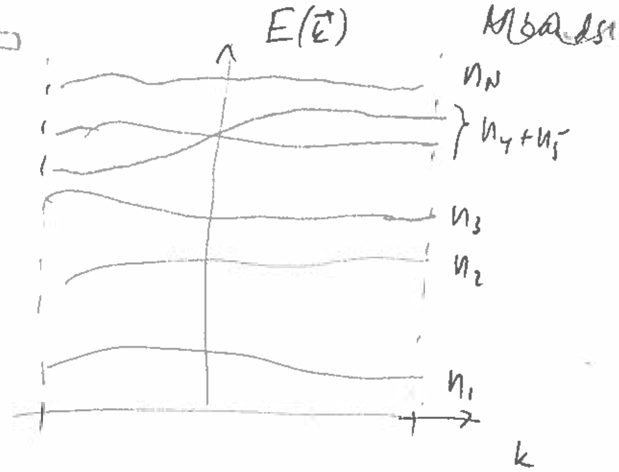
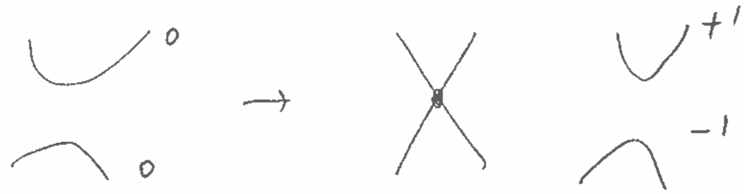
LECTURE 9

Chern insulator summary



N bands separated by gaps (2D)

- Each band has a unique integer Chern # n_j.
- [When bands are touching only the TOTAL Chern # is meaningful, i.e. n_4+n_5.]
- $\sigma_{xy} = \frac{e^2}{h}\sum_{j \in occ.} n_j$ Quantized Hall conductivity
- $\sum_{j=1}^{N} n_j = 0$ $\Leftarrow$ equivalent to no bands filled in terms of holes
- Chern # of a band can only change when it crosses with another band. Specifically $\Delta n = \pm 1$ in a Dirac crossing



- Chiral edge states

$$N_R - N_L = \Delta n$$

TOPOLOGICAL INSULATORS IN 2D

• Time reversal for spin-$\frac{1}{2}$ particles

Spin is a form of angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\mathcal{T}: \quad \vec{L} \to -\vec{L}$$

$$\mathcal{T}: \quad \vec{S} \to -\vec{S}$$

For spin $\frac{1}{2}$ particles: $\quad \vec{S} = \frac{\hbar}{2} \overbrace{(s_x, s_y, s_z)}^{\text{Pauli matrices in spin space}}$

$$\boxed{\mathcal{T}: \quad \vec{S} \to \Theta \vec{S} \Theta^{-1} = -\vec{S} \qquad \Theta = i s_y K}$$

$$\Theta^{-1} = -i s_y K \qquad \Theta\Theta^{-1} = (i s_y K)(-i s_y K) = (i s_y)(-i s_y) = \mathbb{1}$$

$$\Theta^2 = \Theta\Theta = (i s_y K)(i s_y K) = (i s_y)(i s_y) = -1$$

• Kramers theorem for spin-$\frac{1}{2}$ particles

"In a $\mathcal{T}$-invariant spin-$\frac{1}{2}$ system, $[H, \Theta] = 0$, all eigenstates are at least doubly degenerate"

Proof:

$$H|\chi\rangle = E|\chi\rangle$$

$$\Theta|\chi\rangle = c|\chi\rangle$$

Suppose a non-degenerate eigenstate $|\chi\rangle$ existed.

$$\Theta^2|\chi\rangle = \Theta c|\chi\rangle = c^*\Theta|\chi\rangle = c^* c|\chi\rangle = |c|^2|\chi\rangle$$

$-1 = |c|^2$ – contradiction.

• Time reversal for Bloch Hamiltonians (spin-$\frac{1}{2}$ particles)

As before we have

$$\Theta\, \mathcal{H}(\vec{k})\, \Theta^{-1} = \mathcal{H}(-\vec{k})$$

$$\boxed{s_y\, \mathcal{H}^*(\vec{k})\, s_y = \mathcal{H}(-\vec{k})}$$

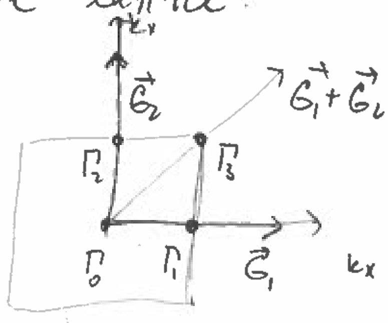
• Time-reversal-invariant momenta: Γ_a

$$\vec{\Gamma} = -\vec{\Gamma} + \vec{G} \qquad \vec{G} \in \text{Reciprocal lattice}$$

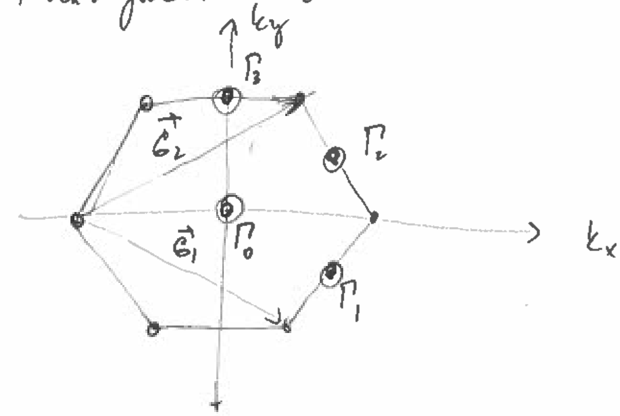
2D: There is 4 such vectors per BZ (3D: 8 Γ's)

$$\vec{\Gamma} = 0,\ \tfrac{1}{2}\vec{G}_1,\ \tfrac{1}{2}\vec{G}_2,\ \tfrac{1}{2}(\vec{G}_1 \pm \vec{G}_2)$$

Square lattice:



Triangular latt.



