PHYS 525
Solution to HW 3

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PROBLEM 1

a) After the transformation

$$H = \sum_{\boldsymbol{k}} \sum_{\mu\nu} U^*_{\mu\alpha}(\boldsymbol{k}) \mathcal{H}_{\alpha\beta}(\boldsymbol{k}) U_{\beta\nu}(\boldsymbol{k}) d_{-\boldsymbol{k}\mu} d^\dagger_{-\boldsymbol{k},\nu} \tag{1}$$

Using $\left\{d_{\boldsymbol{k}\mu}, d^\dagger_{\boldsymbol{k}\nu}\right\} = \delta_{\mu\nu}$ we obtain

$$H = \sum_{\boldsymbol{k}} \sum_{\mu\nu} \left[U^\dagger(-\boldsymbol{k}) \mathcal{H}(-\boldsymbol{k}) U(-\boldsymbol{k})\right]_{\nu\mu} d^\dagger_{\boldsymbol{k},\mu} d_{\boldsymbol{k},\nu} + \left[U^*(-\boldsymbol{k}) \mathcal{H}(-\boldsymbol{k}) U(-\boldsymbol{k})\right]_{\mu\mu} \tag{2}$$

The last term reduces to $\text{Tr}\left(\mathcal{H}(-\boldsymbol{k})\right)$. For H to be the same as the original we thus require

$$\boxed{\begin{aligned} &1)\ U^\dagger(\boldsymbol{k}) \mathcal{H}(\boldsymbol{k}) U(\boldsymbol{k}) = -H^T(-\boldsymbol{k}) = -H^*(-\boldsymbol{k}) && (3)\\ &2)\ \text{Tr}\left(\mathcal{H}(-\boldsymbol{k})\right) = 0. && (4)\end{aligned}}$$

For condition (1) we used the fact that H is Hermitian.

b) Consider additional time-reversal symmetry:

$$\mathcal{T}: \quad \mathcal{H}^*(\boldsymbol{k}) = \mathcal{H}(-\boldsymbol{k}). \tag{5}$$

The eigenstates are defined by

$$\mathcal{H}\Psi(\boldsymbol{k}) = E_{\boldsymbol{k}} \Psi(\boldsymbol{k}). \tag{6}$$

Inserting identity $UU^\dagger = 1$, we get

$$\left[U(\boldsymbol{k}) \mathcal{H}(\boldsymbol{k}) U^\dagger(\boldsymbol{k})\right] \left[U(\boldsymbol{k}) \Psi(\boldsymbol{k})\right] = E_{\boldsymbol{k}} \Psi(\boldsymbol{k}) \tag{7}$$

Using p-h symmetry condition 1) (Eq. 3) and TR (Eq. 5), we get

$$\mathcal{H}^*(-\boldsymbol{k}) \tilde{\Psi}(\boldsymbol{k}) = -E_{\boldsymbol{k}} \tilde{\Psi}(\boldsymbol{k}) \tag{8}$$

$$\boxed{\mathcal{H}(\boldsymbol{k}) \tilde{\Psi}(\boldsymbol{k}) = -E_{\boldsymbol{k}} \tilde{\Psi}(\boldsymbol{k}).} \tag{9}$$

For each eigenstate $\Psi(\boldsymbol{k})$ with energy $E_{\boldsymbol{k}}$ there exists an eigenstate $\tilde{\Psi}(\boldsymbol{k}) = U(\boldsymbol{k})\Psi(\boldsymbol{k})$ with energy $-E_{\boldsymbol{k}}$.

c) For spinless graphene we have $\mathcal{H}(\boldsymbol{k}) = \boldsymbol{\sigma} \cdot \boldsymbol{d}(\boldsymbol{k})$, where

$$d_x(\boldsymbol{k}) = +d_x(-\boldsymbol{k}), \quad d_y(\boldsymbol{k}) = -d_y(-\boldsymbol{k}), \text{and} \quad d_z(\boldsymbol{k}) = 0 \tag{10}$$

Clearly, $\text{Tr}\left(\mathcal{H}(\boldsymbol{k})\right) = 0$, so condition (2) is satisfied. It is easy to check that

$$\boxed{U(\boldsymbol{k}) = e^{i\phi_{\boldsymbol{k}}} \sigma_z} \tag{11}$$

with $\phi_{\boldsymbol{k}}$ an arbitrary phase which can be taken equal to zero.

PROBLEM 2

Recall that the low-energy Hamiltonian for graphene with a Semenoff mass is

$$\mathcal{H}(x,k_y) = v_F\left(\sigma_x\tau_z(-i\partial_x) + k_y\sigma_y\right) + m_s(x)\sigma_z, \tag{12}$$

where we assumed translational invariance along y and $\lim_{x\to\pm\infty} m_s(x) = \pm|m_0|$. For $k_y = 0$ we want $\mathcal{H}(x)\Psi(x) = 0$. After a rotation $e^{i(\pi/4)\sigma_x}$ which sends $\sigma_z \to \sigma_y, \sigma_y \to -\sigma_z$, we get

$$\begin{pmatrix} 0 & -i\left[v_F\tau_z\partial_x + m_s(x)\right] \\ i\left[-v_F\tau_z\partial_x + m_s(x)\right] & 0 \end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 0. \tag{13}$$

For $\tau_z = +1$, we find normalizable solutions $\Psi_1 = 0$ and $\Psi_2 = Ae^{-\frac{1}{v_f}\int_0^x m(x')dx'}$. Similarly, for $\tau_z = -1$, we find $\Psi_2 = 0$ and $\Psi_1 = Ae^{-\frac{1}{v_f}\int_0^x m(x')dx'}$.

Altogether there are two gapless modes which, after rotating back, have the form

$$\boxed{\Psi^+ = \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} Ae^{-\frac{1}{v_f}\int_0^x m(x')dx'}, \qquad \Psi^- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix} Ae^{-\frac{1}{v_f}\int_0^x m(x')dx'}.} \tag{14}$$

The two have spectra $\epsilon_k^\pm = \langle\Psi^\pm|v_F k_y\sigma_y|\Psi^\pm\rangle = \pm v_F k_y$. Therefore we may conclude that a domain wall in the Semenoff mass hosts a pair of gapless, counter-propagating one-dimensional modes with a linear dispersion.