l potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|\mathcal{M}(p_i \to p_f)|^2 (2\pi)\delta(E_f - E_i),$$

; is the particle's initial velocity. This formula is a natural modification
. Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.
e to the case of electron scattering from a Coulomb potential (A^0 =
. Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2v^4\sin^4(\theta/2)}.$$

few calculational tricks from Section 5.1, you will have no difficulty
g the general cross section in the relativistic case; see Problem 5.1.)

Chapter 5

Elementary Processes of Quantum Electrodynamics

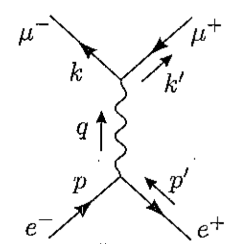
Finally, after three long chapters of formalism, we are ready to perform some real relativistic calculations, to begin working out the predictions of Quantum Electrodynamics. First we will return to the process considered in Chapter 1, the annihilation of an electron-positron pair into a pair of heavier fermions. We will study this paradigm process in extreme detail in the next three sections, then do a few more simple QED calculations in Sections 5.4 and 5.5. The problems at the end of the chapter treat several additional QED processes. More complete surveys of QED can be found in the books of Jauch and Rohrlich (1976) and of Berestetskii, Lifshitz, and Pitaevskii (1982).

5.1 $e^+e^- \to \mu^+\mu^-$: Introduction

The reaction $e^+e^- \to \mu^+\mu^-$ is the simplest of all QED processes, but also one of the most important in high-energy physics. It is fundamental to the understanding of all reactions in e^+e^- colliders, and is in fact used to calibrate such machines. The related process $e^+e^- \to q\bar{q}$ (a quark-antiquark pair) is extraordinarily useful in determining the properties of elementary particles.

In this section we will compute the *unpolarized* cross section for $e^+e^- \to \mu^+\mu^-$, to lowest order. In Chapter 1 we used elementary arguments to guess the answer (Eq. (1.8)) in the limit where all the fermions are massless. We now relax that restriction and retain the muon mass in the calculation. Retaining the electron mass as well would be easy but pointless, since the ratio $m_e/m_\mu \approx 1/200$ is much smaller than the fractional error introduced by neglecting higher-order terms in the perturbation series.

Using the Feynman rules from Section 4.8, we can at once draw the diagram and write down the amplitude for our process:



$$= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k').$$

Rearranging this slightly and leaving the spin superscripts implicit, we have

$$i\mathcal{M}\big(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\big) = \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \quad (5.1)$$

This answer for the amplitude $\mathcal{M}$ is simple, but not yet very illuminating.

To compute the differential cross section, we need an expression for $|\mathcal{M}|^2$, so we must find the complex conjugate of $\mathcal{M}$. A bi-spinor product such as $\bar{v}\gamma^\mu u$ can be complex-conjugated as follows:

$$(\bar{v}\gamma^\mu u)^* = u^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger v = u^\dagger(\gamma^\mu)^\dagger\gamma^0 v = u^\dagger\gamma^0\gamma^\mu v = \bar{u}\gamma^\mu v.$$

(This is another advantage of the 'bar' notation.) Thus the squared matrix element is

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\Big). \quad (5.2)$$

At this point we are still free to specify any particular spinors $u^s(p)$, $\bar{v}^{s'}(p')$, and so on, corresponding to any desired spin states of the fermions. In actual experiments, however, it is difficult (though not impossible) to retain control over spin states; one would have to prepare the initial state from polarized materials and/or analyze the final state using spin-dependent multiple scattering. In most experiments the electron and positron beams are unpolarized, so the measured cross section is an *average* over the electron and positron spins s and s'. Muon detectors are normally blind to polarization, so the measured cross section is a *sum* over the muon spins r and r'.

The expression for $|\mathcal{M}|^2$ simplifies considerably when we throw away the spin information. We want to compute

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'}\big|\mathcal{M}(s,s' \to r,r')\big|^2.$$

The spin sums can be performed using the completeness relations from Section 3.3:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m; \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m. \quad (5.3)$$

Working with the first half of (5.2), and writing in spinor indices so we can freely move the v next to the $\bar{v}$, we have

$$\sum_{s,s'}\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p') = (\not{p}' - m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}$$
$$= \text{trace}\big[(\not{p}' - m)\gamma^\mu(\not{p}+m)\gamma^\nu\big].$$

Evaluating the second half of (5.2) in the same way, we arrive at the desired simplification:

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\,\text{tr}\Big[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\Big]\,\text{tr}\Big[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\Big]. \quad (5.4)$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very general: Any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices.

Trace Technology

This last step would hardly be an improvement if the traces had to be laboriously computed by brute force. But Feynman found that they could be worked out easily by appealing to the algebraic properties of the γ matrices. Since the evaluation of such traces occurs so often in QED calculations, it is worthwhile to pause and attack the problem systematically, once and for all.

We would like to evaluate traces of products of n gamma matrices, where $n = 0, 1, 2, \ldots$. (For the present problem we need $n = 2, 3, 4$.) The $n = 0$ case is fairly easy: $\text{tr}\,\mathbf{1} = 4$. The trace of one γ matrix is also easy. From the explicit form of the matrices in the chiral representation, we have

$$\text{tr}\,\gamma^\mu = \text{tr}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = 0.$$

It is useful to prove this result in a more abstract way, which generalizes to an arbitrary odd number of γ matrices:

$$\begin{aligned}\text{tr}\,\gamma^\mu &= \text{tr}\,\gamma^5\gamma^5\gamma^\mu && \text{since } (\gamma^5)^2 = 1 \\ &= -\,\text{tr}\,\gamma^5\gamma^\mu\gamma^5 && \text{since } \{\gamma^\mu,\gamma^5\} = 0 \\ &= -\,\text{tr}\,\gamma^5\gamma^5\gamma^\mu && \text{using cyclic property of trace} \\ &= -\,\text{tr}\,\gamma^\mu.\end{aligned}$$

Since the trace of γ^μ is equal to minus itself, it must vanish. For n γ-matrices we would get n minus signs in the second step (as we move the second γ^5 all the way to the right), so the trace must vanish if n is odd.

To evaluate the trace of two γ matrices, we again use the anticommutation properties and the cyclic property of the trace:

$$\begin{aligned}\text{tr}\,\gamma^\mu\gamma^\nu &= \text{tr}\big(2g^{\mu\nu}\cdot\mathbf{1} - \gamma^\nu\gamma^\mu\big) && \text{(anticommuatation)} \\ &= 8g^{\mu\nu} - \text{tr}\,\gamma^\mu\gamma^\nu && \text{(cyclicity)}\end{aligned}$$

Thus $\text{tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$. The trace of any even number of γ matrices can be evaluated in the same way: Anticommute the first γ matrix all the way to the right, then cycle it back to the left. Thus for the trace of four γ matrices, we have

$$\begin{aligned}\text{tr}\big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\big) &= \text{tr}\big(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\big) \\ &= \text{tr}\big(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\big).\end{aligned}$$

Using the cyclic property on the last term and bringing it to the left-hand side, we find

$$\begin{aligned}\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= g^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma - g^{\mu\rho}\operatorname{tr}\gamma^\nu\gamma^\sigma + g^{\mu\sigma}\operatorname{tr}\gamma^\nu\gamma^\rho \\ &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).\end{aligned}$$

In this manner one can always reduce a trace of n γ-matrices to a sum of traces of $(n-2)$ γ-matrices. The case $n = 6$ is easy to work out, but has fifteen terms (the number of ways of grouping the six indices in pairs to make terms of the form $g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}$). Fortunately, we will not need it in this book. (If you ever do need to evaluate such complicated traces, it may be easier to learn to use one of the several computer programs that can perform symbolic manipulations on Dirac matrices.)

Starting in Section 5.2, we will often need to evaluate traces involving γ^5. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the trace of γ^5 times any odd number of other γ matrices is zero. It is also easy to show that the trace of γ^5 itself is zero:

$$\operatorname{tr}\gamma^5 = \operatorname{tr}(\gamma^0\gamma^0\gamma^5) = -\operatorname{tr}(\gamma^0\gamma^5\gamma^0) = -\operatorname{tr}(\gamma^0\gamma^0\gamma^5) = -\operatorname{tr}\gamma^5.$$

The same trick works for $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^5)$, if we insert two factors of γ^α for some α different from both μ and ν. The first nonvanishing trace involving γ^5 contains four other γ matrices. In this case the trick still works unless every γ matrix appears, so $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 0$ unless $(\mu\nu\rho\sigma)$ is some permutation of (0123). From the anticommutation rules it also follows that interchanging any two of the indices simply changes the sign of the trace, so $\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$. The overall constant turns out to be $-4i$, as you can easily check by plugging in $(\mu\nu\rho\sigma) = (0123)$.

Here is a summary of the trace theorems, for convenient reference:

$$\begin{aligned}\operatorname{tr}(\mathbf{1}) &= 4 \\ \operatorname{tr}(\text{any odd \# of }\gamma\text{'s}) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ \operatorname{tr}(\gamma^5) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0 \\ \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}\end{aligned} \tag{5.5}$$

Expressions resulting from use of the last formula can be simplified by means of the identities

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= -24 \\ \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu{}_\nu \\ \epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} &= -2(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho)\end{aligned} \tag{5.6}$$

All of these can be derived by first appealing to symmetry arguments, then evaluating one special case to determine the overall constant.

Another useful identity allows one to reverse the order of all the γ matrices inside a trace:

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \operatorname{tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu). \tag{5.7}$$

To prove this relation, consider the matrix $C \equiv \gamma^0\gamma^2$ (essentially the charge-conjugation operator). This matrix satisfies $C^2 = 1$ and $C\gamma^\mu C = -(\gamma^\mu)^T$. Thus if there are n γ-matrices inside the trace,

$$\begin{aligned}\operatorname{tr}(\gamma^\mu\gamma^\nu\cdots) &= \operatorname{tr}(C\gamma^\mu C\, C\gamma^\nu C\cdots) \\ &= (-1)^n \operatorname{tr}[(\gamma^\mu)^T(\gamma^\nu)^T\cdots] \\ &= \operatorname{tr}(\cdots\gamma^\nu\gamma^\mu),\end{aligned}$$

since the trace vanishes unless n is even. It is easy to show that the reversal identity (5.7) is also valid when the trace contains one or more factors of γ^5.

When two γ matrices inside a trace are dotted together, it is easiest to eliminate them before evaluating the trace. For example,

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = g_{\mu\nu}g^{\mu\nu} = 4. \tag{5.8}$$

The following *contraction identities*, all easy to prove using the anticommutation relations, can be used when other γ matrices lie in between:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu\end{aligned} \tag{5.9}$$

Note the reversal of order in the last identity.

All of the γ matrix identities proved in this section are collected for reference in the Appendix.