At TRI momenta we have

$$s_y\, \mathcal{H}^*(\vec{\Gamma})\, s_y = \mathcal{H}(\vec{\Gamma}) \qquad \text{or} \qquad [\mathcal{H}(\vec{\Gamma}), \Theta] = 0$$

$\Rightarrow$ Eigenstates of $\mathcal{H}(\vec{\Gamma})$ are at least doubly degenerate

– Kramers theorem for spin-$\frac{1}{2}$ Bloch Hamiltonian.

Remark: Often systems obey both $\mathcal{T}$ and $\mathcal{P}$

$\mathcal{P}$: $\quad P\,\mathcal{H}(\vec{k})\,P^{-1} = \mathcal{H}(-\vec{k}) \qquad P^2 = 1$

$\mathcal{T}$: $\quad s_y\,\mathcal{H}^*(\vec{k})\,s_y = \mathcal{H}(-\vec{k}) \qquad (P\theta)^2 = -1$

$$\theta P\,\mathcal{H}(\vec{k})\,P^{-1}\theta^{-1} = \mathcal{H}(\vec{k}) \quad , \quad \text{or} \quad [\mathcal{H}(\vec{k}), \theta P] = 0$$

$\Rightarrow$ $\boxed{\text{Eigenstates of } \mathcal{P} \text{ and } \mathcal{T}\text{-symmetric Bloch Hamiltonian } \mathcal{H}(\vec{k}) \text{ are (at least) doubly degenerate at all } \vec{k}.}$

KANE-MELE MODEL [PRL, 95, 226801 (2005)]

Consider graphene for spin $-\frac{1}{2}$ electrons

$$H_0 = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix}$$

$$\mathcal{H}_0(\vec{k}) = \begin{pmatrix} \mathcal{H}_\uparrow(\vec{k}) & 0 \\ 0 & \mathcal{H}_\downarrow(\vec{k}) \end{pmatrix}$$

$\mathcal{H}_{\uparrow,\downarrow}$ – Bloch Hamiltonian for graphene, as before

Consider the Dirac approx.:

$\vec{\sigma}$: sublattice
$\vec{\tau}$: valley
$\vec{S}$: spin

$$\mathcal{H}_{\uparrow,\downarrow}(\vec{q}) = v_F(\tau_z\sigma_x q_x + \sigma_y q_y)$$

spin

$$\mathcal{H}_0(\vec{q}) = v_F(\mathbb{1}\tau_z\sigma_x q_x + \mathbb{1}\mathbb{1}\sigma_y q_y)$$

Kane - Mele mass term

= Haldane mass with opposite sign for ↑ and ↓ spin.

$$\boxed{\delta\mathcal{H}_{KM} = \lambda_{SO}\, S_z\tau_z\sigma_z}$$

- $\delta\mathcal{H}_{KM}$ anticommutes with $\mathcal{H}_0(\vec{q})$

$$\Rightarrow \quad E(\vec{q}) = \pm\sqrt{v_F^2 q^2 + \lambda_{SO}^2}$$

(it is a legitimate mass term)

- $\delta\mathcal{H}_{KM}$ is both $\mathcal{P}$ and $\mathcal{T}$ invariant

$\mathcal{T}$: $s_y \tau_x \, \delta\mathcal{H}^*_{KM} \, \tau_x s_y = \delta\mathcal{H}_{KM}$ ✓

$\mathcal{P}$: $\tau_x \sigma_x \, \delta\mathcal{H}_{KM} \, \sigma_x \tau_x = \delta\mathcal{H}_{KM}$ ✓

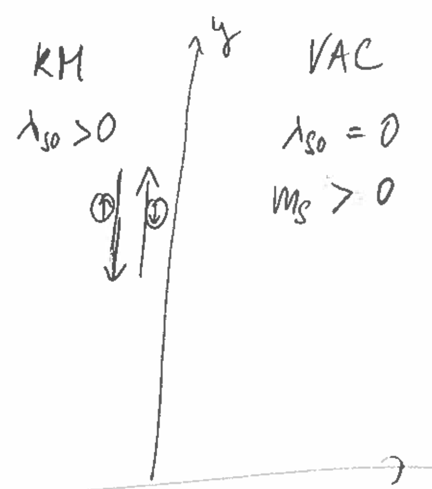
In fact, the Kane-Mele mass obeys all the physical symmetries of graphene and therefore SHOULD BE PRESENT.

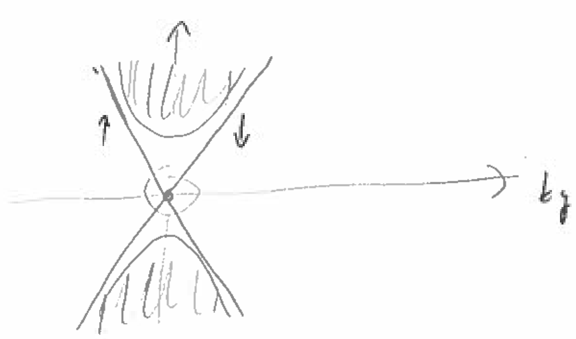
- it is realized through the spin-orbit coupling,

$$V_{so} = \frac{\hbar}{4m^2c^2} \vec{S} \cdot (\vec{\nabla} V \times \vec{p})$$

(relativistic effect, mag. moment moving in $\vec{E}$-field experiences a torque.)

- Edge states and quantum spin Hall effect





- A pair of non-degenerate gapless, spin-filtered states.