we map the spherical case to a hyperboloid $H^3 \times S^1$:

\[
ds_4^2 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + g_{ij} dy^i dy^j \right] + d\tau^2, \quad g_{ij} dy^i dy^j = d\Omega_2^2
\]

We can map the case of arbitrary $\Sigma$ in arbitrary background to a deformed hyperboloid:

\[
ds_4^2 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + (g_{ij} + 2K_{aij} x^a + Q_{abij} x^a x^b) dy^i dy^j \right]
\]
\[+ (1 + T \rho^2) d\tau^2 + 2U_i d\tau dy^i + \text{(higher orders)}, \quad x^{1,2} \equiv \rho e^{\pm i\tau}\]
\[ ds^2_4 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + (g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b) \, dy^i dy^j \right] \]
\[ + (1 + T\rho^2) d\tau^2 + 2U_i d\tau dy^i + (\text{higher orders}), \quad x^{1,2} \equiv \rho e^{\pm i\tau} \]

Write it as the undeformed metric plus a perturbation:

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \]

The CFT partition function is

\[ \log Z = \log Z^{(0)} + \int \delta g_{\mu\nu} \langle T^{\mu\nu} \rangle + \frac{1}{2} \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T^{\mu\nu} T^{\rho\sigma} \rangle + (\text{higher orders}) \]

- \( \log Z^{(0)} \sim \text{Volume } (H^3) \) with cutoff \( \rho > \epsilon \) has quadratic and logarithmic divergences.
- Our goal is to extract logarithmic divergences in the perturbation.
\[ ds_4^2 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + (g_{ij} + 2K_{a(i}x^{a} + Q_{ab}x^{a}x^{b})\, dy^{i} dy^{j} \right] \]
\[ + (1 + T\rho^2)d\tau^2 + 2U_i d\tau dy^i + \text{(higher orders)} , \quad x^{1,2} \equiv \rho e^{\pm i\tau} \]

\[
\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu} \langle T^{\mu\nu} \rangle + \frac{1}{2} \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T^{\mu\nu} T^{\rho\sigma} \rangle + \text{(higher orders)}
\]

- \[ \int \delta g_{\mu\nu} \langle T^{\mu\nu} \rangle \text{ produces } \log \epsilon \text{ for terms in } \delta g_{\mu\nu} \text{ quadratic in } \rho. \text{ The coefficient of } \log \epsilon \text{ is schematically} \]

\[
- f_c(n) \int_\Sigma (T + Q^a_a) = - f_c(n) \int_\Sigma \left[ C^{ab}_{ab} + \text{Tr} K^2 + \frac{8}{3} U^2 \right]
\]

- \[ f_c(n) \text{ is determined by } \langle T^{\mu\nu} \rangle \text{ on the hyperboloid with } \beta = 2\pi n, \text{ which can be computed holographically.} \]

- Indeed it is related to \( f_a(n) \) by

\[
f_c(n) = \frac{n}{n-1} \left[ a - f_a(n) - (n-1)f_a'(n) \right].
\]
\[ ds_4^2 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + (g_{ij} + 2K_{aij}x^a + Q_{abij}x^ax^b) \, dy^i dy^j \right] + (1 + T\rho^2)d\tau^2 + 2U_i d\tau dy^i + \text{(higher orders)}, \quad x^{1,2} \equiv \rho e^{\pm i\tau} \]

\[ \log Z = \log Z^{(0)} + \int \delta g_{\mu\nu} \langle T^{\mu\nu} \rangle + \frac{1}{2} \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T^{\mu\nu} T^{\rho\sigma} \rangle \] + \text{(higher orders)}

