PATH INTEGRALS IN QFT

Key object in QFT: evolution operator

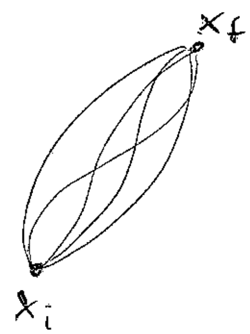
$$\langle \psi_f | e^{-iHt} | \psi_i \rangle$$

action

Claim: $$= \mathcal{N} \int_{\psi(x,t_i)=\psi_i}^{\psi(x,t_f)=\psi_f} [d\psi(x,t)]\, e^{iS[\psi]/\hbar}$$

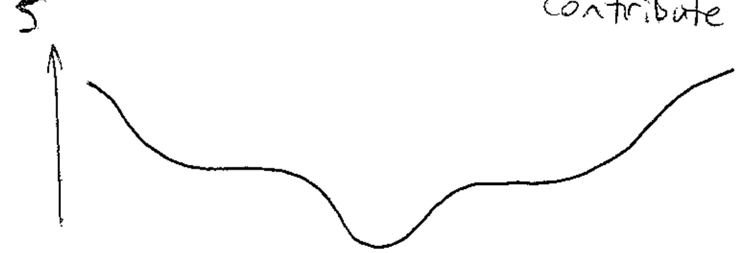
e.g. 1 particle in QM

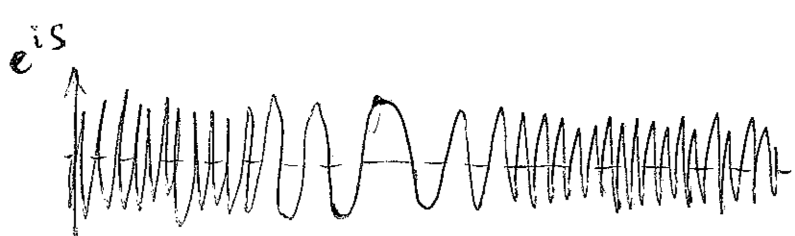
$$\langle x_f | e^{-i(t_f-t_i)H} | x_i \rangle = \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} [dx(t)]\, e^{iS[x(t)]/\hbar}$$



Quantum amplitude is sum/average over all possible histories weighted by e^{iS}

Classical physics $\Rightarrow \hbar \to 0$ $\therefore$ integrand rapidly oscillates
- only minimum action trajectories contribute





Derivation: set $\hbar = 1$ work with $T \equiv t_f - t_i$

- Break time evolution up into many steps & insert complete basis of states

$$\mathcal{U}(x_f, x_i, T) \equiv \langle x_f | e^{-iHT} | x_i \rangle$$
$$= \int \prod dx_n \langle x_f | e^{-iH\varepsilon} | x_N \rangle \langle x_N | e^{-iH\varepsilon} | x_{N-1} \rangle \ldots \langle x_1 | e^{-iH\varepsilon} | x_i \rangle$$
$$\varepsilon \equiv \frac{T}{N}$$

Q: Calculate $\langle x | e^{-iH\varepsilon} | y \rangle$ to $\mathcal{O}(\varepsilon)$ for $H = \frac{p^2}{2m} + V(x)$

$$= \int \frac{dp}{(2\pi)} \langle x | e^{-i\hat{H}\varepsilon} | p \rangle \langle p | y \rangle$$

ordinary fn.

$$= \int \frac{dp}{(2\pi)} \langle x | e^{-iH(p,x)\varepsilon} | p \rangle \langle p | y \rangle$$
$$= \int \frac{dp}{(2\pi)} e^{-iH(p,x)\varepsilon} e^{ip(x-y)}$$

$$\therefore \mathcal{U} = \int \prod_{n=1}^{N-1} dx_n \prod \frac{dp_n}{(2\pi)} e^{i\varepsilon\left(\sum p_j \frac{(x_j - x_i)}{\varepsilon} - H(p_j, x_j)\right)}$$

$\varepsilon \to 0$ Define $x(\tau)$ s.t. $x\left(t_i + \frac{T}{N} n\right) = x_n$

