LECTURE 8

Clarify Berry curvature:

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \qquad \text{(any dimension)}$$

2D $\quad \mathcal{A}_x, \mathcal{A}_y \rightarrow \mathcal{F}_{xy} = \partial_x \mathcal{A}_y - \partial_y \mathcal{A}_x \equiv \mathcal{B}_z$

$$n = \frac{1}{2\pi} \int_{BZ} dk_x dk_y \, \mathcal{F}_{xy}$$

2-level system

$$\mathcal{F}_{ij} = \frac{1}{2} \hat{d} \cdot (\partial_i \hat{d} \times \partial_j \hat{d}) = \frac{1}{2d^3} \vec{d} \cdot (\partial_i \vec{d} \times \partial_j \vec{d})$$

Chern # of a single Dirac fermion

$$n = \frac{1}{2} \operatorname{sgn}(v_x v_y) \int_0^\Lambda dq \frac{mq}{(q^2+m^2)^{3/2}} \longrightarrow \frac{1}{2} \operatorname{sgn}(v_x v_y m) \quad \begin{matrix}\text{for} \\ \Lambda \to \infty\end{matrix}$$

- Semenoff insulator

Valley	v_x	v_y	m	n	
1	v_F	v_F	m_s	$+\frac{1}{2}\operatorname{sign}(m_s)$	$n_{tot} = 0$
2	$-v_F$	v_F	m_s	$-\frac{1}{2}\operatorname{sign}(m_s)$	
1	v_F	v_F	m_H	$\frac{1}{2}\operatorname{sign}(m_H)$	$n_{tot} = \operatorname{sign} m_H$
2	$-v_F$	v_F	$-m_H$	$\frac{1}{2}\operatorname{sign}(m_H)$	

$n_{tot} = 0, \quad \sigma_{xy} = 0$

- Haldane insulator

$n_{tot} = \operatorname{sign}(m_H)$ or "Chern insulator"

$\sigma_{xy} = \frac{e^2}{h} \operatorname{sign}(m_H) \Leftarrow$ quantum Hall state (in 0 mag. field)

• Consider more carefully transition between the Semenoff & Haldane insulators, i.e. $m \to -m$ at valley 2 (for $m_s, m_H > 0$) Λ - cutoff.

$$\Delta n = \frac{1}{2}\,\mathrm{sgn}(v_x v_y)\left[\int_0^\Lambda dq\,\frac{m_s q}{(q^2+m_s^2)^{3/2}} - \int_0^\Lambda dq\,\frac{m_H q}{(q^2+m_H^2)^{3/2}}\right]$$

$m_s = m$, $m_H = -m$, $v_x = v_y$

$$\Delta n = \frac{1}{2}\cdot 2\int_0^\Lambda dq\,\frac{mq}{(q^2+m^2)^{3/2}} = \left[-\frac{m}{(q^2+m^2)^{1/2}}\right]_0^\Lambda$$

$$= \frac{m}{\sqrt{m^2}} - \frac{m}{\sqrt{\Lambda^2+m^2}} = \mathrm{sign}(m) - \frac{m}{\sqrt{\Lambda^2+m^2}}$$

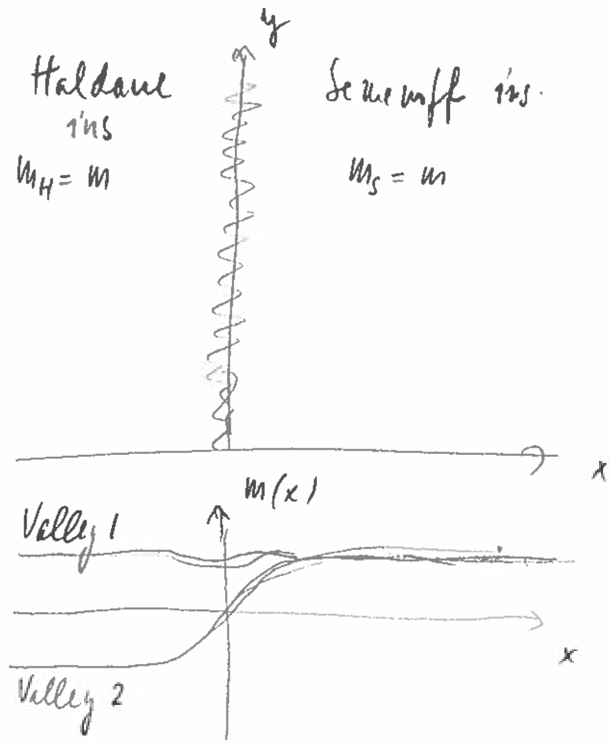
$\longrightarrow$ $\underline{\mathrm{sign}(m) \quad \text{as } |m| \ll \Lambda}$