- \[ \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T^{\mu\nu} T^{\rho\sigma} \rangle \] produces \( \log \epsilon \) for terms in \( \delta g_{\mu\nu} \) linear in \( \rho \).
- They produces terms involving \( K^2 \) (and \( U^2 \)).
- Computing \( \langle T^{\mu\nu} T^{\rho\sigma} \rangle \) holographically gives the conjectured relation \( f_b(n) = f_c(n) \) in the universal structure:

\[ S_n \sim \log \epsilon \left[ \frac{f_a(n)}{2\pi} \int_{\Sigma} R_{\Sigma} + \frac{f_b(n)}{2\pi} \int_{\Sigma} \left( \text{Tr}K^2 - \frac{1}{2}(\text{Tr}K)^2 \right) - \frac{f_c(n)}{2\pi} \int_{\Sigma} C_{ab}^{ab} \right] \]
\[ ds_4^2 = \frac{1}{\rho^2} \left[ \frac{d\rho^2}{1 + \rho^2} + (g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b) \, dy^i \, dy^j \right] \]

\[ + (1 + T\rho^2) \, d\tau^2 + 2U_i \, d\tau \, dy^i + \text{(higher orders)}, \quad x^{1,2} \equiv \rho e^{\pm i\tau} \]

\[
\log Z = \log Z^{(0)} + \int \delta g_{\mu\nu} \langle T_{\mu\nu} \rangle + \frac{1}{2} \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T_{\mu\nu} T_{\rho\sigma} \rangle + \text{(higher orders)}
\]

- \[ \int \delta g_{\mu\nu} \delta g_{\rho\sigma} \langle T_{\mu\nu} T_{\rho\sigma} \rangle \] produces \( \log \epsilon \) for terms in \( \delta g_{\mu\nu} \) linear in \( \rho \).
- They produces terms involving \( K^2 \) (and \( U^2 \)).
- Computing \( \langle T_{\mu\nu} T_{\rho\sigma} \rangle \) holographically gives the conjectured relation \( f_b(n) = f_c(n) \) in the universal structure:

\[
S_n \sim \log \epsilon \left[ \frac{f_a(n)}{2\pi} \int \Sigma R_\Sigma + \frac{f_b(n)}{2\pi} \int \Sigma \left( \text{Tr}K^2 - \frac{1}{2} (\text{Tr}K)^2 \right) - \frac{f_c(n)}{2\pi} \int \Sigma C_{ab}^{ab} \right]
\]

Quick “derivation”: \( \langle T_{ij} T^{kl} \rangle \propto \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{d} g_{ij} g^{kl} \). Contracting with \( K_{aij} K_{kli} \), we get \( \tilde{f}_b(n) \left[ \text{Tr}K^2 - \frac{1}{4} (\text{Tr}K)^2 \right] \).

Combining this with \( -f_c(n) \text{Tr}K^2 \), and requiring it to be \( \propto \left[ \text{Tr}K^2 - \frac{1}{2} (\text{Tr}K)^2 \right] \), we find \( \tilde{f}_b(n) = 2f_c(n) \Rightarrow f_b(n) = f_c(n) \).
There is a general formula that, evaluated on the conical defect $C_1$, gives the holographic EE in higher derivative gravity:

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \left\{ \frac{\partial L}{\partial R_{\bar{z} z \bar{z} z}} + \sum_{\alpha} \left( \frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right) \frac{8K_{zij} K_{\bar{z}kl}}{q_{\alpha} + 1} \right\}$$

Wald’s formula

“Anomaly” from extrinsic curvature

Logarithmic terms in Renyi entropies for 4D CFTs have a universal structure that can be computed at least holographically:

$$S_n \sim \log \epsilon \left[ \frac{f_a(n)}{2\pi} \int_{\Sigma} R_{\Sigma} + \frac{f_b(n)}{2\pi} \int_{\Sigma} \left( \text{Tr} K^2 - \frac{1}{2} (\text{Tr} K)^2 \right) - \frac{f_c(n)}{2\pi} \int_{\Sigma} C_{ab}^{ab} \right]$$

Many open questions.

