PHYS 525
Solution to HW 3

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PROBLEM 1

a) A possible choice of anticommuting hermitian 4×4 matrices reads

$$\Gamma_1=\tau_z\sigma_x,\ \ \Gamma_2=\tau_z\sigma_y,\ \ \Gamma_3=\tau_z\sigma_z,\ \ \Gamma_4=\tau_x\sigma_0,\ \ \Gamma_5=\tau_y\sigma_0. \tag{1}$$

Here and hereafter σ_0 is used to denote a 2×2 unit matrix and we use the tensor product notation with $\otimes$ sign ommitted for the sake of brevity. The matrices defined above are often called Dirac Γ matrices and are used to formulate the relativistic version of the Schrödinger equation.

b) The calculation of the energy spectrum follows as in the 2×2 case discussed in class. We square the Hamiltonian

$$H^2=\sum_{i,j}\underbrace{d_i\left(\mathbf{k}\right)d_j\left(\mathbf{k}\right)}_{\text{symmetric in } i\leftrightarrow j}\underbrace{\Gamma_i\Gamma_j}_{\frac{1}{2}\{\Gamma_i,\Gamma_j\}}=\left|\mathbf{d}\left(\mathbf{k}\right)\right|^2\mathbb{I}$$

Taking a square root then gives, as before

$$E_\pm\left(\mathbf{k}\right)=\pm\left|\mathbf{d}\left(\mathbf{k}\right)\right|. \tag{2}$$

c) Extending the construction in part (a), a possible choice of 8×8 matrices reads

$$\rho_z\tau_z\sigma_x,\ \ \rho_z\tau_z\sigma_y,\ \ \rho_z\tau_z\sigma_z,\ \ \rho_z\tau_y\sigma_0,\ \ \rho_z\tau_x\sigma_0,\ \ \rho_x\tau_0\sigma_0,\ \ \rho_y\tau_0\sigma_0, \tag{3}$$

The structure is self-explanatory. In general, one can extend this scheme to construct a set of anticommuting $2^n\times 2^n$ matrices based on the knowledge of $2^{n-1}\times 2^{n-1}$ such matrices. At every step two new matrices are added which adds up to a total of $3+2\times(n-1)=2n+1$ matrices.

Mathematically, this corresponds to the Clifford algebra $Cl_{d-1}\left(\mathbb{R}\right)$ for even d, plus the chiral matrix. The Clifford algebra always contains d matrices Γ_i with $i=0,...,d-1$ and for even d it admits the chiral matrix $\Gamma_c=i^{d/2-1}\Gamma_0...\Gamma_{d-1}$ which anticommutes with all Γ_i. For $2^{d/2}\times 2^{d/2}\equiv 2^n\times 2^n$we have $d=2n$ and adding Γ_c one arrives at $2n+1$ again.

PROBLEM 2

$$H=-t\sum_{\langle ij\rangle}\left(e^{i\phi_{ij}}c_i^\dagger c_j+h.c.\right) \tag{4}$$

a) In the gauge $\vec{A}=B(0,x,0)$, the phase factors along the x-bonds remain 1 but the phase factors along the y-bonds are x-dependent. We denote $\phi=2\pi/q$, and illustrate the situation in Fig. **??**.

The smalles unit cell contains q sites for

$$B=\frac{p}{q}\frac{\Phi_0}{a^2} \tag{5}$$

The band structure thus consists of q bands for each $\vec{k}$ in the 1st BZ, the latter being $k_x\in(-\frac{\pi}{qa},\frac{\pi}{qa}), k_y\in(-\frac{\pi}{a},\frac{\pi}{a})$.

b) For $p=1,q=2$ the unit cell has 2 sites (Fig. **??**), denoted by $a_{\vec{r}},b_{\vec{r}}$

$$H=-t\sum_{ij}\left[\left(a_{ij}^\dagger b_{ij}+a_{ij}^\dagger b_{i-1j}\right)+\left(a_{ij}^\dagger a_{ij+1}+b_{ij}^\dagger b_{ij+1}\right)+h.c.\right]. \tag{6}$$

Fourier transforming, we get $H = \sum_k \Psi^\dagger_{\vec{k}} \mathcal{H}(\vec{k}) \Psi_{\vec{k}}$ with $\Psi_{\vec{k}} = \left(a_{\vec{k}}, b_{\vec{k}}\right)^T$ and

$$\mathcal{H}(\vec{k}) = -2t \begin{pmatrix} \cos k_y & e^{-ik_x}\cos k_x \\ e^{ik_x}\cos k_x & -\cos k_y \end{pmatrix}, \tag{7}$$

$$E_{\vec{k}} = \pm 2t\sqrt{\cos^2 k_x + \cos^2 k_y}. \tag{8}$$

The Dirac points are at $\vec{k} = \left(\frac{\pi}{2a}, \pm\frac{\pi}{2a}\right)$, as shown in Fig. **??**.