Unpolarized Cross Section

We now return to the evaluation of the squared matrix element, Eq. (5.4). The electron trace is

$$\operatorname{tr}[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu] = 4[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)].$$

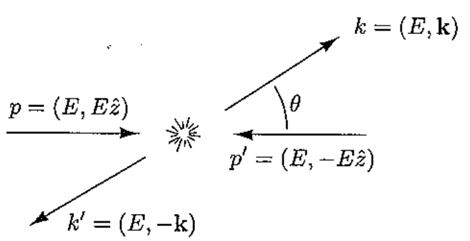
The terms with only one factor of m vanish, since they contain an odd number of γ matrices. Similarly, the muon trace is

$$\operatorname{tr}[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu] = 4[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)].$$

From now on we will set $m_e = 0$, as discussed at the beginning of this section. Dotting these expressions together and collecting terms, we get the simple result

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\Big]. \tag{5.10}$$

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, p', k, k', and q in terms of the basic kinematic variables—energies and angles—in that frame. In practice, the choice of frame will be dictated by the experimental conditions. In this book, we will usually make the simplest choice of evaluating cross sections in the center-of-mass frame. For this choice, the initial and final 4-momenta for $e^+e^- \to \mu^+\mu^-$ can be written as follows:



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos\theta$$

To compute the squared matrix element we need

$$q^2 = (p+p')^2 = 4E^2; \qquad p\cdot p' = 2E^2;$$
$$p\cdot k = p'\cdot k' = E^2 - E|\mathbf{k}|\cos\theta; \qquad p\cdot k' = p'\cdot k = E^2 + E|\mathbf{k}|\cos\theta.$$

We can now rewrite Eq. (5.10) in terms of E and θ:

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\Big[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\mu^2E^2\Big]\\ &= e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{5.11}$$

All that remains is to plug this expression into the cross-section formula derived in Section 4.5. Since there are only two particles in the final state and we are working in the center-of-mass frame, we can use the simplified formula (4.84). For our problem $|v_A - v_B| = 2$ and $E_A = E_B = E_{\rm cm}/2$, so we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{\rm cm}^2}\frac{|\mathbf{k}|}{16\pi^2E_{\rm cm}}\cdot\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\\ &= \frac{\alpha^2}{4E_{\rm cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].\end{aligned} \tag{5.12}$$

Integrating over $d\Omega$, we find the total cross section:

$$\sigma_{\rm total} = \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right). \tag{5.13}$$

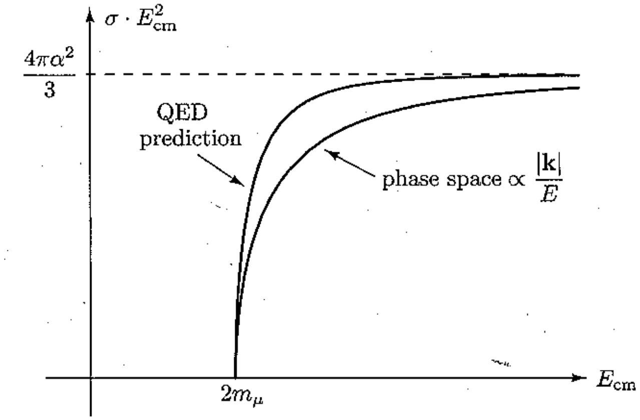


Figure 5.1. Energy dependence of the total cross section for $e^+e^- \to \mu^+\mu^-$, compared to "phase space" energy dependence.

In the high-energy limit where $E \gg m_\mu$, these formulae reduce to those given in Chapter 1:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\xrightarrow[E\gg m_\mu]{} \frac{\alpha^2}{4E_{\rm cm}^2}(1+\cos^2\theta);\\ \sigma_{\rm total} &\xrightarrow[E\gg m_\mu]{} \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\left(1-\frac{3}{8}\left(\frac{m_\mu}{E}\right)^4 - \cdots\right).\end{aligned} \tag{5.14}$$

Note that these expressions have the correct dimensions of cross sections. In the high-energy limit, $E_{\rm cm}$ is the only dimensionful quantity in the problem, so dimensional analysis dictates that $\sigma_{\rm total} \propto E_{\rm cm}^{-2}$. Since we knew from the beginning that $\sigma_{\rm total} \propto \alpha^2$, we only had to work to get the factor of $4\pi/3$.

The energy dependence of the total cross-section formula (5.13) near threshold is shown in Fig. 5.1. Of course the cross section is zero for $E_{\rm cm} < 2m_\mu$. It is interesting to compare the shape of the actual curve to the shape one would obtain if $|\mathcal{M}|^2$ did not depend on energy, that is, if all the energy dependence came from the phase-space factor $|\mathbf{k}|/E$. To test Quantum Electrodynamics, an experiment must be able to resolve deviations from the naive phase-space prediction. Experimental results from pair production of both μ and τ leptons confirm that these particles behave as QED predicts. Figure 5.2 compares formula (5.13) to experimental measurements of the $\tau^+\tau^-$ threshold.

Before discussing our result further, let us pause to summarize how we obtained it. The method extends in a straightforward way to the calculation of unpolarized cross sections for other QED processes. The general procedure is as follows:

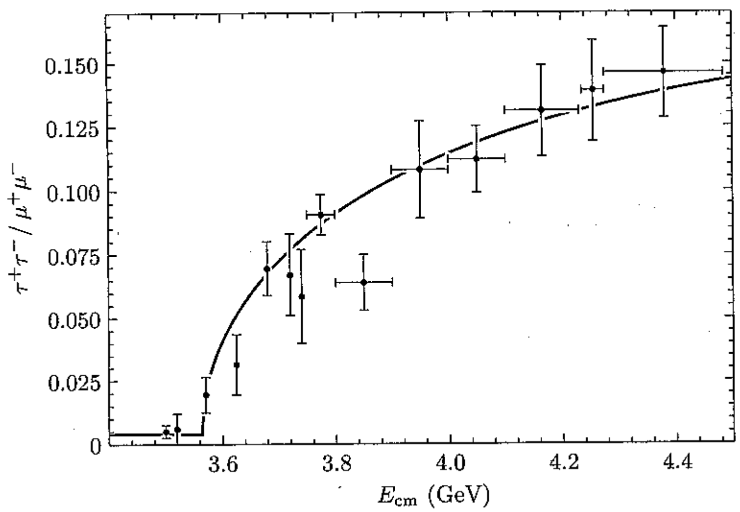


Figure 5.2. The ratio $\sigma(e^+e^- \to \tau^+\tau^-)/\sigma(e^+e^- \to \mu^+\mu^-)$ of measured cross sections near the threshold for $\tau^+\tau^-$ pair-production, as measured by the DELCO collaboration, W. Bacino, et. al., *Phys. Rev. Lett.* **41**, 13 (1978). Only a fraction of τ decays are included, hence the small overall scale. The curve shows a fit to the theoretical formula (5.13), with a small energy-independent background added. The fit yields $m_\tau = 1782^{+2}_{-7}$ MeV.

1. Draw the diagram(s) for the desired process.
2. Use the Feynman rules to write down the amplitude $\mathcal{M}$.
3. Square the amplitude and average or sum over spins, using the completeness relations (5.3). (For processes involving photons in the final state there is an analogous completeness relation, derived in Section 5.5.)
4. Evaluate traces using the trace theorems (5.5); collect terms and simplify the answer as much as possible.
5. Specialize to a particular frame of reference, and draw a picture of the kinematic variables in that frame. Express all 4-momentum vectors in terms of a suitably chosen set of variables such as E and θ.
6. Plug the resulting expression for $|\mathcal{M}|^2$ into the cross-section formula (4.79), and integrate over phase-space variables that are not measured to obtain a differential cross section in the desired form. (In our case these integrations were over the constrained momenta $\mathbf{k}'$ and $|\mathbf{k}|$, and were performed in the derivation of Eq. (4.84).)

While other calculations (especially those involving loop diagrams) often require additional tricks, nearly every QED calculation will involve the basic procedures outlined here.

Production of Quark-Antiquark Pairs

The asymptotic energy dependence of the $e^+e^- \to \mu^+\mu^-$ cross-section formula sets the scale for all e^+e^- annihilation cross sections. A particularly important example is the cross section for

$$e^+e^- \to \text{hadrons},$$

that is, the total cross section for production of any number of strongly interacting particles.

In our current understanding of the strong interactions, given by the theory called Quantum Chromodynamics (QCD), all hadrons are composed o Dirac fermions called *quarks*. Quarks appear in a variety of types, called *flavors*, each with its own mass and electric charge. A quark also carries ar additional quantum number, *color*, which takes one of three values. Colo serves as the "charge" of QCD, as we will discuss in Chapter 17.

According to QCD, the simplest e^+e^- process that ends in hadrons is

$$e^+e^- \to q\bar{q},$$

the annihilation of an electron and a positron, through a virtual photon, into a quark-antiquark pair. After they are created, the quarks interact with one another through their strong forces, producing more quark pairs. Eventually the quarks and antiquarks combine to form some number of mesons and baryons

To adapt our results for muon production to handle the case of quarks we must make three modifications:

1. Replace the muon charge e with the quark charge $Q|e|$.
2. Count each quark three times, one for each color.
3. Include the effects of the strong interactions of the produced quark anc antiquark.

The first two changes are easy to make. For the first, it is simply necessary t know the masses and charges of each flavor of quark. For u, c, and t quark we have $Q = 2/3$, while for d, s, and b quarks we have $Q = -1/3$. The cross section formulae are proportional to the square of the charge of the final-stat particle, so we can simply insert a factor of Q^2 into any of these formula to obtain the cross section for production of any particular variety of quark Counting colors is necessary because experiments measure only the total cros section for production of all three colors. (The hadrons that are actually de tected are colorless.) In any case, this counting is easy: Just multiply th answer by 3.

If you know a little about the strong interaction, however, you migh think this is all a big joke. Surely the third modification is extremely difficul to make, and will drastically alter the predictions of QED. The amazing trut is that in the high-energy limit, the effect of the strong interaction on th quark production process can be completely neglected. As we will discuss i Part III, the only effect of the strong interaction (in this limit) is to dres up the final-state quarks into bunches of hadrons. This simplification is due to a phenomenon called *asymptotic freedom*; it played a crucial role in the identification of Quantum Chromodynamics as the correct theory of the strong force.

Thus in the high-energy limit, we expect the cross section for the reaction $e^+e^- \to q\bar{q}$ to approach $3 \cdot Q^2 \cdot 4\pi\alpha^2/3E_{cm}^2$. It is conventional to define

$$1 \text{ unit of R} \equiv \frac{4\pi\alpha^2}{3E_{cm}^2} = \frac{86.8\,\text{nbarns}}{(E_{cm} \text{ in GeV})^2}. \tag{5.15}$$

The value of a cross section in units of R is therefore its ratio to the asymptotic value of the $e^+e^- \to \mu^+\mu^-$ cross section predicted by Eq. (5.14). Experimentally, the easiest quantity to measure is the total rate for production of all hadrons. Asymptotically, we expect

$$\sigma(e^+e^- \to \text{hadrons}) \underset{E_{cm}\to\infty}{\longrightarrow} 3 \cdot \Big(\sum_i Q_i^2\Big)\, \text{R}, \tag{5.16}$$

where the sum runs over all quarks whose masses are smaller than $E_{cm}/2$. When $E_{cm}/2$ is in the vicinity of one of the quark masses, the strong interactions cause large deviations from this formula. The most dramatic such effect is the appearance of *bound states* just below $E_{cm} = 2m_q$, manifested as very sharp spikes in the cross section.

Experimental measurements of the cross section for e^+e^- annihilation to hadrons between 2.5 and 40 GeV are shown in Fig. 5.3. The data shows three distinct regions: a low-energy region in which u, d, and s quark pairs are produced; a region above the threshold for production of c quark pairs; and a region also above the threshold for b quark pairs. The prediction (5.16) is shown as a set of solid lines; it agrees quite well with the data in each region, as long as the energy is well away from the thresholds where the high-energy approximation breaks down. The dotted curves show an improved theoretical prediction, including higher-order corrections from QCD, which we will discuss in Section 17.2. This explanation of the e^+e^- annihilation cross section is a remarkable success of QCD. In particular, experimental verification of the factor of 3 in (5.16) is one piece of evidence for the existence of color.

The angular dependence of the differential cross section is also observed experimentally.* At high energy the hadrons appear in *jets*, clusters of several hadrons all moving in approximately the same direction. In most cases there are two jets, with back-to-back momenta, and these indeed have the angular dependence $(1 + \cos^2\theta)$.

*The basic features of hadron production in high-energy e^+e^- annihilation are reviewed by P. Duinker, *Rev. Mod. Phys.* **54**, 325 (1982).

