ANGULAR MOMENTUM IN QUANTUM MECHANICS

$$J_x,\ J_y,\ J_z$$

satisfy $[J_i, J_j] = i\varepsilon_{ijk} J_k$

angular momentum operators

describe infinitesimal rotations:

$$\delta|\psi\rangle = \delta\theta \frac{J_z}{i\hbar}|\psi\rangle$$

Can choose basis of states $|j\ \ m\rangle$

J^2 eigenvalue is $j(j+1)$

J^z eigenvalue $m = -j, -j+1, \ldots, j$

Have:

$$J^z|j\ \ m\rangle = m|j\ \ m\rangle$$

$$J^\pm = J^x \pm i J^y$$

$$J^+|j\ \ m\rangle = \sqrt{j(j+1) - m(m+1)}\ |j\ \ m+1\rangle$$

$$J^-|j\ \ m\rangle = \sqrt{j(j+1) - m(m-1)}\ |j\ \ m-1\rangle$$

In matrix notation: $\sum_m a_m|j\ m\rangle \sim \begin{pmatrix} a_j \\ a_{j-1} \\ \vdots \\ a_{-j} \end{pmatrix}$

$$J^z \sim \begin{pmatrix} j & & & \\ & j-1 & & \\ & & \ddots & \\ & & & -j \end{pmatrix} \qquad J^+ \sim \begin{pmatrix} 0 & \sqrt{2j} & & & \\ & 0 & \sqrt{4j-2} & & \\ & & 0 & \ddots & \\ & & & \ddots & \sqrt{2j} \\ & & & & 0 \end{pmatrix} \sim (J^-)^\dagger$$

These give $(2j+1)\times(2j+1)$ representation of rotation generators.

examples:

spin 0 : $J^z = 0 \quad J^x = 0 \quad J^y = 0$

spin $\frac{1}{2}$: $J^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

spin 1 : $J^x = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ +1 & 0 & 1 \\ 0 & +1 & 0 \end{pmatrix} \quad J^y = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & 0 & 0 \end{pmatrix} \quad J^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

spin $\frac{3}{2}$: $J^x = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \quad J^y = \begin{pmatrix} 0 & -i\frac{\sqrt{3}}{2} & 0 & 0 \\ i\frac{\sqrt{3}}{2} & 0 & -i & 0 \\ 0 & i\frac{\sqrt{3}}{2} & 0 & -i\frac{\sqrt{3}}{2} \\ 0 & 0 & i\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \quad J^z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$

Adding spins:

States of system w. 2 spins: basis $|j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$ ↖ dimension $(2j_1+1)\times(2j_2+1)$

- These aren't eigenstates of J^2_{TOT} (i.e. don't have definite total spin)

Definite total spin states are linear combinations:

$$|J\, M\rangle = \sum_{m_1} C^{JM}_{j_1 m_1\, j_2\, M-m_1} |j_1\, m_1\rangle \otimes |j_2\, m_2\rangle$$

↖ Clebsch-Gordon coefficient

possibilities for J : $|j_1 - j_2| \leq J \leq j_1 + j_2$

In $|J\, M\rangle$ basis, $$g^i = \begin{pmatrix} g^i_{\text{spin}\,(j_1+j_2)} & & & \\ & g^i_{\text{spin}\,j_1+j_2-1} & & \\ & & \ddots & \\ & & & g^i_{\text{spin}\,|j_1-j_2|} \end{pmatrix}$$