Notes on Quantization

The standard way to define a quantum system from a classical one is[1]

- Identify the physical variables q_a
- Define conjugate momenta $p_a \equiv \frac{\delta L}{\delta \dot{q}_a}$
- Write all physical observables in terms of q_a and p_a, eliminating $\dot{q}_a$
- Reinterpret these expressions as quantum operators acting on the Hilbert space by defining q_a and p_a as operators with commutation relations

  $$[q_a, p_b] = i\delta_{ab}$$

  .
- To find the Hamiltonian for the system, we can perform a Legendre transformation on the Lagrangian

  $$H = p_a \dot{q}_a - L$$

  where we should rewrite H completely in terms of q_a and p_a, eliminating $\dot{q}_a$

Precisely this procedure can be applied in field theory if we work at finite volume so that the fields can be written in terms of an infinite number of quantum variables. This can be shown to be equivalent to the following procedure, known as CANONICAL QUANTIZATION

- Identify the physical variables $\phi_a(\vec{x})$
- Define conjugate momentum fields $\pi_a(\vec{x}) \equiv \frac{\delta L}{\delta \dot{\phi}_a(\vec{x})}$
- Write all physical observables in terms of $\phi_a(\vec{x})$ and $\pi_a(\vec{x})$, eliminating $\dot{\phi}_a$
- Reinterpret these expressions as quantum operators acting on the Hilbert space by defining $\phi_a(\vec{x})$ and $\pi_a(\vec{x})$ as operators with commutation relations

  $$[\phi_a(x_1), \pi_b(x_2)] = i\delta_{ab}\delta^d(x_1 - x_2)$$

  .
- To find the Hamiltonian for the system, we can perform a Legendre transformation on the Lagrangian

  $$H = \int d^d\vec{x}\pi_a(\vec{x})\dot{\phi}_a(\vec{x}) - L$$

  where the integral is over spatial coordinates and we should rewrite H completely in terms of $\pi_a(\vec{x})$ and $\phi_a(\vec{x})$, eliminating $\dot{q}_a$

[1]We did not discuss these formal steps when we were discussing the harmonic oscillator, but you can verify that they are equivalent to the steps we took.

When there is time translation invariance in the system, this expression for the Hamiltonian is equivalent to what we would get by deriving the conserved quantity associated with time translation invariance.

The main difficulty with this procedure comes when we have constraints, for example when the fields are not allowed to take arbitrary values at some initial time, either because we impose a restriction, or because one of the equations of motion imposes a restriction. In the simplest cases, the equations of motion allow us to solve for the constrained field in terms of the other fields and conjugate momenta, so that we can simply eliminate it. More generally, we can follow a general procedure worked out by Dirac that is described in detail in chapter 7.6 of Weinberg volume 1.