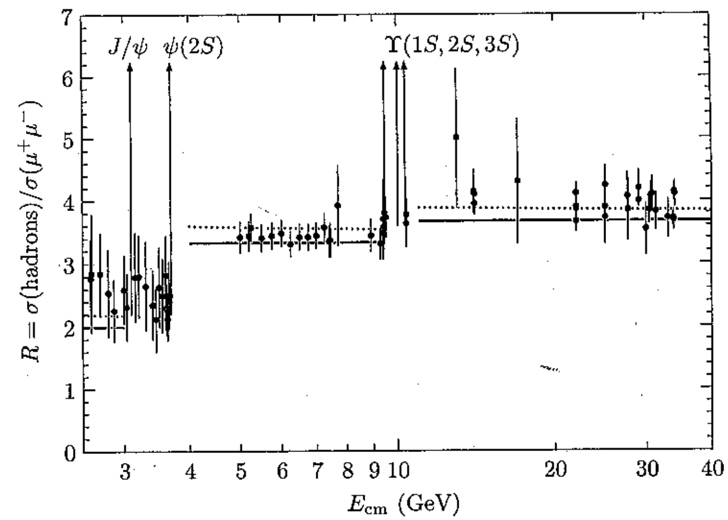


Figure 5.3. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, *Phys. Rev.* **D** (to appear). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the simple prediction (5.16).

5.2 $e^+e^- \to \mu^+\mu^-$: Helicity Structure

The unpolarized cross section for a reaction is generally easy to calculat (and to measure) but hard to understand. Where does the $(1 + \cos^2\theta)$ angu lar dependence come from? We can answer this question by computing th $e^+e^- \to \mu^+\mu^-$ cross section for each set of spin orientations separately.

First we must choose a basis of polarization states. To get a simple answe in the high-energy limit, the best choice is to quantize each spin along th direction of the particle's motion, that is, to use states of definite helicit Recall that in the massless limit, the left- and right-handed helicity state of a Dirac particle live in different representations of the Lorentz group. W might therefore expect them to behave independently, and in fact they do.

In this section we will compute the polarized $e^+e^- \to \mu^+\mu^-$ cross section using the helicity basis, in two different ways: first, by using trace technolog but with the addition of helicity projection operators to project out the desire left- or right-handed spinors; and second, by plugging explicit expressions fc these spinors directly into our formula for the amplitude $\mathcal{M}$. Throughout th section we work in the high-energy limit where all fermions are effective massless. (The calculation can be done for lower energy, but it is much more difficult and no more instructive.)†

Our starting point for both methods of calculating the polarized cross section is the amplitude

$$i\mathcal{M}\big(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\big) = \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \quad (5.1)$$

We would like to use the spin sum identities to write the squared amplitude in terms of traces as before, even though we now want to consider only one set of polarizations at a time. To do this, we note that for massless fermions, the matrices

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (5.17)$$

are *projection operators* onto right- and left-handed spinors, respectively. Thus if in (5.1) we make the replacement

$$\bar{v}(p')\gamma^\mu u(p) \longrightarrow \bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p),$$

the amplitude for a right-handed electron is unchanged while that for a left-handed electron becomes zero. Note that since

$$\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p) = v^\dagger(p')\Big(\frac{1+\gamma^5}{2}\Big)\gamma^0\gamma^\mu u(p), \qquad (5.18)$$

this same replacement imposes the requirement that $v(p')$ also be a right-handed spinor. Recall from Section 3.5, however, that the right-handed spinor $v(p')$ corresponds to a *left*-handed positron. Thus we see that the annihilation amplitude vanishes when both the electron and the positron are right-handed. In general, the amplitude vanishes (in the massless limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.

Having inserted this projection operator, we are now free to sum over the electron and positron spins in the squared amplitude; of the four terms in the sum, only one (the one we want) is nonzero. The electron half of $|\mathcal{M}|^2$, for a right-handed electron and a left-handed positron, is then

$$\begin{aligned}\sum_{\text{spins}}\Big|\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p)\Big|^2 &= \sum_{\text{spins}}\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p)\,\bar{u}(p)\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)v(p') \\ &= \text{tr}\Big[\not{p}'\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)\not{p}\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)\Big] \\ &= \text{tr}\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\Big(\frac{1+\gamma^5}{2}\Big)\Big]\end{aligned}$$

†The general formalism for S-matrix elements between states of definite helicity is presented in a beautiful paper of M. Jacob and G. C. Wick, *Ann. Phys.* **7**, 404 (1959).

$$= 2\big(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p\cdot p' - i\epsilon^{\alpha\mu\beta\nu}p'_\alpha p_\beta\big). \qquad (5.19)$$

The indices in this expression are to be dotted into those of the muon half of the squared amplitude. For a right-handed μ^- and a left-handed μ^+, an identical calculation yields

$$\sum_{\text{spins}}\Big|\bar{u}(k)\gamma_\mu\Big(\frac{1+\gamma^5}{2}\Big)v(k')\Big|^2 = 2\big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' - i\epsilon_{\rho\mu\sigma\nu}k^\rho k'^\sigma\big). \quad (5.20)$$

Dotting (5.19) into (5.20), we find that the squared matrix element for $e^-_R e^+_L \to \mu^-_R\mu^+_L$ in the center-of-mass frame is

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{4e^4}{q^4}\Big[2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) - \epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}p'_\alpha p_\beta k^\rho k'^\sigma\Big] \\ &= \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) - (p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k)\Big] \\ &= \frac{16e^4}{q^4}(p\cdot k')(p'\cdot k) \\ &= e^4(1+\cos\theta)^2. \end{aligned} \qquad (5.21)$$

Plugging this result into (4.85) gives the differential cross section,

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_R\mu^+_L) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2. \qquad (5.22)$$

There is no need to repeat the entire calculation to obtain the other three nonvanishing helicity amplitudes. For example, the squared amplitude for $e^-_R e^+_L \to \mu^-_L\mu^+_R$ is identical to (5.20) but with γ^5 replaced by $-\gamma^5$ on the left-hand side, and thus $\epsilon_{\rho\mu\sigma\nu}$ replaced by $-\epsilon_{\rho\mu\sigma\nu}$ on the right-hand side. Propagating this sign though (5.21), we easily see that

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_L\mu^+_R) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2. \qquad (5.23)$$

Similarly,

$$\begin{aligned}\frac{d\sigma}{d\Omega}(e^-_L e^+_R \to \mu^-_R\mu^+_L) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2; \\ \frac{d\sigma}{d\Omega}(e^-_L e^+_R \to \mu^-_L\mu^+_R) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2.\end{aligned} \qquad (5.24)$$

(These two results actually follow from the previous two by parity invariance. The other twelve helicity cross sections (for instance, $e^-_L e^+_R \to \mu^-_L\mu^+_L$) are zero, as we saw from Eq. (5.18). Adding up all sixteen contributions, and dividing by 4 to average over the electron and positron spins, we recover the unpolarized cross section in the massless limit, Eq. (5.14).

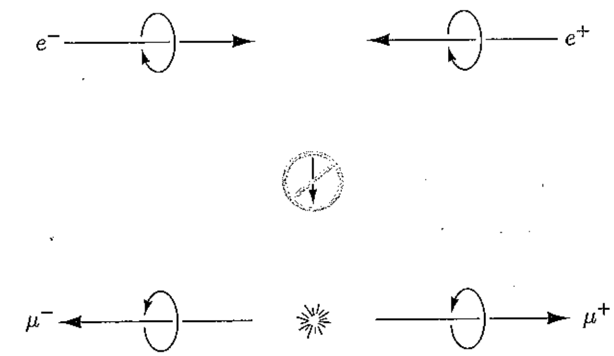


Figure 5.4. Conservation of angular momentum requires that if the z-component of angular momentum is measured, it must have the same value as initially.

Note that the cross section (5.22) for $e^-_R e^+_L \to \mu^-_R \mu^+_L$ vanishes at $\theta = 180°$. This is just what we would expect, since for $\theta = 180°$, the total angular momentum of the final state is opposite to that of the initial state (see Figure 5.4).

This completes our first calculation of the polarized $e^+e^- \to \mu^+\mu^-$ cross sections. We will now redo the calculation in a manner that is more straightforward, more enlightening, and no more difficult. We will calculate the amplitude $\mathcal{M}$ (rather than the squared amplitude) directly, using explicit values for the spinors and γ matrices. This method does have its drawbacks: It forces us to specialize to a particular frame of reference much sooner, so manifest Lorentz invariance is lost. More pragmatically, it is very cumbersome except in the nonrelativistic and ultra-relativistic limits.

Consider again the amplitude

$$\mathcal{M} = \frac{e^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \tag{5.25}$$

In the high-energy limit, our general expressions for Dirac spinors become

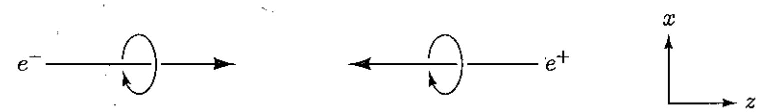
$$\begin{aligned} u(p) &= \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E}\begin{pmatrix}\frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi \\ \frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi\end{pmatrix}; \\ v(p) &= \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E}\begin{pmatrix}\frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi \\ -\frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi\end{pmatrix}. \end{aligned} \tag{5.26}$$

A right-handed spinor satisfies $(\hat{p}\cdot\boldsymbol{\sigma})\xi = +\xi$, while a left-handed spinor has $(\hat{p}\cdot\boldsymbol{\sigma})\xi = -\xi$. (Remember once again that for antiparticles, the handedness of the spinor is the opposite of the handedness of the particle.) We must evaluate expressions of the form $\bar{v}\gamma^\mu u$, so we need

$$\gamma^0\gamma^\mu = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} = \begin{pmatrix}\bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu\end{pmatrix}. \tag{5.27}$$

Thus we see explicitly that the amplitude is zero when one of the spinors is left-handed and the other is right-handed. In the language of Chapter 1, the Clebsch-Gordan coefficients that couple the vector photon to the product of such spinors are zero; those coefficients are just the off-block-diagonal elements of the matrix $\gamma^0\gamma^\mu$ (in the chiral representation).

Let us choose p and p' to be in the $\pm z$-directions, and first consider the case where the electron is right-handed and the positron is left-handed:



Thus for the electron we have $\xi = \binom{1}{0}$, corresponding to spin up in the z-direction, while for the positron we have $\xi = \binom{0}{1}$, also corresponding to (physical) spin up in the z-directon. Both particles have $(\hat{p}\cdot\boldsymbol{\sigma})\xi = +\xi$, so the spinors are

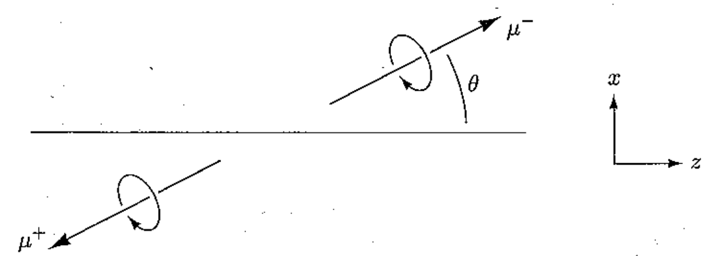
$$u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \qquad v(p') = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}. \tag{5.28}$$

The electron half of the matrix element is therefore

$$\bar{v}(p')\gamma^\mu u(p) = 2E\,(0,\ -1)\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix} = -2E\,(0,1,i,0). \tag{5.29}$$

We can interpret this expression by saying that the virtual photon has circular polarization in the $+z$-direction; its polarization vector is $\epsilon_+ = (1/\sqrt{2})(\hat{x}+i\hat{y})$.

