FEYNMAN PROPAGATOR IN MOMENTUM SPACE

In calculating transition amplitudes, we often have time-ordered fields contracted together, giving factors of

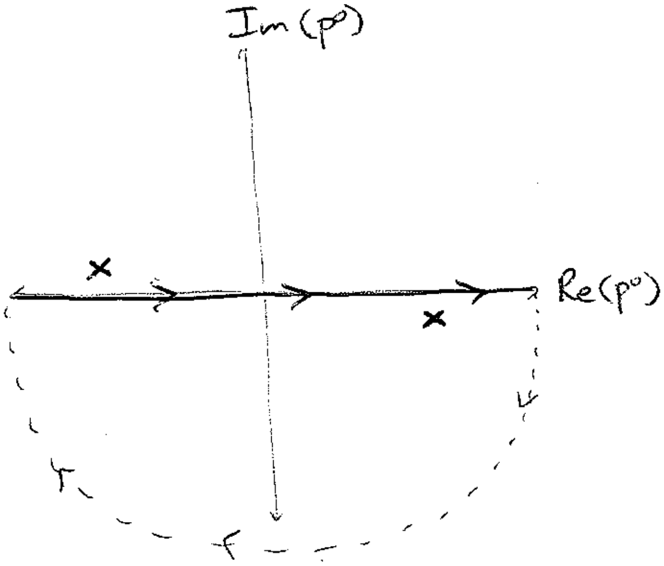
$$\langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = D_F(x-y)$$
$$= \theta(x^0-y^0) D(x-y) + \theta(y^0-x^0) D(y-x)$$

where $D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip\cdot(x-y)}$

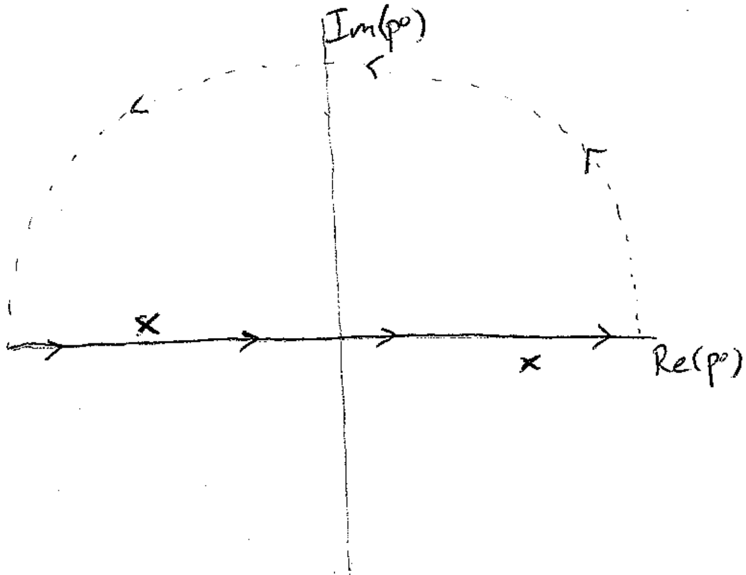
It is usually convenient to perform all of the integrals over positions of the interactions to get the simplest form for amplitudes, and this is made simple if we write

$$D_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot x} \qquad (\star)$$
$$= \int \frac{d^4p}{(2\pi)^4} D_F(p)\, e^{-ip\cdot x}$$

Here $D_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$ is the momentum-space Feynman propagator. The constant ε is a small number which is to be taken to zero at the end of the calculation. We will now check that $(\star)$ gives the correct result for $D_F(x)$. To see this, note that the $i\varepsilon$ changes the poles in the integrand (places where the denominator goes to zero) from $p^0 = \pm\sqrt{m^2+p^2}$ to $p^0 = \pm\sqrt{m^2+p^2} \mp i\varepsilon$, as shown. To evaluate



the p^0 integral in (★) we can use complex contour integration methods. For $x^0 > 0$, the integral is equivalent to the contour integral to the left, since the contribution from the semicircle vanishes as we take it infinitely large. Similarly,



for $x^0 < 0$, the integral (★) is equivalent to the integral on the complex contour shown in the second diagram. In either case, we can use Cauchy's theorem to evaluate the integral,

and we find:

$$x^0 > 0 \qquad (\star) = -2\pi i \int \frac{d^3p}{(2\pi)^4} \frac{i}{2\left(\sqrt{m^2+p^2} - i\varepsilon\right)} e^{-i\sqrt{m^2+p^2}\,x^0 + i\vec{p}\cdot\vec{x}} \underset{\varepsilon\to 0}{=} D(x)$$

$$x^0 < 0 \qquad (\star) = 2\pi i \int \frac{d^3p}{(2\pi)^4} \frac{i}{-2\left(\sqrt{m^2+p^2} - i\varepsilon\right)} e^{i\sqrt{m^2+p^2}\,x^0 + i\vec{p}\cdot\vec{x}} = D(-x)$$

Thus $(\star) = \theta(x^0)D(x) + \theta(-x^0)D(-x)$ as promised