How do we “enjoy” these results (in the big picture)?
Back Up Slides
Details on the Decomposition of the Riemann Tensor

\[ ds^2 = e^{2A} \left[ dzd\bar{z} + e^{2A} T(zdz - zd\bar{z})^2 \right] + (g_{ij} + 2K_{aij}x^a + Q_{abij}x^ax^b) \, dy^i dy^j \]
\[ + 2ie^{2A} (U_i + V_{ai}x^a)(\bar{z}dz - zd\bar{z}) \, dy^i + \cdots . \quad (1) \]

\[ R_{abcd} = 12e^{4A} T \hat{\varepsilon}_{ab} \hat{\varepsilon}_{cd} , \]
\[ R_{abci} = 3e^{2A} \hat{\varepsilon}_{ab} V_{ci} , \]
\[ R_{abij} = 2e^{2A} \hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i) + g^{kl}(K_{ajk}K_{bil} - K_{aik}K_{bjl}) , \]
\[ R_{ai bj} = e^{2A} [\hat{\varepsilon}_{ab}(\partial_i U_j - \partial_j U_i) + 4\hat{g}_{ab} U_i U_j] + g^{kl} K_{ajk}K_{bil} - Q_{abij} , \]
\[ R_{aijk} = \partial_l K_{aij} - \gamma^l_{ik} K_{ajl} + 2\hat{\varepsilon}_{ab} \hat{g}^{bc} K_{cij} U_k - (j \leftrightarrow k) , \]
\[ R_{ikjl} = r_{ikjl} + e^{-2A} \hat{g}^{ab}(K_{ail} K_{bjk} - K_{aij} K_{bkl}) , \]
\[ ds^2 = e^{2A} \left[ dz d\bar{z} + e^{2A} T(zdz - zd\bar{z})^2 \right] + \left( g_{ij} + 2K_{aij} x^a + Q_{abij} x^a x^b \right) dy^i dy^j + 2i e^{2A} (U_i + V_{ai} x^a) (zdz - zd\bar{z}) dy^i + \cdots. \quad (2) \]

\[ R_{abij} = \tilde{R}_{abij} + g^{kl} (K_{ajk} K_{bil} - K_{aik} K_{bjl}), \]
\[ R_{aibj} = \tilde{R}_{aibj} + g^{kl} K_{ajk} K_{bil} - Q_{abij}, \]
\[ R_{ikjl} = r_{ikjl} + \hat{g}^{ab} (K_{ail} K_{bjk} - K_{aik} K_{bjl}), \]

\[ \tilde{R}_{abij} \equiv 2 e^{2A} \hat{\epsilon}_{ab} (\partial_i U_j - \partial_j U_i), \]
\[ \tilde{R}_{aibj} \equiv e^{2A} \left[ \hat{\epsilon}_{ab} (\partial_i U_j - \partial_j U_i) + 4 \hat{g}_{ab} U_i U_j \right]. \]
Details on the Squashed Cone

\[ ds^2 = d\tilde{\rho}^2 + \tilde{\rho}^2 \left[ 1 + \tilde{\rho}^2 O (1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}) \right] d\tilde{\tau}^2 \]
\[ + \left[ g_{ij} + O (\tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}) \right] dy^i dy^j + \tilde{\rho}^2 O (1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}) d\tilde{\tau} dy^i . \]  

\[ O (1, \tilde{\rho}^2, \tilde{\rho}^n e^{\pm in\tilde{\tau}}) \equiv \sum_{k=-\infty}^{\infty} \left( \sum_{m=0}^{\infty} \tilde{c}_{km} \tilde{\rho}^{2m} \right) \tilde{\rho}^{|k|n} e^{\pm ikn\tilde{\tau}} , \]
\[ \rho \equiv \left( \frac{\tilde{\rho}}{n} \right)^n , \quad \tau \equiv n\tilde{\tau} , \]

\[ ds^2 = \rho^{-2\epsilon} \left\{ d\rho^2 + \rho^2 \left[ 1 + \rho^{2-2\epsilon} O (1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}) \right] d\tau^2 \right\} \]
\[ + \left[ g_{ij} + O (\rho^{2-2\epsilon}, \rho e^{\pm i\tau}) \right] dy^i dy^j + \rho^{2-2\epsilon} O (1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}) d\tau dy^i . \]  