Next we must calculate the muon half of the matrix element. Let the μ^- be emitted at an angle θ to the z-axis, and consider first the case where it is right-handed (and the μ^+ is therefore left-handed):



To calculate $\bar{u}(k)\gamma^\mu v(k')$ we could go back to expressions (5.26), but then it would be necessary to find the correct spinors ξ corresponding to polarization along the muon momentum. It is much easier to use a trick: Since any expression of the form $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector, we can just rotate the result

(5.29). Rotating that vector by an angle θ in the xz-plane, we find

$$\begin{aligned}\bar{u}(k)\gamma^\mu v(k') &= \left[\bar{v}(k')\gamma^\mu u(k)\right]^* \\ &= \left[-2E\,(0,\cos\theta,i,\sin\theta)\right]^* \\ &= -2E\,(0,\cos\theta,-i,\sin\theta).\end{aligned} \tag{5.30}$$

This vector can also be interpreted as the polarization of the virtual photon; when it has a nonzero overlap with (5.29), we get a nonzero amplitude. Plugging (5.29) and (5.30) into (5.25), we see that the amplitude is

$$\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) = \frac{e^2}{q^2}(2E)^2(-\cos\theta - 1) = -e^2(1+\cos\theta), \tag{5.31}$$

in agreement (up to a sign) with (1.6), and also with (5.21). The differential cross section for this set of helicities can now be obtained in the same way as above, yielding (5.22).

We can calculate the other three nonvanishing helicity amplitudes in an analogous manner. For a left-handed electron and a right-handed positron, we easily find

$$\bar{v}(p')\gamma^\mu u(p) = -2E\,(0,1,-i,0) \equiv -2E\cdot\sqrt{2}\,\epsilon^\mu_-.$$

Perform a rotation to get the vector corresponding to a left-handed μ^- and a right-handed μ^+:

$$\bar{u}(k)\gamma^\mu v(k') = -2E\,(0,\cos\theta,i,\sin\theta).$$

Putting the pieces together in various ways yields the remaining amplitudes,

$$\begin{aligned}\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) &= -e^2(1+\cos\theta); \\ \mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) = \mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) &= -e^2(1-\cos\theta).\end{aligned} \tag{5.32}$$

5.3 $e^+e^- \to \mu^+\mu^-$: Nonrelativistic Limit

Now let us go to the other end of the energy spectrum, and discuss the reaction $e^+e^- \to \mu^+\mu^-$ in the extreme nonrelativistic limit. When E is barely larger than m_μ, our previous result (5.12) for the unpolarized differential cross section becomes

$$\frac{d\sigma}{d\Omega} \underset{|\mathbf{k}|\to 0}{\longrightarrow} \frac{\alpha^2}{2E^2_{\rm cm}}\sqrt{1-\frac{m^2_\mu}{E^2}} = \frac{\alpha^2}{2E^2_{\rm cm}}\frac{|\mathbf{k}|}{E}. \tag{5.33}$$

We can recover this result, and also learn something about the spin dependence of the reaction, by evaluating the amplitude with explicit spinors. Once again we begin with the matrix element

$$\mathcal{M} = \frac{e^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big).$$

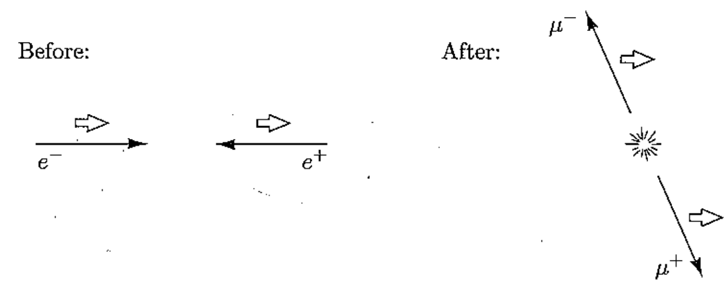


Figure 5.5. In the nonrelativistic limit the total spin of the system is conserved, and thus the muons are produced with both spins up along the z-axis.

The electron and positron are still very relativistic, so this expression will be simplest if we choose them to have definite helicity. Let the electron be right-handed, moving in the $+z$-direction, and the positron be left-handed, moving in the $-z$-direction. Then from Eq. (5.29) we have

$$\bar{v}(p')\gamma^\mu u(p) = -2E\,(0,1,i,0). \tag{5.34}$$

In the other half of the matrix element we should use the nonrelativistic expressions

$$u(k) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \qquad v(k') = \sqrt{m}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}. \tag{5.35}$$

Keep in mind, in the discussion of this section, that the spinor ξ' gives the flipped spin of the antiparticle. Leaving the muon spinors ξ and ξ' undetermined for now, we can easily compute

$$\begin{aligned}\bar{u}(k)\gamma^\mu v(k') &= m(\xi^\dagger,\xi^\dagger)\begin{pmatrix}\bar\sigma^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix} \\ &= \begin{cases}0 & \text{for } \mu = 0,\\ -2m\xi^\dagger\sigma^i\xi' & \text{for } \mu = i.\end{cases}\end{aligned} \tag{5.36}$$

To evaluate $\mathcal{M}$, we simply dot (5.34) into (5.36) and multiply by $e^2/q^2 = e^2/4m^2$. The result is

$$\mathcal{M}(e^-_R e^+_L \to \mu^+\mu^-) = -2e^2\xi^\dagger\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\xi'. \tag{5.37}$$

Since there is no angular dependence in this expression, the muons are equally likely to come out in any direction. More precisely, they are emitted in an s-wave; their orbital angular momentum is zero. Angular momentum conservation therefore requires that the total spin of the final state equal 1, and indeed the matrix product gives zero unless both the muon and the antimuon have spin up along the z-axis (see Fig. 5.5).

To find the total rate for this process, we sum over muon spins to obtain $\mathcal{M}^2 = 4e^4$, which yields the cross section

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^+\mu^-) = \frac{\alpha^2}{E^2_{\text{cm}}}\frac{|\mathbf{k}|}{E}. \tag{5.38}$$

The same expression holds for a left-handed electron and a right-handed positron. Thus the spin-averaged cross section is just $2 \cdot (1/4)$ times this expression, in agreement with (5.33).

Bound States

Until now we have considered the initial and final states of scattering processes to be states of isolated single particles. Very close to threshold, however, the Coulomb attraction of the muons should become an important effect. Just below threshold, we can still form $\mu^+\mu^-$ pairs in electromagnetic bound states.

The treatment of bound states in quantum field theory is a rich and complex subject, but one that lies mainly beyond the scope of this book.‡ Fortunately, many of the familiar bound systems in Nature can be treated (at least to a good first approximation) as nonrelativistic systems, in which the internal motions are slow. The process of creating the constituent particles out of the vacuum is still a relativistic effect, requiring quantum field theory for its proper description. In this section we will develop a formalism for computing the amplitudes for creation and annihilation of two-particle, nonrelativistic bound states. We begin with a computation of the cross section for producing a $\mu^+\mu^-$ bound state in e^+e^- annihilation.

Consider first the case where the spins of the electron and positron both point up along the z-axis. From the preceding discussion we know that the resulting muons both have spin up, so the only type of bound state we can produce will have total spin 1, also pointing up. The amplitude for producing free muons in this configuration is

$$\mathcal{M}(\uparrow\uparrow \to \mathbf{k}_1\uparrow, \mathbf{k}_2\uparrow) = -2e^2, \tag{5.39}$$

independent of the momenta (which we now call $\mathbf{k}_1$ and $\mathbf{k}_2$) of the muons.

Next we need to know how to write a bound state in terms of free-particle states. For a general two-body system with equal constituent masses, the center-of-mass and relative coordinates are

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{5.40}$$

These have conjugate momenta

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2, \qquad \mathbf{k} = \tfrac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2). \tag{5.41}$$

The total momentum $\mathbf{K}$ is zero in the center-of-mass frame. If we know the force between the particles (for $\mu^+\mu^-$, it is just the Coulomb force), we can

‡Reviews of this subject can be found in Bodwin, Yennie, and Gregorio, *Rev. Mod. Phys.* **57**, 723 (1985), and in Sapirstein and Yennie, in Kinoshita (1990).

solve the nonrelativistic Schrödinger equation to find the Schrödinger wavefunction, $\psi(\mathbf{r})$. The bound state is just a linear superposition of free states of definite $\mathbf{r}$ or $\mathbf{k}$, weighted by this wavefunction. For our purposes it is more convenient to build this superposition in momentum space, using the Fourier transform of $\psi(\mathbf{r})$:

$$\widetilde{\psi}(\mathbf{k}) = \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{r}); \qquad \int \frac{d^3k}{(2\pi)^3}\left|\widetilde{\psi}(\mathbf{k})\right|^2 = 1. \tag{5.42}$$

If $\psi(r)$ is normalized conventionally, $\widetilde{\psi}(\mathbf{k})$ gives the amplitude for finding a particular value of $\mathbf{k}$. An explicit expression for a bound state with mass $M \approx 2m$, momentum $\mathbf{K} = 0$, and spin 1 oriented up is then

$$|B\rangle = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\widetilde{\psi}(\mathbf{k})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}|\mathbf{k}\uparrow, -\mathbf{k}\uparrow\rangle. \tag{5.43}$$

The factors of $(1/\sqrt{2m})$ convert our relativistically normalized free-particle states so that their integral with $\widetilde{\psi}(\mathbf{k})$ is a state of norm 1. (The factors should involve $\sqrt{2E_{\pm\mathbf{k}}}$, but for a nonrelativistic bound state, $|\mathbf{k}| \ll m$.) The outside factor of $\sqrt{2M}$ converts back to the relativistic normalization assumed by our formula for cross sections. These normalization factors could easily be modified to describe a bound state with nonzero total momentum $\mathbf{K}$.

Given this expression for the bound state, we can immediately write down the amplitude for its production:

$$\mathcal{M}(\uparrow\uparrow \to B) = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\widetilde{\psi}^*(\mathbf{k})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\mathcal{M}(\uparrow\uparrow \to \mathbf{k}\uparrow, -\mathbf{k}\uparrow). \tag{5.44}$$

Since the free-state amplitude from (5.39) is independent of the momenta of the muons, the integral over $\mathbf{k}$ gives $\psi^*(0)$, the position-space wavefunction evaluated at the origin. It is quite natural that the amplitude for creation of a two-particle state from a pointlike virtual photon should be proportional to the value of the wavefunction at zero separation. Assembling the pieces, we find that the amplitude is simply

$$\mathcal{M}(\uparrow\uparrow \to B) = \sqrt{\frac{2}{M}}(-2e^2)\psi^*(0). \tag{5.45}$$

In a moment we will compute the cross section from this amplitude. First, however, let us generalize this discussion to treat bound states with more general spin configurations. The analysis leading up to (5.37) will cast any S-matrix element for the production of nonrelativistic fermions with momenta $\mathbf{k}$ and $-\mathbf{k}$ into the form of a spin matrix element

$$i\mathcal{M}(\text{something} \to \mathbf{k}, \mathbf{k}') = \xi^\dagger[\Gamma(\mathbf{k})]\xi', \tag{5.46}$$

where $\Gamma(\mathbf{k})$ is some 2×2 matrix. We now must replace the spinors with a normalized spin wavefunction for the bound state. In the example just completed, we replaced

$$\xi'\xi^\dagger \to \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}. \tag{5.47}$$

More generally, a spin-1 state is obtained by the replacement

$$\xi'\xi^\dagger \to \frac{1}{\sqrt{2}}\mathbf{n}^*\cdot\boldsymbol{\sigma}, \tag{5.48}$$

where $\mathbf{n}$ is a unit vector. Choosing $\mathbf{n} = (\hat{x}+i\hat{y})/\sqrt{2}$ gives back (5.47), while the choices $\mathbf{n} = (\hat{x}-i\hat{y})/\sqrt{2}$ and $\mathbf{n} = \hat{z}$ give the other two spin-1 states $\downarrow\downarrow$ and $(\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}$. (The relative minus sign in (5.48) for this last case comes from the rule (3.135) for the flipped spin.) Similarly, the spin-zero state $(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}$ is given by the replacement

$$\xi'\xi^\dagger \to \frac{1}{\sqrt{2}}\mathbf{1}, \tag{5.49}$$

involving the 2×2 unit matrix. With these rules, we can convert an S-matrix element of the form (5.46) quite generally into an S-matrix element for production of a bound state at rest:

$$i\mathcal{M}(\text{something} \to B) = \sqrt{\frac{2}{M}}\int\frac{d^3k}{(2\pi)^3}\,\widetilde{\psi}^*(\mathbf{k})\,\mathrm{tr}\Big(\frac{\mathbf{n}^*\cdot\boldsymbol{\sigma}}{\sqrt{2}}\,\Gamma(\mathbf{k})\Big), \tag{5.50}$$

where the trace is taken over 2-component spinor indices. For a spin-0 bound state, replace $\mathbf{n}\cdot\boldsymbol{\sigma}$ by the unit matrix.

