Representations of the Lorentz Group

For a field theory with fields ϕ_a, $(a = 1..N)$ a general possibility for how the fields change under a Lorentz transformation is

$$\tilde{\phi}_a(\Lambda x) = M_{ab}(\Lambda)\phi_b(x) \ . \tag{1}$$

where M_{ab} is an $N \times N$ matrix that depends on Λ.

The map M from Lorentz transformations to $N \times N$ matrices must be chosen so that the fields after performing two successive Lorentz transformations using the rule (1) are the same as the fields we get after performing the single combined transformation. This requires that:

$$M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2) \tag{2}$$

In other words, the matrix associated with the product of two Lorentz transformations must equal the product of the matrices associated with the individual transformations. When this holds, we say that the matrices M provide a REPRESENTATION of the Lorentz group. Our goal now is to determine all the possible representations of the Lorentz group (for finite-sized matrices M). To start, let's consider a simpler example, the group of rotations around a single axis

Q: In this example, the group elements can be labeled by the rotation angle θ, where rotations by angles differing by a multiple of 2π are equivalent. If the rotation by angle θ acting on some field components is described by a matrix $M(\theta)$, what conditions must this $M(\theta)$ satisfy to be a valid representation?

Q: Suppose that an infinitesimal rotation acts on the field components as

$$\delta\phi_a = \delta\theta L_{ab}\phi_b \ .$$

How does the field transform under a rotation by angle θ (i.e. what is $M(\theta)$)? How is the matrix L related to $M(\theta)$? *Here, we're only worrying about how the components of the field get mixed up, since we already know about how the spacetime points transform. In other words, for this question, we are defining $\delta\phi_i \equiv \tilde{\phi}_i(\Lambda x) - \phi_i(x)$.*

Q: If we demand that $M(2\pi) = M(1)$, what condition does this place on L?

Q: What is $M(\theta)$ for the cases $L = i$ and $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$?

For the second case, it may help to know that, $L = UDU^{-1}$ where $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $U^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$.

These two cases provide examples of two different representations of the group of rotations around a single axis. The crucial point is that if we know how the infinitesimal transformations act, we can figure out the whole representation.

Multiple generators

For more complicated symmetry groups such as the Lorentz group, we have more than one type of infinitesimal transformation. If $\Lambda(\alpha)$ represents a family of Lorentz transformations with $\Lambda(0) = 0$, then for infinitesimal α, we will have

$$\Lambda(\epsilon) = \mathbb{1} + \epsilon\omega + \mathcal{O}(\epsilon^2)$$

It is easy to show that ω is always a linear combination of six matrices:

$$\omega = i(a_i J^i + b_i K^i)$$

as you have read in the notes on special relativity. The matrices $J^1, J^2, J^3, K^1, K^2, K^3$ correspond to infinitesimal rotations around the three axes and boosts in the three directions. The finite rotations or boosts are related to these infinitesimal ones in the same way that you derived above (e.g. $\Lambda = e^{i\theta J^2}$ for a rotation by angle θ in the y-direction).

Q: If M is some representation of the Lorentz group, and we take $\Lambda = \mathbb{1}_{4\times4} + i\epsilon Q + \mathcal{O}(\epsilon^2)$, where Q is one of the generators, argue that $M(\Lambda) = \mathbb{1}_{N\times N} + i\epsilon\mathcal{Q} + \mathcal{O}(\epsilon^2)$ for some $N \times N$ matrix $\mathcal{Q}$.

Thus, given any $N \times N$ representation $M(\Lambda)$, we can define $N \times N$ matrices $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$, $\mathcal{K}^1$, $\mathcal{K}^2$, $\mathcal{K}^3$ corresponding to the six generators. **Q: If $\mathcal{Q}$ is the matrix in a particular representation corresponding to a generator Q, what is $M(e^{iaQ})$ in terms of a and $\mathcal{Q}$?**

Commutators

Q: Consider two infinitesimal Lorentz transformations $\Lambda_1 = e^{\epsilon Q_1}$ and $\Lambda_2 = e^{\epsilon Q_2}$. For the combined transformation $\Lambda_3 = \Lambda_1 \Lambda_2 \Lambda_1^{-1} \Lambda_2^{-1}$, what is $\Lambda_3 - \mathbb{1}$ to the first non-vanishing order in ϵ? *Check your answer to this one with me.*

Q: For some representation, the action on some field components for the previous transformation are specified by the matrices $M_1 = e^{\epsilon \mathcal{Q}_1}$ and $M_2 = e^{\epsilon \mathcal{Q}_2}$. What is the first non-zero term in $M(\Lambda_3) - 1$ in an expansion in powers of ϵ?

Q: **From your results on the previous page, argue that the commutator of any two generators Q_1 and Q_2 must be another generator Q_3 (i.e. a linear combination of the Js and Ks), and that the commutator of the matrices representing these generators must be the matrix representing Q_3.**

Q: **Suppose we consider the set of transformations acting on two-dimensional space which include stretches in the x-direction and rotations in the $x-y$ plane, and all the other transformations that we can get by multiplying these. Determine the number of independent generators of this group of symmetries and work out the set of commutation relations between the generators.**