\[ O (1, \rho^{2-2\epsilon}, \rho e^{\pm i\tau}) \equiv \sum_{k=-\infty}^{\infty} \left( \sum_{m=0}^{\infty} c_{km} \rho^{(2-2\epsilon)m} \right) \rho^{|k|} e^{\pm ik\tau} , \]
Caveat: in the prescription for $q_\alpha$, there might be an ambiguity about how to count $Q_{zz\bar{z}ij}$. We made a particular assumption about the analytic continuation of the $Z_n$-symmetric metric to non-integer $n$:

$$G_{ij} = g_{ij} + 2K_{zij}z + 2K_{\bar{z}ij}\bar{z} + Q_{zz\bar{z}ij}z^2 + 2Q_{z\bar{z}ij}(z\bar{z})^{1/n} + \cdots$$

In parent space: $w^n \quad \bar{w}^n \quad w^{2n} \quad w\bar{w}$

But can there be a term $2\tilde{Q}_{zz\bar{z}ij}z\bar{z} \sim w^n\bar{w}^n$? Answer should come from EOM.
Details on Lovelock gravity

\[ S^{(2p)} = \int d^D x \sqrt{G} L^{(2p)} \]

\[ L^{(2p)} = \frac{1}{2^p} \delta_{\mu_1 \rho_1 \mu_2 \rho_2 \ldots \mu_p \rho_p} \delta_{\nu_1 \sigma_1 \nu_2 \sigma_2 \ldots \nu_p \sigma_p} R_{\mu_1 \rho_1}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}^{\nu_2 \sigma_2} \ldots R_{\mu_p \rho_p}^{\nu_p \sigma_p} \]

\[ E^{(2p)}_{\mu \nu} = \frac{1}{\sqrt{G}} \frac{\delta S^{(2p)}}{\delta G_{\mu \nu}} = \frac{1}{2} G^{\mu \nu} L^{(2p)} - L^{(2p)}_{4 \mu \rho_1 \nu_1 \sigma_1} R^{\nu}_{\rho_1 \nu_1 \sigma_1} \]

\[ E^{(2p)}_{\mu \nu} = \frac{1}{2^{p+1}} \delta^{\mu \mu_1 \rho_1 \mu_2 \rho_2 \ldots \mu_p \rho_p} \delta_{\nu \nu_1 \sigma_1 \nu_2 \sigma_2 \ldots \nu_p \sigma_p} R_{\mu_1 \rho_1}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}^{\nu_2 \sigma_2} \ldots R_{\mu_p \rho_p}^{\nu_p \sigma_p} \]

\[ L_4^{\mu \rho \nu \sigma} = \frac{1}{\sqrt{G}} \frac{\delta S}{\delta R_{\mu \rho \nu \sigma}} \]
Details on Minimization: $f(R)$ Gravity

$$S_{EE} = -4\pi \int d^d y \sqrt{g} \frac{\partial L}{\partial R}$$

Claim: Minimizing this gives the location of $C_n$ at $n = 1$

- **Proof 1:** Transform to Einstein gravity + scalar.
- **Proof 2:** Cosmic brane method:

$$S_{\text{total}} = \lambda \int d^D x \sqrt{G} R^p - 4\pi p \lambda \epsilon \int d^d y \sqrt{g} R^{p-1}$$

$$= \lambda \int d^D x \sqrt{G} R^p - 4\pi p \lambda \epsilon \int d^D x \sqrt{g} R^{p-1} \delta(x^1, x^2)$$

Solve the most singular terms in EOM:

$$\frac{\delta S_{\text{total}}}{\delta G_{\mu\nu}} \sim p \nabla_a \nabla_b R^{p-1} - 4\pi p(p - 1) R^{p-2} \nabla_a \nabla_b \delta(x^1, x^2) + \cdots$$