Near the Dirac points, the low-E Hamiltonian can be written as

$$\boxed{\mathcal{H}_{eff}(\vec{q}) = v_F\left(\tau_z\sigma_z q_y + \sigma_y q_x\right)} \tag{9}$$

where $\tau_z = \pm 1$ labels the two ”valleys”.

c) Time reversal - this is a spinless Hamiltonian so $\mathcal{T}$ is represented as

$$\boxed{\mathcal{T}: \quad \Theta = K, \quad \mathcal{H}^*(\vec{k}) = \mathcal{H}(-\vec{k})} \tag{10}$$

Inversion is more complicated beacuse the system is not inversion symmetric about the midpoint between a and b sites. However, either site a or site b can be taken as the center of inversion. This is generated by

$$\boxed{\mathcal{P}: \quad P\mathcal{H}(\vec{k})P^{-1} = \mathcal{H}(-\vec{k}), \text{ where } P = \begin{pmatrix} e^{-2i\vec{k}} & 0 \\ 0 & 1 \end{pmatrix}} \tag{11}$$

In the low-E theory, valleys get exchanged under both τ and $\mathcal{P}$:

$$\mathcal{T}: \quad \tau_x \mathcal{H}^*_{eff}(\vec{q})\tau_x = \mathcal{H}_{eff}(-\vec{q}) \tag{12}$$

$$\mathcal{P}: \quad \tau_x\sigma_z\mathcal{H}_{eff}(\vec{q})\sigma_z\tau_x = \mathcal{H}_{eff}(-\vec{q}) \tag{13}$$

$\mathcal{T}$-invariance of H with $q = 2$ is special. It holds because the smallest loop that an electron can traverse (one square plaquette) contains a flux $(1/2)\Phi_0$ and thus gives rise to a phase factor $e^{i\pi} = -1$. This is real and the electron cannot differentiate between clockwise and counter-clockwise hopping.

d) Possible mass terms are: $\sigma_x\tau_z$, σ_x, $\sigma_z\tau_x$, and $\sigma_z\tau_y$ (these are the only terms that anticommute with $\mathcal{H}_{eff}(\vec{q})$).

Mass term	$\mathcal{P}$	$\mathcal{T}$
$\sigma_x\tau_z$	✓	✗
$\sigma_z\tau_x$	✓	✓
$\sigma_z\tau_y$	✗	✓
σ_x	✗	✓

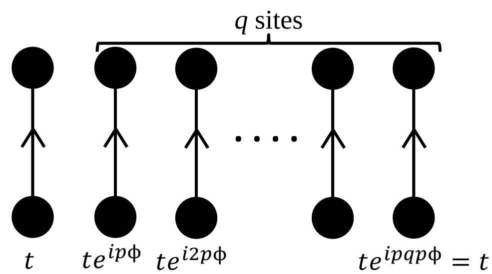

FIG. 1. Hopping parameters along y direction.

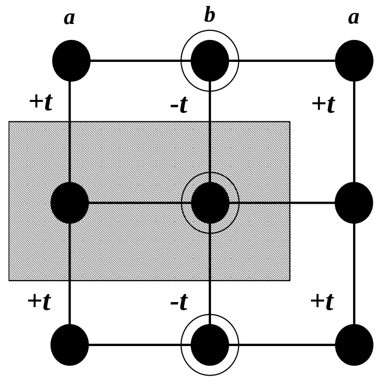

FIG. 2. Unit cell for $p = 1, q = 2$

The $\mathcal{T}$-breaking mass $\sigma_x \tau_z$ describes second-neighbor hopping with imaginary amplitude.
The $\mathcal{P}$-breaking mass σ_x can be realized by dimerizing the lattice along the x-direction, i.e. alternating $t + \delta t$ and $t - \delta t$.

e) With the $\mathcal{T}$-breaking mass we have

$$\mathcal{H}_{eff}(\vec{q}) = v_F\left(\tau_z \sigma_z q_y + \sigma_y q_x\right) + m\sigma_x \tau_z \tag{14}$$

We can calculate the Chern number for the two valleys in analogy with graphene

$$C_1 = \frac{1}{2}\text{sgn}\left[(+v_F)(+v_F)(+m)\right] = \frac{1}{2}\text{sgn}(m) \quad (\tau_z = +1) \tag{15}$$

$$C_2 = \frac{1}{2}\text{sgn}\left[(-v_F)(+v_F)(-m)\right] = \frac{1}{2}\text{sgn}(m) \quad (\tau_z = -1) \tag{16}$$

Thus, $C = C_1 + C_2 = \text{sgn}(m) = \pm 1$. The system is a Chern insulator.

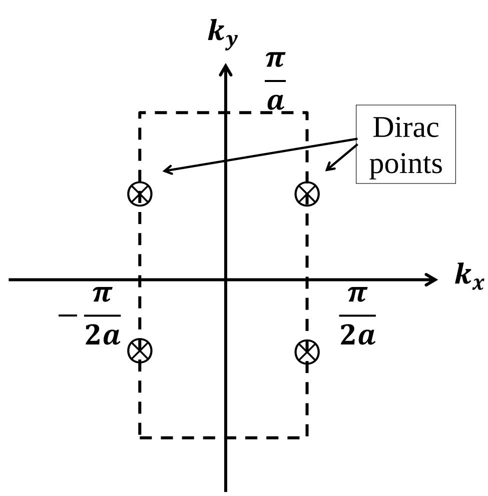

FIG. 3. Brillouin Zone and Dirac points.