PHYS 525
Solutions to HW 5

(Dated: March 1, 2019)

PROBLEM 1

a) The Hamiltonian is

$$\mathcal{H}(\vec{q}) = \vec{d}\cdot\vec{\sigma}, \quad \text{with} \quad \vec{d} = (q_x^2 - q_y^2, 2q_xq_y, m), \tag{1}$$

with an energy spectrum

$$E(\vec{q}) = \pm\left|\vec{d}\right| = \pm\sqrt{q^4+m^2}. \tag{2}$$

At the critical point ($m=0$) we have

$$\boxed{E(\vec{q}, m=0) = \pm q^2,} \tag{3}$$

i.e. quadratic band crossing.

b) Calculate the change in Chern number of the occupied band (negative energy)

$$\begin{aligned}
\Delta n =& n(m<0) - n(m>0) && (4)\\
=& \frac{1}{4\pi}\int d^2q \frac{1}{d^3}\vec{d}\cdot\left(\partial_x\vec{d}\times\partial_y\vec{d}\right)\Big|_{m>0}^{m<0} && (5)\\
=& \frac{1}{4\pi}\int d^2q \frac{4q^2m}{(q^4+m^2)^{3/2}}\Big|_{m>0}^{m<0} && (6)\\
=& \frac{1}{2}\int_0^{\Lambda} qdq \frac{4q^2m}{(q^4+m^2)^{3/2}}\Big|_{m>0}^{m<0} && (7)\\
&\boxed{= \left[\text{sgn}(m) - \frac{m}{\sqrt{\Lambda^4+m^2}}\right]_{m>0}^{m<0} \xrightarrow{\Lambda\gg|m|} -2} && (8)
\end{aligned}$$

Thus the Chern number of the occupied band is -2 while that of the empty band is +2.

PROBLEM 2

a) The spectrum is

$$\mathcal{H}^2(\vec{k}) = \sin^2 k_x + \sin^2 k_y + M_k^2 \tag{9}$$

$$\boxed{E(\vec{k}) = \pm\sqrt{\sin^2 k_x + \sin^2 k_y + M_k^2}} \tag{10}$$

<u>Time reversal</u>:

$$\begin{aligned}
s_y\mathcal{H}^*(\vec{k})s_y =& -\sigma_z\left(s_x\sin k_y - s_y\sin k_x\right) + \sigma_x M_k && (11)\\
=& \mathcal{H}(-\vec{k}). && (12)
\end{aligned}$$

<u>Inversion</u>:

$$\begin{aligned}
\sigma_x\mathcal{H}(\vec{k})\sigma_x =& -\sigma_z\left(s_x\sin k_y - s_y\sin k_x\right) + \sigma_x M_k && (13)\\
=& \mathcal{H}(-\vec{k}). && (14)
\end{aligned}$$

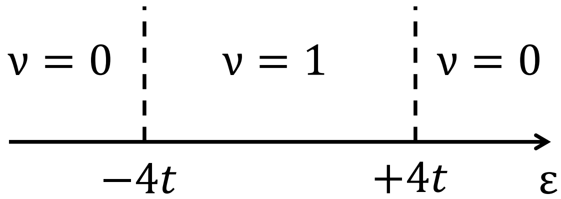

FIG. 1. Phase diagram for (b).

Therefore, $\mathcal{H}(\vec{k})$ respects both $\mathcal{T}$ and $\mathcal{P}$.

b) At the time-reversal invariant momenta (TRIM) $\vec{\Gamma} = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$ the Hamiltonian is

$$\mathcal{H}(\Gamma) = \sigma_x M_k. \tag{15}$$

The eigenstates of the filled bands are eigenstates of σ_x with eigenvalue $-\mathrm{sgn}\,(M_\Gamma)$. Since the inversion symmetry is implemented by σ_x as well, we have

$$\boxed{\xi(\Gamma) = -\mathrm{sgn}\,(H_\Gamma)\,.} \tag{16}$$

According to the Fu-Kane parity criteria we have

$$(-1)^\nu = \prod_j \left[-\mathrm{sgn}\left(M_{\Gamma_j}\right)\right] = \mathrm{sgn}\left[M_{(0,0)}M_{(0,\pi)}M_{(\pi,0)}M_{(\pi,\pi)}\right]. \tag{17}$$

Thus,

$$(-1)^\nu = \mathrm{sgn}\left[(\epsilon-4t)\epsilon^2(\epsilon+4t)\right] = \mathrm{sgn}\left(|\epsilon|-4t\right). \tag{18}$$

For $t > 0$ we thus have

$$\nu = 1 \quad \text{when} \quad |\epsilon| < 4t \tag{19}$$

$$\nu = 0 \quad \text{when} \quad |\epsilon| > 4t. \tag{20}$$

In addition, ν is not defined at $\epsilon = 0$ because the system is gapless at that point. The phase diagram is given in Fig. **??**.

c) The spectrum for $\mathcal{H}(\vec{k}) + \delta\mathcal{H}$ is

$$E(\vec{k}) = \pm\sqrt{\left(\sqrt{\sin^2 k_x + \sin^2 k_x} \pm |m|\right)^2 + M_k^2} \tag{21}$$

For small $|m|$, the phase diagram will be modified only close to the critical points $\epsilon = -4t, 0, 4t$. Also, the nature of the phase (i.e. its topological class) will be unchanged far away from the points of adiabatic continuity.

Let us analyze the spectrum near the critical points by expanding $E(\vec{k})$ close to the relevant gapless points.

(i) $\underline{\epsilon \approx 4t}$

The gap closes at $\vec{k} = (0,0)$ so expand

$$E(\vec{k}) \approx \pm\sqrt{\left(|\vec{k}| \pm |m|\right)^2 + (\epsilon - 4t + tk^2)^2}. \tag{22}$$

The energy vanishes when $k^2 = m^2$ and $\epsilon - 4t + tk^2 = 0$. This happens only when

$$\epsilon = t(4 - m^2), \tag{23}$$

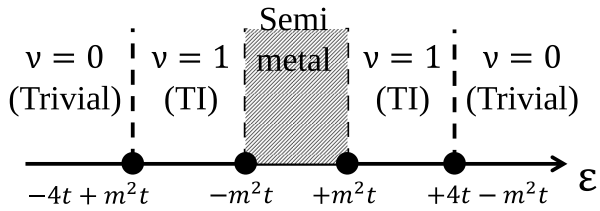

FIG. 2. Phase diagram for (c). Not drawn to scale.

which defines a new (shifted) critical point.

(ii) $\epsilon \approx 0$
The gap closes at $\vec{k} = (0, \pi)$ and $(\pi, 0)$.

$$E(\vec{k}) \approx \pm\sqrt{\left(|\vec{k}| \pm |m|\right)^2 + \left[\epsilon + t(k_x^2 - k_y^2)\right]^2}. \tag{24}$$

The energy vanishes when $k_x^2 + k_y^2 = m^2$ and $\epsilon + t(k_x^2 - k_y^2) = 0$.
These two equations have a solution for (k_x, k_y) if and only if $|\epsilon| < m^2 t$. Thus, in the range $-m^2 t < \epsilon < m^2 t$ the spectrum is gapless.

(iii) $\epsilon \approx -4t$
This case is analyzed as in (i) and one finds a shifted critical point $\epsilon = -t(4 - m^2)$.
The resulting phase diagram is sketched in Fig. 2.