Problem set 2

1. Draw all distinct proper self-energy diagrams of 3rd order, and find their multiplicities, for disorder averaged Green's functions.

2. Use path integral techniques to find the propagator $\mathcal{G}(xt, x_0t_0)$ for a charged particle moving in a uniform electric field, described by $\hat{H} = \frac{\hat{p}^2}{2m} - qE\hat{x}$. Are any approximations needed?

3. In class, we derived the propagator for a free particle to be

$$\mathcal{G}_0(xt, x_0t_0) = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left(\frac{im}{2\hbar\Delta t}(x - x_0)^2\right)$$

where $\Delta t = t - t_0$. Consider now a free particle moving on a finite chain of length L, which has periodic boundary conditions (in other words, a circle).

(a) Find the exact eigenspectrum and eigenvalues and derive an expression for the propagator starting from the definition

$$\mathcal{G}(xt, x_0t_0) = \langle x| \exp\left(-\frac{i}{\hbar}\hat{H}(t - t_0)\right)|x_0\rangle$$

(b) use path integral techniques to derive the propagator. Hint: in a periodic space there may be more than one way to get from "here" to "there".

You might find this useeful: the Jacobi theta function is defined by

$$\theta_3(z, t) = \sum_{n=-\infty}^{\infty} \exp\left(i\pi tn^2 + i2nz\right)$$

and satisfies the properties

$$\theta_3(z, t) = \theta_3(z + \pi, t)$$
$$\theta_3(z + \pi t, t) = e^{-i\pi t - 2iz}\theta_3(z, t)$$
$$\theta_3(z, t) = (-it)^{\frac{-1}{2}} e^{z^2/i\pi t}\theta_3\left(\frac{z}{t}, -\frac{1}{t}\right)$$

4. Find the ground-state wavefunction for a harmonic oscillator $\langle x|n = 0\rangle = \phi_0(x)$ starting from $a|0\rangle = 0$. Write a in terms of $\hat{x}$ and $\hat{p}_x$, and project the equation on $\langle x|$ to get a differential equation for $\phi_0(x)$. Solve it and normalize the solution. Note: you can now use the same method starting from $a^\dagger|n - 1\rangle = \sqrt{n}|n\rangle$ to find all eigenstates $\langle x|n\rangle$.

5. Use the same approach as in (4) to find (up to an overall phase shift) $\langle x|z\rangle$, starting from the fact that $a|z\rangle = z|z\rangle$. Check your result by computing this wavefunction directly from definition

$$\langle x|z\rangle = e^{\frac{-|z|^2}{2}}\langle x|e^{\frac{z}{\sqrt{2}l}(\hat{x} - i\hat{p}/(m\omega))}|0\rangle$$

Remember that $T(a) = \exp(ia\hat{p}/\hbar)$ is simply the translation operator: $\langle x|T(a)|\psi\rangle = \psi(x + a)$ (you should be able to check this). Using this result and $\exp(-\frac{i}{\hbar}\hat{H}t)|z\rangle = \exp(-i\omega t/2)|z\exp(-i\omega t)\rangle$, check the equations written directly above Eq. (1.64) in the notes.