Therefore $R \sim -2 \nabla^2 A$ needs to produce $4\pi \delta(x^1, x^2) \Rightarrow A = -\epsilon \log \rho$. 
\[ L = \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \]

\[ S_{EE} = -4\pi \int d^d y \sqrt{g} \left\{ 2\lambda_1 R + \lambda_2 \left( R^a_a - \frac{1}{2} K_a K^a \right) + 2\lambda_3 \left( R^{ab}_{\ ab} - K_{aij} K^{aij} \right) \right\} \]

- Can also show minimizing this gives the location of \( C_n \) at \( n = 1 \).
- Use the cosmic brane method.
- Extrinsic curvature terms show up to compensate

\[ \frac{\delta S}{\delta G_{\mu\nu}} \supset \nabla_z \nabla_z R_{\bar{z}i\bar{z}j} \supset (\nabla_z \nabla_z \nabla_{\bar{z}} A) K_{\bar{z}ij} \]

in EOM by providing e.g. \( \nabla_z \delta(x^1, x^2) K_{\bar{z}ij} \).
Details on Minimization: General Lovelock Gravity

\[ L_{2p}(R) = \frac{1}{2^p} \delta^{\mu_1 \rho_1 \mu_2 \rho_2 \cdots \mu_p \rho_p}_{\nu_1 \sigma_1 \nu_2 \sigma_2 \cdots \nu_p \sigma_p} R_{\mu_1 \rho_1}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}^{\nu_2 \sigma_2} \cdots R_{\mu_p \rho_p}^{\nu_p \sigma_p} \]

\[ S_{EE} = -4\pi p \int d^d y \sqrt{g} L_{2p-2}(r) \]

**Cosmic brane method**

- Lovelock is simple because EOM is 2-derivative, no \( \nabla R \).
- Simply match coefficients of \( \delta(x^1, x^2) \) to linear order in \( \epsilon \).
- “Explains” why \( S_{EE} \) depends only on \( d \)-dim’l intrinsic curvature \( r \).

**Boundary condition method (generalizing [Bhattacharyya Sharma Sinha 1308.5748])**

The \( zz \) component of “Einstein equation” is potentially divergent:

\[ E^{\tilde{z}}_{\tilde{z}} = \frac{1}{2^{p+1}} \delta^{\tilde{z} \mu_1 \rho_1 \mu_2 \rho_2 \cdots \mu_p \rho_p}_{z \nu_1 \sigma_1 \nu_2 \sigma_2 \cdots \nu_p \sigma_p} R_{\mu_1 \rho_1}^{\nu_1 \sigma_1} R_{\mu_2 \rho_2}^{\nu_2 \sigma_2} \cdots R_{\mu_p \rho_p}^{\nu_p \sigma_p} \]

\[ \sim \frac{\epsilon}{Z} \delta^{ij}_{ij} \delta^{k_1 k_2 \cdots i_{p-1} k_{p-1}} R_{i_1 k_1}^{j_1 l_1} R_{i_2 k_2}^{j_2 l_2} \cdots R_{i_p k_p}^{j_{p-1} l_{p-1}} \]

Precisely the equation \( \frac{\delta S_{EE}}{\delta g_{ij}} K_{zij} = 0 \) from minimizing \( S_{EE} \)!
Details on One-Loop Bulk Correction

- Given by the functional determinant of the operator describing quadratic fluctuations of all the bulk fields.
- For $AdS_3/\Gamma$ there is an elegant expression. [Giombi, Maloney & Yin 0804.1773; Yin 0710.2129]

For metric fluctuations:

$$\log Z|_{\text{one-loop}} = - \sum_{\gamma \in \mathcal{P}} \sum_{m=2}^{\infty} \log |1 - q_\gamma^m|$$