Vector Meson Production and Decay

Equation (5.45) can be straightforwardly converted into a cross section for production of $\mu^+\mu^-$ bound states in e^+e^- annihilation. To make it easier to extract all the physics in this equation, let us introduce polarization vectors for the initial and final spin configurations: $\boldsymbol{\epsilon}_+ = (\hat{x}+i\hat{y})/\sqrt{2}$, from Eq. (5.29), and $\mathbf{n}$, from Eq. (5.48). Then (5.45) can be rewritten in a more invariant form as

$$\mathcal{M}(e^-_R e^+_L \to B) = \sqrt{\frac{2}{M}}\,(-2e^2)\,(\mathbf{n}^*\cdot\boldsymbol{\epsilon}_+)\,\psi^*(0). \tag{5.51}$$

The bound state spin polarization $\mathbf{n}$ is projected parallel to $\boldsymbol{\epsilon}_+$. Note that if the electrons are initially unpolarized, the cross section for production of B will involve the polarization average

$$\frac{1}{4}\big(|\mathbf{n}^*\cdot\boldsymbol{\epsilon}_+|^2 + |\mathbf{n}^*\cdot\boldsymbol{\epsilon}_-|^2\big) = \frac{1}{4}\big((n^x)^2 + (n^y)^2\big). \tag{5.52}$$

Thus, the bound states produced will still be preferentially polarized along the e^+e^- collision axis.

Assuming an unpolarized electron beam, and summing (5.52) over the three possible directions of $\mathbf{n}$, we find the following expression for the total cross section for production of the bound state:

$$\sigma(e^+e^- \to B) = \frac{1}{2}\frac{1}{2m}\frac{1}{2m}\int\frac{d^3K}{(2\pi)^3}\frac{1}{2E_{\mathbf{K}}}(2\pi)^4\delta^{(4)}(p+p'-K)\cdot\frac{2}{M}(4e^4)\frac{1}{2}|\psi(0)|^2 \tag{5.53}$$

Notice that the 1-body phase space integral can remove only three of the four delta functions. It is conventional to rewrite the last delta function using $\delta(P^0 - K^0) = 2K^0\delta(P^2 - K^2)$. Then

$$\sigma(e^+e^- \to B) = 64\pi^3\alpha^2\frac{|\psi(0)|^2}{M^3}\delta(E^2_{\text{cm}} - M^2). \tag{5.54}$$

The last delta function enforces the constraint that the total center-of-mass energy must equal the bound-state mass; thus, the bound state is produced as a resonance in e^+e^- annihilation. If the bound state has a finite lifetime this delta function will be broadened into a resonance peak. In practice, the intrinsic spread of the e^+e^- beam energy is often a more important broadening mechanism. In either case, (5.54) correctly predicts the area under the resonance peak.

If the bound state B can be produced from e^+e^-, it can also annihilate back to e^+e^-, or to any other sufficiently light lepton pair. According to (4.86) the total width for this decay mode is given by

$$\Gamma(B \to e^+e^-) = \frac{1}{2M}\int d\Pi_2\,|\mathcal{M}|^2, \tag{5.55}$$

where $\mathcal{M}$ is just the complex conjugate of the matrix element (5.51) we used to compute B production. Thus

$$\Gamma = \frac{1}{2M}\int\Big(\frac{1}{2\pi}\frac{d\cos\theta}{2}\Big)\frac{8e^4}{M}\,|\psi(0)|^2\big(|\mathbf{n}\cdot\boldsymbol{\epsilon}|^2 + |\mathbf{n}\cdot\boldsymbol{\epsilon}^*|^2\big). \tag{5.56}$$

Now we must sum over electron polarization states and average over the three possible values of $\mathbf{n}$. We thus obtain

$$\Gamma(B \to e^+e^-) = \frac{16\pi\alpha^2}{3}\frac{|\psi(0)|^2}{M^2}. \tag{5.57}$$

The formula for the decay width of B is very similar to that for the production cross section, and this is no surprise: Both calculations involve the square of the same matrix element, summed over initial and final polarizations. The two calculations differed only in how we formed the polarization averages, and in the phase-space factors. By this logic, the relation we have found between the two quantities,

$$\sigma(e^+e^- \to B) = 4\pi^2\cdot\frac{3\Gamma(B \to e^+e^-)}{M}\cdot\delta(E^2_{\text{cm}} - M^2), \tag{5.58}$$

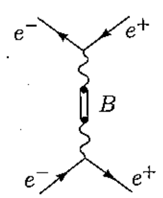
is very general and completely independent of the details of the matrix element computation. The factor 3 in (5.58) came from the orientation average for **n**; for a spin-J bound state, this factor would be $(2J+1)$.

The most famous application of this formalism is to bound states not of muons but of quarks: *quarkonium*. We saw the experimental evidence for $q\bar{q}$ bound states (the J/ψ and Υ, for example) in Fig. 5.2. (The resonance peaks are much too high and too narrow to show in the figure, but their sizes have been carefully measured.) Equations (5.54) and (5.57) must be multiplied by a color factor of 3 to give the production cross section and decay width for a spin-1 $q\bar{q}$ bound state. The value $\psi(0)$ of the $q\bar{q}$ wavefunction at the origin cannot be computed from first principles, but can be estimated from a nonrelativistic model of the $q\bar{q}$ spectrum with a phenomenologically chosen potential. Alternatively, we can use the formula

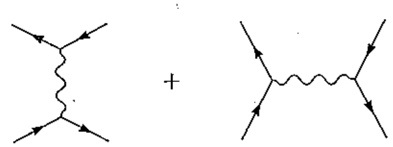
$$\Gamma(B(q\bar{q}) \to e^+e^-) = 16\pi\alpha^2 Q^2 \frac{|\psi(0)|^2}{M^2} \tag{5.59}$$

to measure $\psi(0)$ for a $q\bar{q}$ bound state. For example, the $1S$ spin-1 state of $s\bar{s}$, the ϕ meson, has an e^+e^- partial width of 1.4 keV and a mass of 1.02 GeV. From this we can infer $|\psi(0)|^2 = (1.2\,\text{fm})^{-3}$. This result is physically reasonable, since hadronic dimensions are typically $\sim$1 fm.

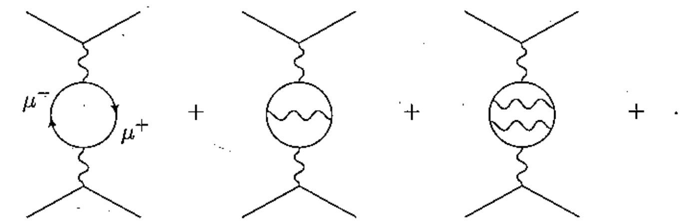
Our viewpoint in this section has been quite different from that of earlier sections: Instead of computing everything from first principles, we have pieced together an approximate formula using a bit of quantum field theory and a bit of nonrelativistic quantum mechanics. In principle, however, we could treat bound states entirely in the relativistic formalism. Consider the annihilation of an e^+e^- pair to form a $\mu^+\mu^-$ bound state, which subsequently decays back into e^+e^-. In our present formalism we might represent this process by the diagram



The net process is simply $e^+e^- \to e^+e^-$ (Bhabha scattering). What would happen if we tried to compute the Bhabha scattering cross section directly in QED perturbation theory? Obviously there is no $\mu^+\mu^-$ contribution in the tree-level diagrams:



As we go to higher orders in the perturbation series, however, we find (among others) the following set of diagrams:

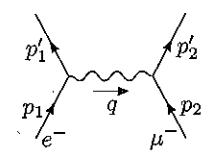


At most values of E_{cm}, these diagrams give only a small correction to the tree-level expression. But when E_{cm} is near the $\mu^+\mu^-$ threshold, the diagrams involving the exchange of photons within the muon loop contain the Coulomb interaction between the muons, and therefore become quite large. One must sum over all such diagrams, and it can be shown that this summation is equivalent to solving the nonrelativistic Schrödinger equation.* The final prediction is that the cross section contains a resonance peak, whose area is given by (5.54) and whose width is given by (5.57).

5.4 Crossing Symmetry

Electron-Muon Scattering

Now that we have completed our discussion of the process $e^+e^- \to \mu^+\mu^-$, let us consider a different but closely related QED process: electron-muon scattering, or $e^-\mu^- \to e^-\mu^-$. The lowest-order Feynman diagram is just the previous one turned on its side:



$$= \frac{ie^2}{q^2}\,\bar{u}(p_1')\gamma^\mu u(p_1)\,\bar{u}(p_2')\gamma_\mu u(p_2).$$

The relation between the processes $e^+e^- \to \mu^+\mu^-$ and $e^-\mu^- \to e^-\mu^-$ becomes clear when we compute the squared amplitude, averaged and summed over spins:

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^4}\,\text{tr}\Big[(\not{p}'_1+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\Big]\,\text{tr}\Big[(\not{p}'_2+m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu\Big]$$

This is exactly the same as our result (5.4) for $e^+e^- \to \mu^+\mu^-$, with the replacements

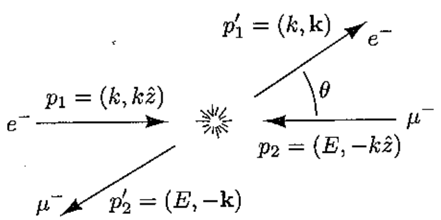
$$p \to p_1, \qquad p' \to -p_1', \qquad k \to p_2', \qquad k' \to -p_2. \tag{5.60}$$

*This analysis is carried out in Berestetskii, Lifshitz, and Pitaevskii (1982).

So instead of evaluating the traces from scratch, we can just make the same replacements in our previous result, Eq. (5.10). Setting $m_e = 0$, we find

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4}\Big[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\Big]. \quad (5.61)$$

To evaluate this expression, we must work out the kinematics, which will be completely different. Working in the center-of-mass frame, we make the following assignments:



$$E^2 = k^2 + m_\mu^2$$
$$\mathbf{k}\cdot\hat{z} = k\cos\theta$$
$$E + k = E_{\text{cm}}$$

The combinations we need are

$$p_1\cdot p_2 = p_1'\cdot p_2' = k(E+k); \qquad p_1'\cdot p_2 = p_1\cdot p_2' = k(E + k\cos\theta);$$
$$p_1\cdot p_1' = k^2(1-\cos\theta); \qquad q^2 = -2p_1\cdot p_1' = -2k^2(1-\cos\theta).$$

Our expression for the squared matrix element now becomes

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2e^4}{k^2(1-\cos\theta)^2}\Big((E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\Big). \quad (5.62)$$

To find the cross section from this expression, we use Eq. (4.84), which in the case where one particle is massless takes the simple form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2(E+k)^2}. \quad (5.63)$$

Thus we have our result for unpolarized electron-muon scattering in the center-of-mass frame:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E+k)^2(1-\cos\theta)^2}\Big((E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\Big), \quad (5.64)$$

where $k = \sqrt{E^2 - m_\mu^2}$. In the high-energy limit where we can set $m_\mu = 0$, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{\text{cm}}^2(1-\cos\theta)^2}\Big(4 + (1+\cos\theta)^2\Big). \quad (5.65)$$

Note the singular behavior

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\theta^4} \qquad \text{as } \theta \to 0 \quad (5.66)$$

of formulae (5.64) and (5.65). This singularity is the same as in the Rutherford formula (Problem 4.4). Such behavior is always present in Coulomb scattering; it arises from the nearly on-shell (that is, $q^2 \approx 0$) virtual photon.

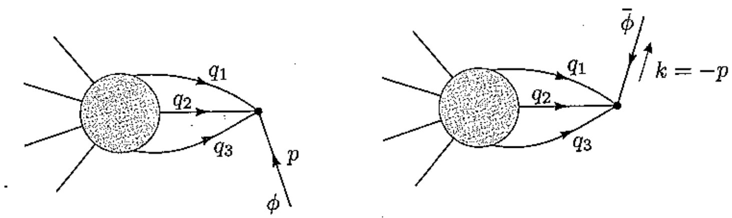
Crossing Symmetry

The trick we made use of here, namely the relation between the two processes $e^+e^- \to \mu^+\mu^-$ and $e^-\mu^- \to e^-\mu^-$, is our first example of a type of relation known as *crossing symmetry*. In general, the S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k = -p$ in the final state. That is,

$$\mathcal{M}\big(\phi(p) + \cdots \to \cdots\big) = \mathcal{M}\big(\cdots \to \cdots + \bar{\phi}(k)\big), \quad (5.67)$$

where $\bar{\phi}$ is the antiparticle of ϕ and $k = -p$. (Note that there is no value of p for which p and k are both physically allowed, since the particle must have $p^0 > 0$ and the antiparticle must have $k^0 > 0$. So technically, we should say that either amplitude can be obtained from the other by analytic continuation.)

Relation (5.67) follows directly from the Feynman rules. The diagrams that contribute to the two amplitudes fall into a natural one-to-one correspondence, where corresponding diagrams differ only by changing the incoming ϕ into the outgoing $\bar{\phi}$. A typical pair of diagrams looks like this:



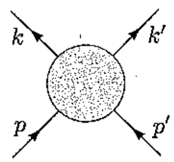
In the first diagram, the momenta q_i coming into the vertex from the rest of the diagram must add up to $-p$, while in the second diagram they must add up to k. Thus the two diagrams are equal, except for any possible difference in the external leg factors, if $p = -k$. If ϕ is a spin-zero boson, there is no external leg factor, so the identity is proved. If ϕ is a fermion, the analysis becomes more subtle, since the relation depends on the relative phase convention for the external spinors u and v. If we simply replace p by $-k$ in the fermion polarization sum, we find

$$\sum u(p)\bar{u}(p) = \not{p} + m = -(\not{k} - m) = -\sum v(k)\bar{v}(k). \quad (5.68)$$

The minus sign can be compensated by changing our phase convention for $v(k)$. In practice, it is easiest to cancel by hand one minus sign for each crossed fermion. With appropriate conventions for the spinors $u(p)$ and $v(k)$, it is possible to prove the identity (5.67) without spin-averaging.

Mandelstam Variables

It is often useful to express scattering amplitudes in terms of variables that make it easy to apply crossing relations. For 2-body $\to$ 2-body processes, this can be done as follows. Label the four external momenta as

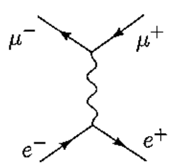


We now define three new quantities, the *Mandelstam variables*:

$$\begin{aligned} s &= (p+p')^2 = (k+k')^2; \\ t &= (k-p)^2 = (k'-p')^2; \\ u &= (k'-p)^2 = (k-p')^2. \end{aligned} \tag{5.69}$$

The definitions of t and u appear to be interchangeable (by renaming $k \to k'$); it is conventional to define t as the squared difference of the initial and final momenta of the most similar particles. For any process, s is the square of the total initial 4-momentum. Note that if we had defined all four momenta to be ingoing, all signs in these definitions would be +.

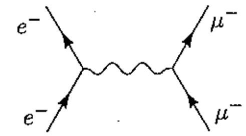
To illustrate the use of the Mandelstam variables, let us first consider the squared amplitude for $e^+e^- \to \mu^+\mu^-$, working in the massless limit for simplicity. In this limit we have $t = -2p\cdot k = -2p'\cdot k'$ and $u = -2p\cdot k' = -2p'\cdot k$, while of course $s = (p+p')^2 = q^2$. Referring to our previous result (5.10), we find



$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]. \tag{5.70}$$

To convert to the process $e^-\mu^- \to e^-\mu^-$, we turn the diagram on its side and make use of the crossing relations, which become quite simple in terms of Mandelstam variables. For example, the crossing relations tell us to change the sign of p', the positron momentum, and reinterpret it as the momentum of the outgoing electron. Therefore $s = (p+p')^2$ becomes what we would now call t, the difference of the outgoing and incoming electron momenta. Similarly, t becomes s, while u remains unchanged. Thus for $e^-\mu^- \to e^-\mu^-$,

we can immediately write down

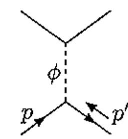


$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]. \tag{5.71}$$

You can easily check that this agrees with (5.61) in the massless limit. Note that while (5.70) and (5.71) look quite similar, they are physically very different: The denominator of the first is just $s^2 = E_{\text{cm}}^4$, but that of the second involves t, which depends on angles and goes to zero as $\theta \to 0$.

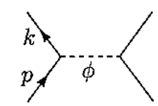
When a 2-body $\to$ 2-body diagram contains only one virtual particle, it is conventional to describe that particle as being in a certain "channel". The channel can be read from the form of the Feynman diagram, and each channel leads to a characteristic angular dependence of the cross section:

s-channel:



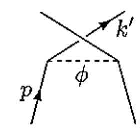
$$\mathcal{M} \propto \frac{1}{s - m_\phi^2}$$

t-channel:



$$\mathcal{M} \propto \frac{1}{t - m_\phi^2}$$

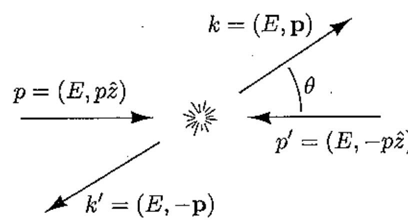
u-channel:



$$\mathcal{M} \propto \frac{1}{u - m_\phi^2}$$

In many cases, a single process will receive contributions from more than one channel; these must be added coherently. For example, the amplitude for *Bhabha scattering*, $e^+e^- \to e^+e^-$, is the sum of s- and t-channel diagrams *Møller scattering*, $e^-e^- \to e^-e^-$, involves t- and u-channel diagrams.

To get a better feel for s, t, and u, let us evaluate them explicitly in the center-of-mass frame for particles all of mass m. The kinematics is as usual:



Thus the Mandelstam variables are

$$\begin{aligned} s &= (p+p')^2 = (2E)^2 = E_{\rm cm}^2; \\ t &= (k-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta-1)^2 = -2p^2(1-\cos\theta); \\ u &= (k'-p)^2 = -p^2\sin^2\theta - p^2(\cos\theta+1)^2 = -2p^2(1+\cos\theta). \end{aligned} \tag{5.72}$$

Thus we see that $t \to 0$ as $\theta \to 0$, while $u \to 0$ as $\theta \to \pi$. (When the masses are not all equal, the limiting values of t and u will shift slightly.)

Note from (5.72) that when all four particles have mass m, the sum of the Mandelstam variables is $s+t+u = 4E^2 - 4p^2 = 4m^2$. This is a special case of a more general relation, which is often quite useful:

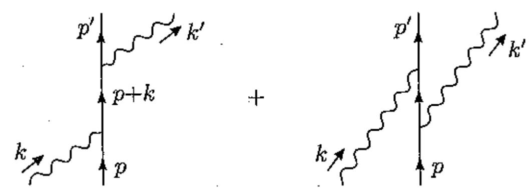
$$s+t+u = \sum_{i=1}^{4} m_i^2, \tag{5.73}$$

where the sum runs over the four external particles. This identity is easy to prove by adding up the terms on the right-hand side of Eqs. (5.69), and applying momentum conservation in the form $(p+p'-k-k')^2 = 0$.

5.5 Compton Scattering

We now move on to consider a somewhat different QED process: *Compton scattering*, or $e^-\gamma \to e^-\gamma$. We will calculate the unpolarized cross section for this reaction, to lowest order in α. The calculation will employ all the machinery we have developed so far, including the Mandelstam variables of the previous section. We will also develop some new technology for dealing with external photons.

This is our first example of a calculation involving two diagrams:



As usual, the Feynman rules tell us exactly how to write down an expression for $\mathcal{M}$. Note that since the fermion portions of the two diagrams are identical, there is no relative minus sign between the two terms. Using $\epsilon_\nu(k)$ and $\epsilon^*_\mu(k')$ to denote the polarization vectors of the initial and final photons, we have

$$\begin{aligned} i\mathcal{M} &= \bar{u}(p')(-ie\gamma^\mu)\epsilon^*_\mu(k')\frac{i(\not p+\not k+m)}{(p+k)^2-m^2}(-ie\gamma^\nu)\epsilon_\nu(k)u(p) \\ &\quad + \bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not p-\not k'+m)}{(p-k')^2-m^2}(-ie\gamma^\mu)\epsilon^*_\mu(k')u(p) \\ &= -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu(\not p+\not k+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not p-\not k'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p). \end{aligned}$$

We can make a few simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$, the denominators of the propagators are

$$(p+k)^2 - m^2 = 2p\cdot k \qquad \text{and} \qquad (p-k')^2 - m^2 = -2p\cdot k'.$$

To simplify the numerators, we use a bit of Dirac algebra:

$$\begin{aligned} (\not p + m)\gamma^\nu u(p) &= (2p^\nu - \gamma^\nu \not p + \gamma^\nu m)u(p) \\ &= 2p^\nu u(p) - \gamma^\nu(\not p - m)u(p) \\ &= 2p^\nu u(p). \end{aligned}$$

Using this trick on the numerator of each propagator, we obtain

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not k'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p). \tag{5.74}$$

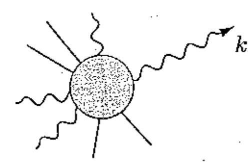
Photon Polarization Sums

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum (or average) over electron and photon polarization states. The sum over electron polarizations can be performed as before, using the identity $\Sigma u(p)\bar{u}(p) = \not p + m$. Fortunately, there is a similar trick for summing over photon polarization vectors. The correct prescription is to make the replacement

$$\sum_{\text{polarizations}} \epsilon^*_\mu \epsilon_\nu \longrightarrow -g_{\mu\nu}. \tag{5.75}$$

The arrow indicates that this is not an actual equality. Nevertheless, the replacement is valid as long as both sides are dotted into the rest of the expression for a QED amplitude $\mathcal{M}$.

To derive this formula, let us consider an arbitrary QED process involving an external photon with momentum k:



$$= i\mathcal{M}(k) \equiv i\mathcal{M}^\mu(k)\epsilon^*_\mu(k). \tag{5.76}$$

Since the amplitude always contains $\epsilon^*_\mu(k)$, we have extracted this factor and defined $\mathcal{M}^\mu(k)$ to be the rest of the amplitude $\mathcal{M}$. The cross section will be proportional to

$$\sum_\epsilon \left|\epsilon^*_\mu(k)\mathcal{M}^\mu(k)\right|^2 = \sum_\epsilon \epsilon^*_\mu\epsilon_\nu\, \mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k).$$

For simplicity, we orient k in the 3-direction: $k^\mu = (k, 0, 0, k)$. Then the two transverse polarization vectors, over which we are summing, can be chosen to be

$$\epsilon_1^\mu = (0,1,0,0); \qquad \epsilon_2^\mu = (0,0,1,0).$$

With these conventions, we have

$$\sum_\epsilon \left|\epsilon_\mu^*(k)\mathcal{M}^\mu(k)\right|^2 = \left|\mathcal{M}^1(k)\right|^2 + \left|\mathcal{M}^2(k)\right|^2. \tag{5.77}$$

Now recall from Chapter 4 that external photons are created by the interaction term $\int d^4x\, ej^\mu A_\mu$, where $j^\mu = \bar\psi\gamma^\mu\psi$ is the Dirac vector current. Therefore we expect $\mathcal{M}^\mu(k)$ to be given by a matrix element of the Heisenberg field j^μ:

$$\mathcal{M}^\mu(k) = \int d^4x\, e^{ik\cdot x} \left\langle f\right| j^\mu(x) \left|i\right\rangle, \tag{5.78}$$

where the initial and final states include all particles except the photon in question.

From the classical equations of motion, we know that the current j^μ is conserved: $\partial_\mu j^\mu(x) = 0$. Provided that this property still holds in the quantum theory, we can dot k_μ into (5.78) to obtain

$$k_\mu \mathcal{M}^\mu(k) = 0. \tag{5.79}$$

The amplitude $\mathcal{M}$ vanishes when the polarization vector $\epsilon_\mu(k)$ is replaced by k_μ. This famous relation is known as the *Ward identity.* It is essentially a statement of current conservation, which is a consequence of the gauge symmetry (4.6) of QED. We will give a formal proof of the Ward identity in Section 7.4, and discuss a number of subtle points skimmed over in this quick "derivation".

It is useful to check explicitly that the Compton amplitude given in (5.74) obeys the Ward identity. To do this, replace $\epsilon_\nu(k)$ by k_ν or $\epsilon_\mu^*(k')$ by k'_μ, and manipulate the Dirac matrix products. In either case (after a bit of algebra) the terms from the two diagrams cancel each other to give zero.

Returning to our derivation of the polarization sum formula (5.75), we note that for $k^\mu = (k, 0, 0, k)$, the Ward identity takes the form

$$k\mathcal{M}^0(k) - k\mathcal{M}^3(k) = 0. \tag{5.80}$$

Thus $\mathcal{M}^0 = \mathcal{M}^3$, and we have

$$\begin{aligned}\sum_\epsilon \epsilon_\mu^*\epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k) &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \\ &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 \\ &= -g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k).\end{aligned}$$

That is, we may sum over external photon polarizations by replacing $\sum \epsilon_\mu^*\epsilon_\nu$ with $-g_{\mu\nu}$.

Note that this proves (pending our general proof of the Ward identity) that the unphysical timelike and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability. The negative norm of the timelike photon state, a property that troubled us in the discussion after Eq. (4.132), plays an essential role in this cancellation.

The Klein-Nishina Formula

The rest of the computation of the Compton scattering cross section is straightforward, although it helps to be somewhat organized. We want to average the squared amplitude over the initial electron and photon polarizations, and sum over the final electron and photon polarizations. Starting with expression (5.74) for $\mathcal{M}$, we find

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{e^4}{4} g_{\mu\rho}\, g_{\nu\sigma}\cdot \operatorname{tr}\Big\{(\not p'+m)\Big[\frac{\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\Big] \\ &\qquad\cdot(\not p+m)\Big[\frac{\gamma^\sigma \not k\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho \not k'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\Big]\Big\} \\ &\equiv \frac{e^4}{4}\Big[\frac{\mathbf{I}}{(2p\cdot k)^2} + \frac{\mathbf{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathbf{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathbf{IV}}{(2p\cdot k')^2}\Big],\end{aligned} \tag{5.81}$$

where **I**, **II**, **III**, and **IV** are complicated traces. Note that **IV** is the same as **I** if we replace k with $-k'$. Also, since we can reverse the order of the γ matrices inside a trace (Eq. (5.7)), we see that **II** = **III**. Thus we must work only to compute **I** and **II**.

The first of the traces is

$$\mathbf{I} = \operatorname{tr}\big[(\not p' + m)(\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu)(\not p + m)(\gamma_\nu \not k\gamma_\mu + 2\gamma_\mu p_\nu)\big].$$

There are 16 terms inside the trace, but half contain an odd number of γ matrices and therefore vanish. We must now evaluate the other eight terms, one at a time. For example,

$$\begin{aligned}\operatorname{tr}\big[\not p'\gamma^\mu \not k\gamma^\nu \not p\gamma_\nu \not k\gamma_\mu\big] &= \operatorname{tr}\big[(-2\not p')\not k(-2\not p)\not k\big] \\ &= \operatorname{tr}\big[4\not p'\not k(2p\cdot k - \not k\not p)\big] \\ &= 8p\cdot k\ \operatorname{tr}[\not p'\not k] \\ &= 32(p\cdot k)(p'\cdot k).\end{aligned}$$

By similar use of the contraction identities (5.8) and (5.9), and other Dirac algebra such as $\not p\not p = p^2 = m^2$, each term in **I** can be reduced to a trace of no more than two γ matrices. When the smoke clears, we find

$$\mathbf{I} = 16\big(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)(p'\cdot k)\big). \tag{5.82}$$

Although it is not obvious, this expression can be simplified further. To see how, introduce the Mandelstam variables:

$$\begin{aligned} s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2; \\ t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k'; \\ u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2. \end{aligned} \tag{5.83}$$

Recall from (5.73) that momentum conservation implies $s+t+u = 2m^2$. Writing everything in terms of s, t, and u, and using this identity, we eventually obtain

$$\mathbf{I} = 16\big(2m^4 + m^2(u-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big). \tag{5.84}$$

Sending $k \leftrightarrow -k'$, we can immediately write

$$\mathbf{IV} = 16\big(2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\big). \tag{5.85}$$

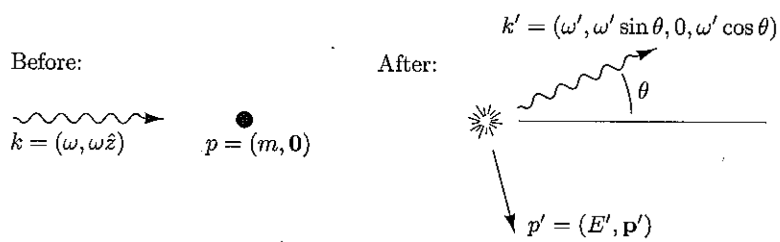
Evaluating the traces in the numerators **II** and **III** requires about the same amount of work as we have just done. The answer is

$$\mathbf{II} = \mathbf{III} = -8\big(4m^4 + m^2(s-m^2) + m^2(u-m^2)\big). \tag{5.86}$$

Putting together the pieces of the squared matrix element (5.81), and rewriting s and u in terms of $p\cdot k$ and $p\cdot k'$, we finally obtain

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]. \tag{5.87}$$

To turn this expression into a cross section we must decide on a frame of reference and draw a picture of the kinematics. Compton scattering is most often analyzed in the "lab" frame, in which the electron is initially at rest:



We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$\begin{aligned} m^2 = (p')^2 &= (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k' \\ &= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta), \end{aligned}$$

hence,
$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta). \tag{5.88}$$

The last line is Compton's formula for the shift in the photon wavelength. For our purposes, however, it is more useful to solve for ω':

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}. \tag{5.89}$$

The phase space integral in this frame is

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^2\delta^{(4)}(k'+p'-k-p) \\ &= \int \frac{(\omega')^2 d\omega'\, d\Omega}{(2\pi)^3}\frac{1}{4\omega' E'} \\ &\qquad \times 2\pi\,\delta\big(\omega' + \sqrt{m^2+\omega^2+(\omega')^2-2\omega\omega'\cos\theta} - \omega - m\big) \\ &= \int \frac{d\cos\theta}{2\pi}\frac{\omega'}{4E'}\frac{1}{\left|1 + \frac{\omega'-\omega\cos\theta}{E'}\right|} \\ &= \frac{1}{8\pi}\int d\cos\theta\, \frac{\omega'}{m+\omega(1-\cos\theta)} \\ &= \frac{1}{8\pi}\int d\cos\theta\, \frac{(\omega')^2}{\omega m}. \end{aligned} \tag{5.90}$$

Plugging everything into our general cross-section formula (4.79) and setting $|v_A - v_B| = 1$, we find

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\cdot\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}\cdot\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right).$$

To evaluate $|\mathcal{M}|^2$, we replace $p\cdot k = m\omega$ and $p\cdot k' = m\omega'$ in (5.87). The shortest way to write the final result is

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right], \tag{5.91}$$

where ω'/ω is given by (5.89). This is the (spin-averaged) *Klein-Nishina formula*, first derived in 1929.†

In the limit $\omega \to 0$ we see from (5.89) that $\omega'/\omega \to 1$, so the cross section becomes

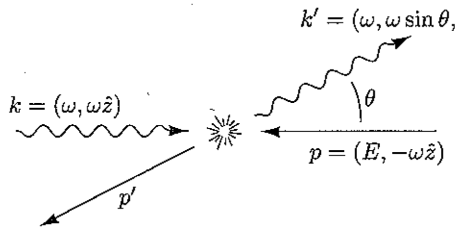
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta); \qquad \sigma_{\text{total}} = \frac{8\pi\alpha^2}{3m^2}. \tag{5.92}$$

This is the familiar Thomson cross section for scattering of classical electromagnetic radiation by a free electron.

† O. Klein and Y. Nishina, *Z. Physik*, **52**, 853 (1929).

High-Energy Behavior

To analyze the high-energy behavior of the Compton scattering cross section, it is easiest to work in the center-of-mass frame. We can easily construct the differential cross section in this frame from the invariant expression (5.87). The kinematics of the reaction now looks like this:



$$p\cdot k = \omega(E+\omega)$$
$$p\cdot k' = \omega(E+\omega\cos\theta)$$
$$E^2 = \omega^2 + m^2$$

Plugging these values into (5.87), we see that for $\theta \approx \pi$, the term $p{\cdot}k/p{\cdot}k'$ becomes very large, while the other terms are all of $\mathcal{O}(1)$ or smaller. Thus for $E \gg m$ and $\theta \approx \pi$, we have

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \approx 2e^4 \cdot \frac{p\cdot k}{p\cdot k'} = 2e^4 \cdot \frac{E+\omega}{E+\omega\cos\theta}. \tag{5.93}$$

The cross section in the CM frame is given by (4.84):

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{2}\cdot\frac{1}{2E}\cdot\frac{1}{2\omega}\cdot\frac{\omega}{(2\pi)4(E+\omega)}\cdot\frac{2e^4(E+\omega)}{E+\omega\cos\theta}\\ &\approx \frac{2\pi\alpha^2}{2m^2+s(1+\cos\theta)}.\end{aligned} \tag{5.94}$$

Notice that, since $s \gg m^2$, the denominator of (5.92) almost vanishes when the photon is emitted in the backward direction ($\theta \approx \pi$). In fact, the electron mass m could be neglected completely in this formula if it were not necessary to cut off this singularity. To integrate over $\cos\theta$, we can drop the electron mass term if we supply an equivalent cutoff near $\theta = \pi$. In this way, we can approximate the total Compton scattering cross section by

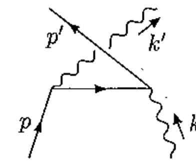
$$\int_{-1}^{1} d(\cos\theta)\,\frac{d\sigma}{d\cos\theta} \approx \frac{2\pi\alpha^2}{s}\int_{-1+2m^2/s}^{1} d(\cos\theta)\,\frac{1}{(1+\cos\theta)}. \tag{5.95}$$

Thus, we find that the total cross section behaves at high energy as

$$\sigma_{\text{total}} = \frac{2\pi\alpha^2}{s}\log\Big(\frac{s}{m^2}\Big). \tag{5.96}$$

The main dependence α^2/s follows from dimensional analysis. But the singularity associated with backward scattering of photons leads to an enhancement by an extra logarithm of the energy.

Let us try to understand the physics of this singularity. The singular term comes from the square of the u-channel diagram,



$$= -ie^2\,\epsilon_\mu(k)\epsilon_\nu^*(k')\bar{u}(p')\gamma^\mu \frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\gamma^\nu u(p). \tag{5.97}$$

The amplitude is large at $\theta \approx \pi$ because the denominator of the propagator is then small ($\sim m^2$) compared to s. To be more precise, define $\chi \equiv \pi - \theta$. We will be interested in values of χ that are somewhat larger than m/ω, but still small enough that we can approximate $1-\cos\chi \approx \chi^2/2$. For χ in this range, the denominator is

$$(p-k')^2 - m^2 = -2p\cdot k' \approx -2\omega^2\Big(\frac{m^2}{2\omega^2}+1-\cos\chi\Big) \approx -(\omega^2\chi^2+m^2). \tag{5.98}$$

This is small compared to s over a wide range of values for χ, hence the enhancement in the total cross section.

Looking back at (5.93), we see that for χ such that $m/\omega \ll \chi \ll 1$, the squared amplitude is proportional to $1/\chi^2$, and hence we expect $\mathcal{M} \propto 1/\chi$. But we have just seen that the denominator of $\mathcal{M}$ is proportional to χ^2, so there must be a compensating factor of χ in the numerator. We can understand the physical origin of that factor by looking at the amplitude for a particular set of electron and photon polarizations.

Suppose that the initial electron is right-handed. The dominant term of (5.97) comes from the term that involves $(\not{p}-\not{k}')$ in the numerator of the propagator. Since this term contains three γ-matrices in (5.97) between the $\bar{u}$ and the u, the final electron must also be right-handed. The amplitude is therefore

$$i\mathcal{M} = -ie^2\epsilon_\mu(k)\epsilon_\nu^*(k')u_R^\dagger(p')\sigma^\mu \frac{\bar{\sigma}\cdot(p-k')}{-(\omega^2\chi^2+m^2)}\sigma^\nu u_R(p), \tag{5.99}$$

where

$$u_R(p) = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \quad\text{and}\quad u_R(p') = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix}. \tag{5.100}$$

If the initial photon is left-handed, with $\epsilon_\mu(k) = (1/\sqrt{2})(0,1,-i,0)$, then

$$\sigma^\mu\epsilon_\mu(k) = \begin{pmatrix}0 & 0\\ \sqrt{2} & 0\end{pmatrix},$$

and the combination $u_R^\dagger(p')\sigma^\mu\epsilon_\mu(k)$ vanishes. The initial photon must therefore be right-handed. Similarly, the amplitude vanishes unless the final photon is right-handed. The kinematic situation for this set of polarizations is shown

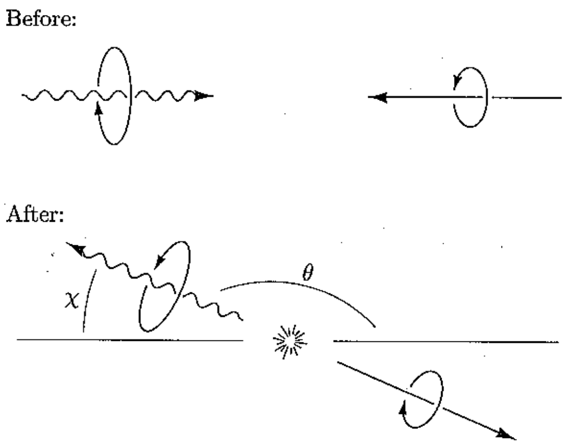


Figure 5.6. In the high-energy limit, the final photon is most likely to be emitted at backward angles. Since helicity is conserved, a unit of spin angular momentum is converted to orbital angular momentum.

in Fig. 5.6. Note that the total spin angular momentum of the final state is one unit less than that of the initial state.

Continuing with our calculation, let us consider the numerator of the propagator in (5.99). For χ in the range of interest, the dominant term is

$$-\sigma^1(p-k')^1 = \sigma^1 \cdot \omega\chi.$$

This is the factor of χ anticipated above. It indicates that the final state is a p-wave, as required by angular momentum conservaton. Assembling all the pieces, we obtain

$$\mathcal{M}(e^-_R\gamma_R \to e^-_R\gamma_R) \approx e^2\sqrt{2E}\sqrt{2}\frac{\omega\chi}{(\omega^2\chi^2+m^2)}\sqrt{2E}\sqrt{2} \approx \frac{4e^2\chi}{\chi^2+m^2/\omega^2}. \tag{5.101}$$

We would find the same result in the case where all initial and final particles are left-handed.

Notice that for directly backward scattering, $\chi = 0$, the matrix element (5.101) vanishes due to the angular momentum zero in the numerator. Thus, at angles very close to backward, we should also take into account the mass term in the numerator of the propagator in (5.97). This term contains only two gamma matrices and so converts a right-handed electron into a left-handed electron. By an analysis similar to the one that led to Eq. (5.101), we can see that this amplitude is nonvanishing only when the initial photon is left-handed and the final photon is right-handed. Following this analysis in more detail, we find

$$\mathcal{M}(e^-_R\gamma_L \to e^-_L\gamma_R) \approx \frac{4e^2 m/\omega}{\chi^2+m^2/\omega^2}. \tag{5.102}$$

The reaction with all four helicities reversed gives the same matrix element.

To compare this result to our previous calculations, we should add the contributions to the cross section from (5.101) and (5.102) and equal contributions for the reactions involving initial left-handed electrons, and divide by 4 to average over initial spins. The unpolarized differential cross section should then be

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{2}\frac{1}{2E}\frac{1}{2\omega}\frac{\omega}{(2\pi)\,4(E+\omega)}\left[\frac{8e^4\chi^2}{(\chi^2+m^2/\omega^2)^2}+\frac{8e^4m^2/\omega^2}{(\chi^2+m^2/\omega^2)^2}\right]\\ &= \frac{4\pi\alpha^2}{s(\chi^2+4m^2/s)},\end{aligned} \tag{5.103}$$

which agrees precisely with Eq. (5.94).

The importance of the helicity-flip process (5.102) just at the kinematic endpoint has an interesting experimental consequence. Consider the process of *inverse* Compton scattering, a high-energy electron beam colliding with a low-energy photon beam (for example, a laser beam) to produce a high-energy photon beam. Let the electrons have energy E and the laser photons have energy ϖ, let the energy of the scattered photon be $E' = yE$, and assume for simplicity that $s = 4E\varpi \gg m^2$. Then the computation we have just done applies to this situation, with the highest energy photons resulting from scattering that is precisely backward in the center-of-mass frame. By computing $2k\cdot k'$ in the center-of-mass frame and in the lab frame, it is easy to show that the final photon energy is related to the center-of-mass scattering angle through

$$y \approx \frac{1}{2}(1-\cos\theta) \approx 1 - \frac{\chi^2}{4}.$$

Then Eq. (5.103) can be rewritten as a formula for the energy distribution of backscattered photons near the endpoint:

$$\frac{d\sigma}{dy} = \frac{2\pi\alpha^2}{s((1-y)+16m^2/s)^2}\left[(1-y)+\frac{16m^2}{s}\right], \tag{5.104}$$

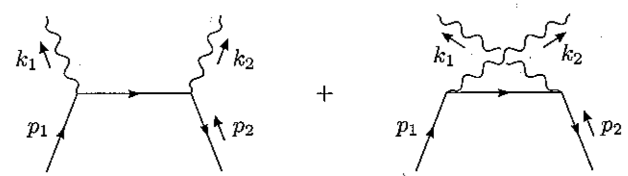
where the first term in brackets corresponds to the helicity-conserving process and the second term to the helicity-flip process. Thus, for example, if a right-handed polarized laser beam is scattered from an unpolarized high-energy electron beam, most of the backscattered photons will be right-handed but the highest-energy photons will be left-handed. This effect can be used experimentally to measure the polarization of an electron beam or to create high-energy photon sources with adjustable energy distribution and polarization.

Pair Annihilation into Photons

We can still obtain one more result from the Compton-scattering amplitude. Consider the annihilation process

$$e^+e^- \to 2\gamma,$$

given to lowest order by the diagrams



This process is related to Compton scattering by crossing symmetry; we can obtain the correct amplitude from the Compton amplitude by making the replacements

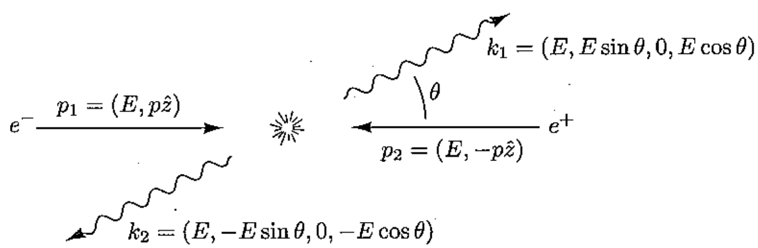
$$p \to p_1 \qquad p' \to -p_2 \qquad k \to -k_1 \qquad k' \to k_2.$$

Making these substitutions in (5.87), we find

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = -2e^4\left[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right) - m^4\left(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\right)^2\right]. \tag{5.105}$$

The overall minus sign is the result of the crossing relation (5.68) and should be removed.

Now specialize to the center-of-mass frame. The kinematics is



A routine calculation yields the differential cross section,

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{E}{p}\right)\left[\frac{E^2+p^2\cos^2\theta}{m^2+p^2\sin^2\theta} + \frac{2m^2}{m^2+p^2\sin^2\theta} - \frac{2m^4}{(m^2+p^2\sin^2\theta)^2}\right]. \tag{5.106}$$

In the high-energy limit, this becomes

$$\frac{d\sigma}{d\cos\theta} \underset{E\gg m}{\longrightarrow} \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right), \tag{5.107}$$

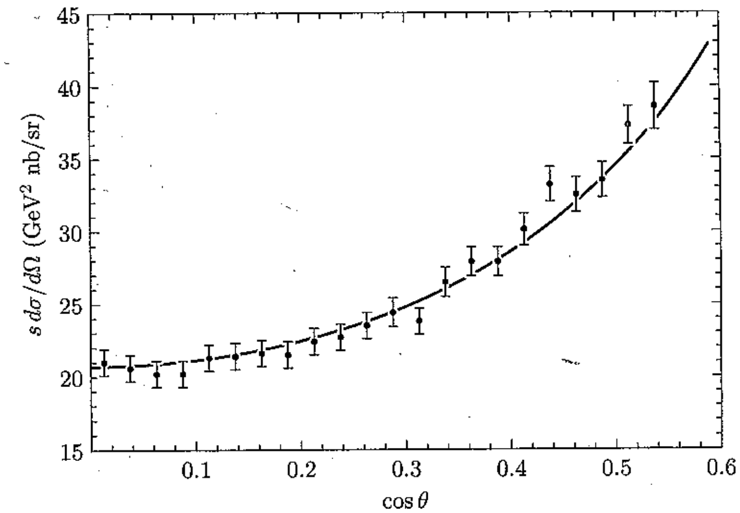


Figure 5.7. Angular dependence of the cross section for $e^+e^- \to 2\gamma$ at $E_{\text{cm}} = 29$ GeV, as measured by the HRS collaboration, M. Derrick, et. al., *Phys. Rev.* **D34**, 3286 (1986). The solid line is the lowest-order theoretical prediction, Eq. (5.107).

except when $\sin\theta$ is of order m/p or smaller. Note that since the two photons are identical, we count all possible final states by integrating only over $0 \le \theta \le \pi/2$. Thus the total cross section is computed as

$$\sigma_{\text{total}} = \int_0^1 d(\cos\theta)\,\frac{d\sigma}{d\cos\theta}. \tag{5.108}$$

Figure 5.7 compares the asymptotic formula (5.107) for the differential cross section to measurements of e^+e^- annihilation into two photons at very high energy.

Problems

5.1 Coulomb scattering. Repeat the computation of Problem 4.4, part (c), this time using the full relativistic expression for the matrix element. You should find, for the spin-averaged cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right),$$

where $\mathbf{p}$ is the electron's 3-momentum and β is its velocity. This is the *Mott formula* for Coulomb scattering of relativistic electrons. Now derive it in a second way, by working