$p(\tau)$ s.t. $p\left(t_i + \frac{T}{N} n\right) = p_n$

$$\mathcal{U}(x_f, x_i, T) = \int_{x(t_i) = x_i}^{x(t_f) = x_f} \prod_\tau [dx(\tau)] \prod_\tau \left[\frac{dp(\tau)}{2\pi}\right] e^{i\int_{t_i}^{t_f} d\tau \{p(\tau)\dot{x}(\tau) - H(p(\tau), x(\tau))\}}$$

If $H(x,p) = \frac{p^2}{2m} + V(x)$

get $\int \prod \frac{dp(\tau)}{2\pi} e^{i\int_{t_i}^{t_f} d\tau \left\{ -\frac{1}{2m}(p(\tau) - m\dot{x}(\tau))^2 \right\}} e^{i\int_{t_i}^{t_f} d\tau \left\{ \frac{m}{2}\dot{x}^2(\tau) - V(x) \right\}}$

$$\therefore \mathcal{U} = N \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \prod [dx(\tau)] \underbrace{e^{i\int_{t_i}^{t_f} d\tau \left[\frac{m}{2}\dot{x}^2(\tau) - V(x(\tau)) \right]}}_{e^{iS}}$$

Correlation functions:

$$\langle x_f, t_f | T\{\mathcal{O}_n(x(t_n)) \dots \mathcal{O}_1(x(t_1))\} | x_i, t_i \rangle$$

$$= \langle x_f\, t_f | \mathcal{O}_n(x(t_n)) \cdots \mathcal{O}_1(x(t_1)) | x_i, t_i \rangle \qquad \text{(assume } t_f \geq t_n \geq t_{n-1} \dots \geq t_i\text{)}$$

$$= \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \prod [dx(t)]\, \mathcal{O}_n(x(t_n)) \cdots \mathcal{O}_1(x(t_1))\, e^{iS}$$

ordinary fns:
order doesn't matter.

Path integral averages automatically give time-ordered correlation fns.

Field theory: want

$$\langle \Omega | T\{\mathcal{O}_1(x(t_1)) \dots \mathcal{O}_n(x(t_n))\} | \Omega \rangle$$ vacuum.

Peskin trick $\quad e^{-iHT}|\psi\rangle = \sum_n e^{-iE_nT} |n\rangle\langle n|\psi\rangle$

$T \to \infty(1-i\varepsilon)$ picks out term w. lowest energy

$$\therefore \lim_{T\to\infty(1-i\varepsilon)} e^{-iHT}|\psi\rangle = C|\Omega\rangle$$

$\langle \Omega | T\{\mathcal{O}_1 \ldots \mathcal{O}_n\} | \Omega$

$$= \lim_{T \to \infty(1-i\varepsilon)} \frac{\int [d\phi_i(\tau)]\, e^{i\int_{-T}^{T} L[\phi_i(\tau)]\,d\tau}\, \mathcal{O}_1(\phi(\tau_1)) \cdots \mathcal{O}_n(\phi(\tau_n))}{\int [d\phi(\tau)]\, e^{i\int_{-T}^{T} L[\phi(\tau)]}}$$

Cancels out normalization factor.

How do we calculate this?

$$\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{QUADRATIC}} + \mathcal{L}_{int}$$

$$\langle \Omega | T\{\theta_1 \ldots \theta_n\} | \Omega \rangle = \lim \frac{\int [dx(\tau)]\, e^{i\int \mathcal{L}_0 d^4x}\, \theta_1 \cdots \theta_n \left(\sum \frac{i}{n!}\left(\int \mathcal{L}_{int}\right)^n\right)}{\int [dx(\tau)]\, e^{i\int \mathcal{L}_0[x(\tau)]} \left(\sum \frac{i}{n!}\left(\int \mathcal{L}_{int}\right)^n\right)}$$

sum of integrals like

$$\int_{-\infty}^{\infty} \prod dx_i\, e^{-\frac{1}{2}M_{ij}x^i x^j}\, x^{i_1} \cdots x^{i_N}$$