Spinor Fields

Last time, we saw that for the matrices $\mathcal{J}_i$ and $\mathcal{K}_i$ representing the infinitesimal Lorentz transformations, we must have

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k \quad (1)$$

$$[\mathcal{J}_i, \mathcal{K}_j] = i\epsilon_{ijk}\mathcal{K}_k \quad (2)$$

$$[\mathcal{K}_i, \mathcal{K}_j] = -i\epsilon_{ijk}\mathcal{J}_k \ . \quad (3)$$

Any $N \times N$ matrices satisfying these commutation relations will lead us to a valid representation of the Lorentz group. So how do we find the general possibilities? Fortunately, the entire problem can be reduced to a familiar one: understanding the possible matrix representations of angular momentum operators in quantum mechanics.[1]

If we focus only on (2) for now, we see that these commutation relations are exactly the same as the commutation relations for the angular momentum operators in quantum mechanics. This is no coincidence; the angular momentum operators in a quantum system represent the action of infinitesimal rotations on the vector space of its quantum states. The matrices above represent the action of infinitesimal rotations on the vector of field components. The commutations relations must be the same in both cases since they are a property of the rotation group itself[2].

Rotations in quantum mechanics

In quantum mechanics, we usually ask the following question: assuming that there are finitely many basis states for a particle (or atom) in its rest frame (or fixed at some location), what are the possible ways that the angular momentum operators can act on these states. The answer is that we can always group the basis elements (or a redefined set of basis elements) into groups that do not mix with each other under rotations; we can label these groups by a number j that we call "spin"; we can label the basis elements in a group with spin j by number $m = -j, -j+1, \cdots, j$, and the action of the rotation operators on the state $|m\rangle$ is given by

$$J^z|m\rangle = m|m\rangle$$

$$(J^x \pm iJ^y)|m\rangle = \sqrt{j(j+1) - m(m \pm 1)}|m \pm 1\rangle \equiv \left((\mathcal{J}^x_j)_{mn} + i(\mathcal{J}^y_j)_{mn}\right)|n\rangle$$

For spin j, these rules define the action of the operators J^i on the $(2j+1)$-dimensional vector space of states. In the basis labeled by m, these linear transformations can be represented by $(2j+1) \times (2j+1)$ matrices

$$\mathcal{J}^i_{mn} = \langle m|J^i|n\rangle \ ,$$

[1] You may want to have a look at the supplementary handout on "Angular momentum in quantum mechanics".

[2] e.g. what rotation do I get if I rotate a little around the x-axis, then the y-axis, then then do the inverse rotations about x and about y?

and these matrices provide the most general IRREDUCIBLE solution to the commutation relations (2), up to similarity transformations associated with a change of basis.[3]

The simplest examples here are the trivial 1×1 representation $\mathcal{J}^i = 0$, and the spin half 2×2 representation $\mathcal{J}^i = \frac{1}{2}\sigma^i$, where σ^i are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Back to Lorentz transformations

Now that we have understood the general possibilities for matrices satisfying the commutation relations for infinitesimal rotations, it is a simple matter to find the most general possible matrix representations for the full set of Lorentz generators. The trick is to define

$$\mathcal{A}_i = \frac{1}{2}(\mathcal{J}^i + i\mathcal{K}^i) \qquad\qquad \mathcal{B}_i = \frac{1}{2}(\mathcal{J}^i - i\mathcal{K}^i)\ . \tag{4}$$

From the commutation relations (2-3), we find that $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfy the commutation relations

$$\begin{aligned} [\mathcal{A}_i, \mathcal{A}_j] &= i\epsilon_{ijk}\mathcal{A}_k \\ [\mathcal{B}_i, \mathcal{B}_j] &= i\epsilon_{ijk}\mathcal{B}_k \\ [\mathcal{A}_i, \mathcal{B}_j] &= 0\ . \end{aligned} \tag{5}$$

We see that both $\mathcal{A}_i$ and $\mathcal{B}_i$ satisfy the commutation relations for rotations matrices, and that the $\mathcal{A}$ matrices and $\mathcal{B}$ matrices commute with one another.