- $\mathcal{P}$ is a set of representatives of the primitive conjugacy classes of $\Gamma$.
- $q_\gamma$ is defined by writing the two eigenvalues of $\gamma \in \Gamma \subset PSL(2, \mathbb{C})$ as $q_\gamma^{\pm 1/2}$ with $|q_\gamma| < 1$.
- Similar expressions exist for other bulk fields.
Nice feature 2: at integer $n$ the sum can be done explicitly in terms of rational functions of $n$:

$$S_n|_{\text{one-loop}} = -\frac{n}{n-1} \sum_{k=1}^{n-1} \left[ \frac{\csc^8}{256n^8} x^4 + \frac{(n^2 - 1) \csc^8 + \csc^{10}}{128n^{10}} x^5 + O(x^6) \right]$$

$$= \frac{(n + 1)(n^2 + 11)(3n^4 + 10n^2 + 227)}{3628800n^7} x^4 + O(x^5)$$

where $\csc \equiv \csc \left( \frac{\pi k}{n} \right)$

(4) Analytically continue the one-loop result to $n \to 1$:

$$S|_{\text{one-loop}} = -\left( \frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234} + \frac{167x^8}{36936} + O(x^9) \right)$$

Exactly agrees with known results at leading order:

$$S = -\mathcal{N} \left( \frac{x}{4} \right)^{2h} \frac{\sqrt{\pi}}{4} \frac{\Gamma(2h + 1)}{\Gamma \left( 2h + \frac{3}{2} \right)} + \cdots$$

[Calabrese, Cardy & Tonni '11]
Details on One-Loop Corrections in the Torus Case

Nice feature: only “single-letter” words \( \{L_i, L_i^{-1}\} \) contribute to the leading order in the low / high \( T \) limit.

**Thermal AdS:**

\[
S_n \big|_{\text{one-loop}} = -\frac{1}{n-1} \left[ \frac{2 \sin^4 \left( \frac{\pi L}{R} \right)}{n^3 \sin^4 \left( \frac{\pi L}{nR} \right)} - 2n \right] e^{-4\pi TR} + O \left( e^{-6\pi TR} \right)
\]

\[
S_{\text{one-loop}} = \left[ -\frac{8\pi L}{R} \cot \left( \frac{\pi L}{R} \right) + 8 \right] e^{-4\pi TR} + O \left( e^{-6\pi TR} \right)
\]

\( \Rightarrow \)

\[
S_A - S_\bar{A} = -8\pi \cot \left( \frac{\pi L_A}{R} \right) e^{-4\pi TR} + O \left( e^{-6\pi TR} \right)
\]

Agrees (morally) with a free field calculation in [Herzog & Spillane 1209.6368].

**BTZ:**

\[
S_n \big|_{\text{one-loop}} = -\frac{1}{n-1} \left[ \frac{2 \sinh^4 \left( \frac{\pi TL}{n} \right)}{n^3 \sinh^4 \left( \frac{\pi TL}{n} \right)} - 2n \right] e^{-4\pi TR} + O \left( e^{-6\pi TR} \right)
\]

\[
S_{\text{one-loop}} = \left[ -8\pi TL \coth(\pi TL) + 8 \right] e^{-4\pi TR} + O \left( e^{-6\pi TR} \right)
\]
Where to Evaluate the Entropy Formula?

**Should evaluate it at the conical defect $C_n$ as $n \to 1$**

- $C_1$ is well-defined in principle but hard to find using its definition.
- Can it be found by minimizing some functional?
- In the cosmic brane method, we ask: What is $S_B$ that creates a conical defect in higher derivative gravity, to linear order in $\epsilon$?

**In particular, can this simply be $S_{EE}$ that we saw?**

**Yes, at least for three classes of examples:**

- $f(R)$ gravity
- General 4-derivative gravity
- Lovelock gravity

[XD 1310.5713]
[Bhattacharyya, Sharma & Sinha 1308.5748]
One-Loop Corrections to Ryu-Takayanagi

\[ I = \frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234} + \frac{167x^8}{36936} + \mathcal{O}(x^9) \]

Exactly agrees with CFT predictions:
[Calabrese, Cardy & Tonni '11; Chen & Zhang 1309.5453]

Can also generalize to finite temperature: