Reading: Work, Force and Potential Energy

The energy of an isolated system is always conserved, but we can transfer energy to a system by interacting with it. One way to do this is by mechanical interactions, that is, by exerting a force.

Let's try to understand how much we change the energy of an object by pushing on it. For simplicity, imagine we're talking about an isolated object in outer space, moving at some velocity $\vec{v} = (v_x, v_y, v_z)$. The energy of this object is

$$K = \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2$$

If we act with a force, this kinetic energy can change. We will now derive a relationship between rate at which K changes and the applied force. First, notice that the time derivative of K is

$$\frac{dK}{dt} = m v_x \frac{dv_x}{dt} + m v_y \frac{dv_y}{dt} + m v_z \frac{dv_z}{dt}.$$

Question 1: which differentiation rules did we have to use to obtain this?

Question 2: show that we can rewrite this as:

$$\frac{dK}{dt} = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt}.$$

By this equation, the rate of change of kinetic energy is related to the rate of change of position and the force. As a result, we can say that the change in energy ΔK during a short time interval will be related to the change in position by

$$\Delta K = F_x \Delta x + F_y \Delta y + F_z \Delta z$$

The expression on the right tells us how much energy we have added to an object by exerting a force on it. We call it the WORK done on the object by the force F

$$W = F_x \Delta x + \; F_y \Delta y + F_z \Delta z$$

In general, the work can increase either kinetic energy or potential energy, or both. It can also decrease them it the right hand side is negative.

Question 3: What would be a situation where exerting a force on something decreases its energy?

Let's mention a few important properties of this expression for work:

1) We only do work (change the energy) if the object actually moves in the direction that the force is exerted in. If the object doesn't move, or if it moves perpendicular to the force (e.g. for a planet orbiting the Sun), there is no work done, so no change in energy.

2) The expression above is only valid if the force is constant over the path $\Delta\vec{r}$. For a changing force, we need to break up the path into little segments and add up $F_x \Delta x + \; F_y \Delta y + F_z \Delta z$ for each segment.

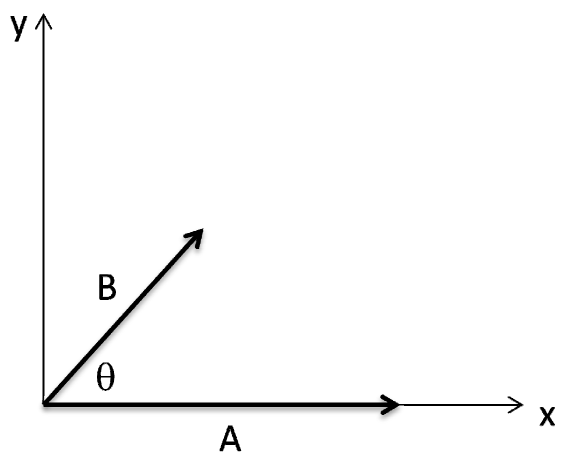
3) The work is obtained by multiplying the common components of two vectors $\vec{F}$ and $\vec{r}$, and adding these up. This operation is known as the DOT PRODUCT $\vec{F} \cdot \Delta\vec{r}$ of the two vectors. Geometrically, it is equal to the product of the lengths of the two vectors, times the angle between them. For two general vectors, we define:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

and we can show that:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta \;.$$

Question 4: Show this by calculating $A_xB_x + A_yB_y + A_zB_z$ for the vectors shown below in terms of the lengths A and B and the angle θ.



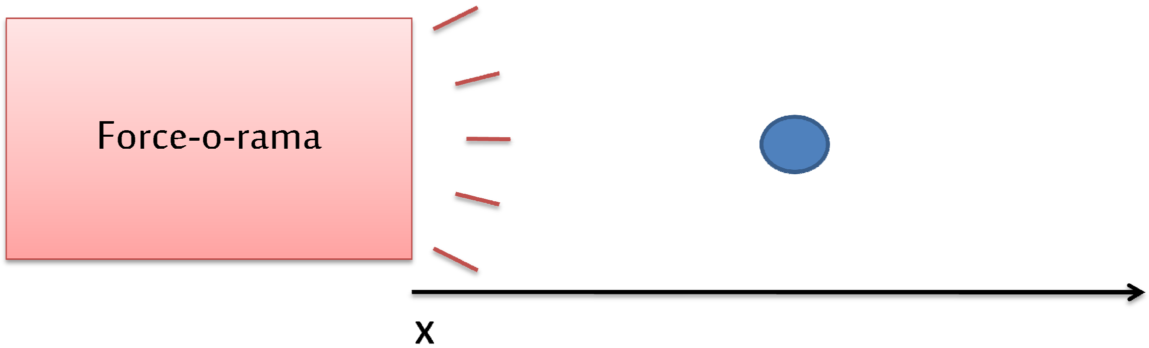
Force and Potentials

Using the dot product, we can write the expression for the work done by a force $\vec{F}$ when it moves an object a distance $\Delta\vec{r}$ as

$$W = \vec{F} \cdot \Delta\vec{r}\,,$$

We will now use this to see that for forces that are not associated with some transfer of energy to heat (so not friction or air drag), there is a potential energy that can be directly related to the force.

For these forces, we can think of the force as being “caused” by the differences in potential energy, and the tendency of an object to move toward places with lower potential energy.



Question 5 a) Force-o-rama produces an attractive force on the ball with a magnitude that depends on the object's position: $|F| = f(x)$. If you push the object a little from x to x + dx with the minimum possible amount of force, how much energy do you transfer to the object (assume that dx is small enough so that f(x) is doesn't change much between x and x + dx)?

b) If the force doesn't involve any transfer of energy to heat, sound waves, etc…, the energy we transfer must go to the object. We used the minimum amount of energy necessary to move the object, so we won't be adding any kinetic energy. Thus, all our transferred energy must go into potential energy associated with the position of the ball relative to Force-o-rama. If the potential energy of the ball at position x is U(x), complete the equation below using your answer to part a):

$$U(x + dx) - U(x) =$$

Now, if the force from Force-o-rama is attractive (i.e. in the -x direction), we have F(x) = -f(x). Rewrite your answer to b) using F(x) instead of f(x), divide both sides by dx and take the limit where dx is small. What relation between U and F do you find?

(you can check with eqn 11.23 in the text)

Congratulations! You have just derived the general relation between CONSERVATIVE FORCES (forces that aren't associated with any loss of energy to heat, etc…) and their associated potential energies.

Question 6

Let's check this out for a few examples.

a) What is the gravitational potential energy U(z) of an object of mass M at a height z?

b) What is $-\frac{dU}{dz}$? Does this match with the force of gravity in the z direction?

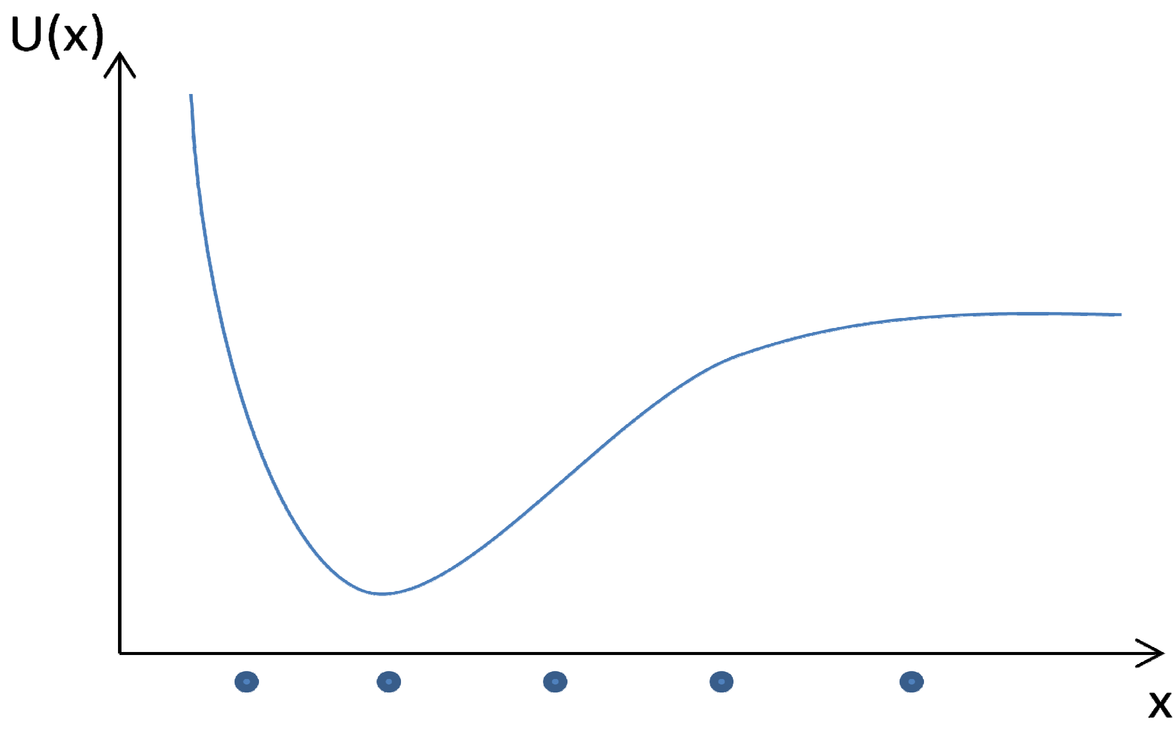
c) A spring with spring constant k and normal length x_0 is stretched/compressed to length x. What is its potential energy U(x)?

d) What is $-\frac{dU}{dx}$? Does this match with the spring force (Hooke's law)?

e) For an electron in a hydrogen atom at distance r from the proton, the Coulomb force from the proton is $F(r) = -ke^2/r^2$, where k is a constant and e is the proton charge. If the potential energy is defined to be 0 at $r = \infty$, find the potential energy $U(r)$ for the electron at position r.

Question 7

A certain force is associated with the potential energy function shown below:



Draw arrows at each of the dots below the x axis to show the direction and magnitude of the force on an object at the position of the dot.