This set of commutation relations is exactly the same as for two separate sets of angular momentum operators, which might correspond to the angular momentum of two completely independent parts of a quantum system (e.g. two separate particles/atoms with spin). For such a system, the general representation for the angular momentum operators will be labeled by a pair (j_1, j_2) where j_1 corresponds to the spin of the first system, and j_2 corresponds to the spin of the second system. The basis states for this system can be written as

$$|m_1 m_2\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

In terms of these basis elements, the matrices $\mathcal{A}_i$ and $\mathcal{B}_i$ will act as

$$\begin{aligned} \mathcal{A}_i|m_1 m_2\rangle &= \mathcal{J}^i_{m_1 n_1}|n_1 m_2\rangle \\ \mathcal{B}_i|m_1 m_2\rangle &= \mathcal{J}^i_{m_2 n_2}|m_1 n_2\rangle \end{aligned}$$

since $\mathcal{A}_i$ acts only on the first spin and $\mathcal{B}_i$ acts only on the second spin.

[3] From the quantum mechanical problem, it is clear that all solutions can be obtained from these by similarity transformation $\mathcal{J}^i \to S^{-1}\mathcal{J}^i S$, corresponding to writing the operators in a different basis, or by building up larger matrices by taking block-diagonal matrices with these solutions (for various j) as the blocks. The latter are known as REDUCIBLE representations.

From this, we can write explicit matrices for $\mathcal{A}_i$ and $\mathcal{B}_j$, which will be square matrices of size $(2j_1+1)(2j_2+1)$

$$\begin{array}{rcl} (\mathcal{A}_i)_{(m_1 m_2),(n_1 n_2)} & = & (\mathcal{J}^i_{j_1})_{m_1 n_1}\delta_{m_2 n_2} \\ (\mathcal{B}_i)_{(m_1 m_2),(n_1 n_2)} & = & \delta_{m_1 n_1}(\mathcal{J}^i_{j_2})_{m_2 n_2} \end{array}$$

where $\mathcal{J}_j$ are the angular momentum matrices in the spin j representation as defined above. More succinctly, we can write

$$\begin{array}{rcl} \mathcal{A}_i & = & \mathcal{J}^i_{j_1}\otimes \mathbb{1} \\ \mathcal{B}_i & = & \mathbb{1}\otimes \mathcal{J}^i_{j_2} \\ . & & \end{array}$$

The matrices (6) give the most general irreducible representation of the commutation relations (5) for Lorentz transformations. While we have used spin systems in quantum mechanics as a tool to find this solution, it is important to emphasize that these spins systems have nothing to do with the field theories we are discussing or the particles that these field theories describe. We have simply been trying to classify the ways in which fields can transform under Lorentz transformations, and now we have found the most general solution. Given a Lorentz transformation

$$\Lambda = e^{ia_i J_i + ib_i K_i}$$

we can have a field with $(2j_1+1)(2j_2+1)$ components that transforms as

$$\tilde{\phi}_M(\Lambda x) = (e^{ia_i\mathcal{J}_i + ib_i\mathcal{K}_i})_{MN}\phi_N(x)$$

where $\mathcal{J}$ and $\mathcal{K}$ are related to $\mathcal{A}$ and $\mathcal{B}$ by (4) and $\mathcal{A}$ and $\mathcal{B}$ are defined by (6).

Examples

Let's look at the simplest examples. For $(j_1,j_2)=(0,0)$ we have a one-component field with $\mathcal{J}_i=\mathcal{K}_i=0$; this is just a scalar field. For $(j_1,j_2)=(\frac{1}{2},\frac{1}{2})$, we have a four-component field with $\mathcal{A}_i=\frac{1}{2}\sigma^i\otimes\mathbb{1}$, $\mathcal{B}_i=\frac{1}{2}\mathbb{1}\otimes\sigma^i$. It turns out that this choice gives exactly the $\mathcal{J}$ and $\mathcal{K}$ matrices for a vector field. More generally, the choice (j,j) corresponds to a traceless symmetric tensor field with j indices.

What about the cases with $j_1\neq j_2$? For $(j_1,j_2)=(\frac{1}{2},0)$ and $(j_1,j_2)=(0,\frac{1}{2})$, the corresponding fields have only two components, so this is definitely something we haven't seen before. In these cases, we find $\mathcal{A}^i=\frac{1}{2}\sigma^i$, $\mathcal{B}_i=0$ and $\mathcal{A}^i=0$, $\mathcal{B}_i=\frac{1}{2}\sigma^i$ respectively, so

$$\mathcal{J}_i=\frac{1}{2}\sigma^i \qquad\qquad \mathcal{K}_i=\mp\frac{i}{2}\sigma^i\ .$$

We see that in either case, the field components are mixed under rotations like the state vector for a spin-half particle. These kinds of fields are known as SPINOR fields.

Parity

So far, we have discussed only the Lorentz transformations that can be built up from infinitesmial Lorentz transformations. However, certain physical theories (such as QED) may also be invariant under the larger group of Lorentz-transformations that include parity. In this case, the action must have a symmetry under parity transformations, so the fields themselves must have some transformation rules under parity, which we can write as

$$\phi_M(Px) \to \mathcal{P}_{MN}\phi_N(x)\ .$$

The way in which parity acts on the fields must again be consistent with the group multiplication rules for the Lorentz group (including parity). In particular, performing a parity transformation, then a boost, then a parity transformation is equivalent to performing the opposite boost, while preforming parity transformation, then a rotation, then a parity transformation is equivalent to performing the same rotation. This is everything we need to know about how parity transformations combine with other Lorentz transformations, and it is captured by the relations

$$\begin{aligned} \mathcal{P}\mathcal{J}^i\mathcal{P} &= \mathcal{J}^i \\ \mathcal{P}\mathcal{K}^i\mathcal{P} &= -\mathcal{K}^i \end{aligned}$$

In terms of $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned} \mathcal{P}\mathcal{A}^i\mathcal{P} &= \mathcal{B}^i \\ \mathcal{P}\mathcal{B}^i\mathcal{P} &= \mathcal{A}^i \end{aligned} \tag{6}$$

so $\mathcal{P}$ switches the role of A and B. Back to the quantum mechanics analogy, we would like to say that the parity transformation corresponds to the switch $|m_1m_2\rangle \to |m_2m_1\rangle$, but this makes sense only in the case $j_1 = j_2$. For the case $j_1 \neq j_2$, no parity transformation consistent with the relations (6) is possible.[4] However, if we have a theory with one field of type (j_1, j_2), and another field of type (j_2, j_1), we can realize a parity transformation by saying that it switches the two fields. Equivalently, we can combine the components of these two fields into a single field with $2(2j_1+1)(2j_2+1)$ components, and then rotations, boosts, and parity transformations all act on this single field.

Let's see how this works for our spinor fields. If we let η_a and χ_a be two-component fields transforming in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations for the infinitesimal Lorentz transformations, then we can define a four-component field

$$\psi_\alpha = \begin{pmatrix} \eta_a \\ \chi_a \end{pmatrix}$$

[4]This is easy to see in the cases $(j_1, j_2) = (\frac{1}{2}, 0)$ and $(j_1, j_2) = (0, \frac{1}{2})$ where only one of $\mathcal{A}_i$ and $\mathcal{B}_i$ is nonzero.

that combines the two. On this field, rotations, boosts, and parity transformations are represented as

$$\mathcal{J}_i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad \mathcal{K}_i = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad \mathcal{P} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

This resulting field, known as a DIRAC SPINOR field, can be used in a parity-invariant field theory, and is the kind of field that represents all the spin half particles in the standard model.