$\boxed{\text{The Chern \# of the two insulators differs by } \pm 1}$

Semenoff insulator is $\mathcal{T}$-invariant, $n_s = 0$.

$$\boxed{\Rightarrow \quad n_H = \mathrm{sign}(m_H) = \pm 1}$$

• The edge state



Valley 1

$$\mathcal{H}^{(1)}_{eff} = v_F (\sigma_x q_x + \sigma_y q_y) + \sigma_z m$$

$$E^{(1)}(\vec{q}) = \pm \sqrt{v_F^2 q^2 + m^2}$$

$\rightarrow$ gapped spectrum

Valley 2

$$\mathcal{H}^{(2)}_{eff} = v_F \left[-\sigma_x (-i \partial_x) + \sigma_y q_y \right) + \sigma_z m(x)$$

Perform rotation around σ_x : $\mathcal{H}^{(2)}_{eff} \rightarrow \tilde{\mathcal{H}}^{(2)}_{eff} = e^{-i\frac{\pi}{4}\sigma_x} \mathcal{H}^{(2)}_{eff} e^{i\frac{\pi}{4}\sigma_x}$

$$\sigma_y \rightarrow \sigma_z$$
$$\sigma_z \rightarrow -\sigma_y$$

$$\tilde{\mathcal{H}}^{(2)}_{eff} = v_F \left[-\sigma_x (-i\partial_x) + \sigma_z q_y \right] - \sigma_y m(x)$$

Consider $q_y = 0$ first

$$\begin{pmatrix} 0 & +i v_F \partial_x + i m(x) \\ +i v_F \partial_x - i m(x) & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

looking for zero-mode

$$\Psi_0(x) = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{1}{v_F}\int_0^x dx' m(x')} \equiv \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Take $q_y \neq 0$ as a perturbation.

$$\delta E(q_y) = \langle \Psi_0 | v_F \sigma_z q_y | \Psi_0 \rangle$$

$$= v_F q_y \,(0, f) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} = - v_F q_y \langle f | f \rangle$$

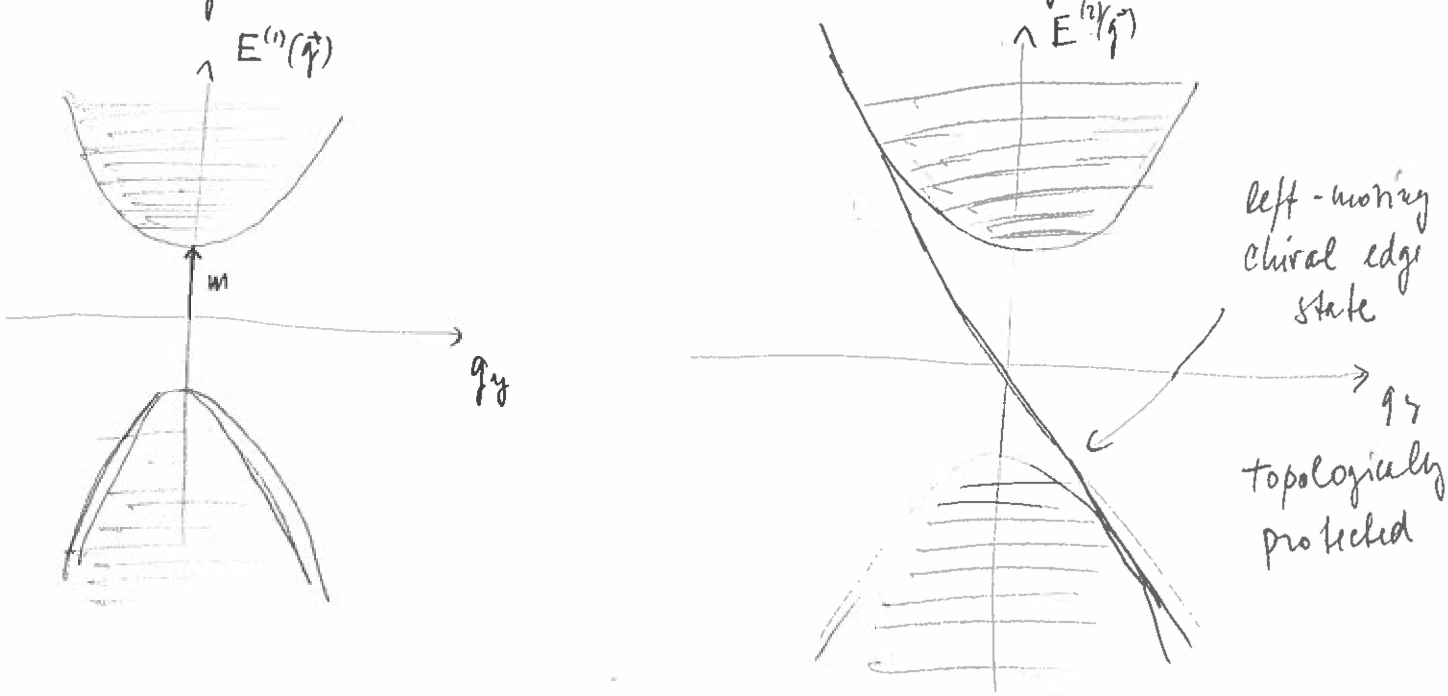
$$\boxed{\delta E(q_y) = - v_F q_y}$$

This is in fact an exact solution, one can check that $\Psi_0(x)$ is an exact eigenstate of $\tilde{\mathcal{H}}^{(2)}_{eff}$ with energy $\underline{- v_F q_y}$

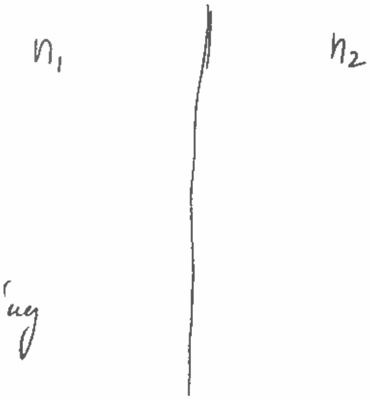
Overall spectrum

Valley 1

Valley 2



Bulk-boundary correspondence

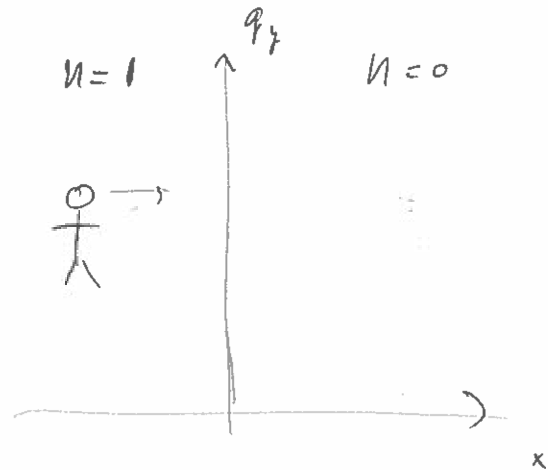


$$\boxed{N_R - N_L = \Delta n}$$

$N_{R,L}$ = # of Right/Left moving edge states

$\Delta n = n_1 - n_2$ (when standing in region 1)

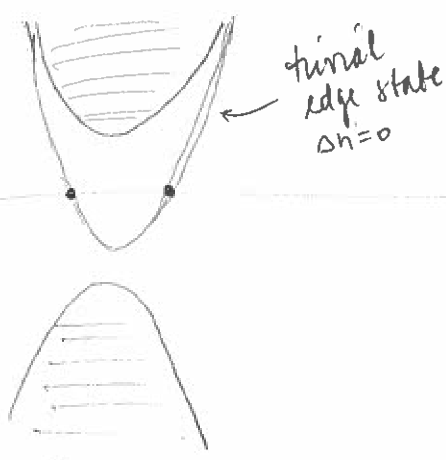
$\Delta n = 1$



$N_R = 1, \ N_L = 0$

$N_e - N_L = \Delta n$ ✓

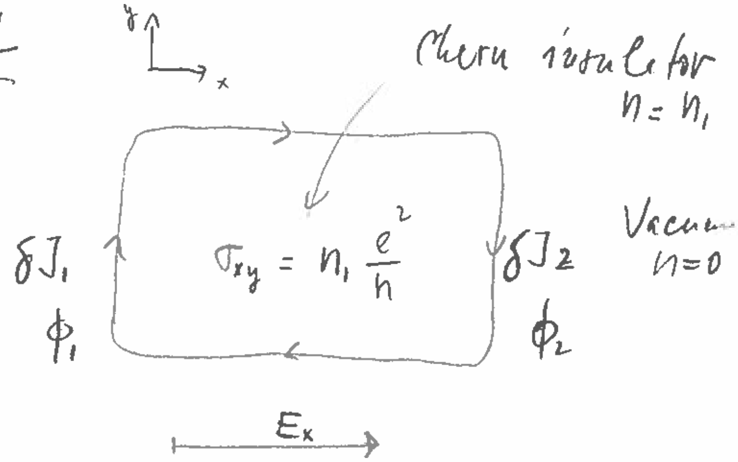
$1 = 1$



<u>Δn does no change</u>

It is robust (topologically protected)

Physical proof



σ_{xy} arises due to the edge states.

Bulk is gapped and thus cannot carry current! (Band theory of solids.)

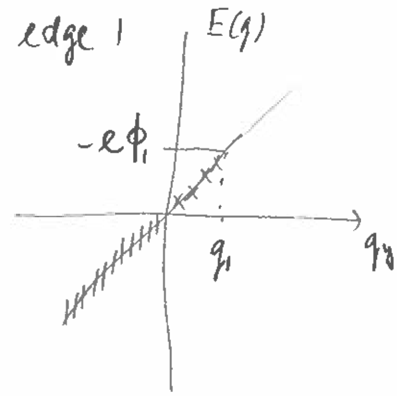
$j_y = j_1 - j_2$ $\qquad \sigma_{yx} = \frac{j_y}{E_x}$

- Current carried by $\underbrace{1d}_{\text{a single}}$ ballistic mode

$$\vec{E}_x = -\vec{\nabla}\phi$$

$$\phi = -E_x x$$

$$\phi_1 - \phi_2 = -E_x L$$



$$\delta J_1 = -e \int_0^{q_1} v(q) \frac{dq}{2\pi} = -e \int_0^{q_1} \frac{1}{\hbar} \frac{dE(q)}{dq} \frac{dq}{2\pi} = -\frac{e}{h} \int_{\varepsilon_F}^{\varepsilon_F - e\phi_1} dE$$

$$= -e \frac{1}{2\pi\hbar} \Big[E \Big]_{\varepsilon_F}^{\varepsilon_F - e\phi_1} = \frac{e^2}{h} \phi_1$$

$$\delta J_2 = + \frac{e^2}{h} \phi_2$$

$$\delta J = \delta J_1 - \delta J_2 = \frac{e^2}{h} (\phi_1 - \phi_2) = \frac{e^2}{h} (-E_x L_x)$$

$$\delta j = \frac{\delta J}{L_x} = -\frac{e^2}{h} E_x \qquad \sigma_{yx} = -\frac{e^2}{h} \qquad \boxed{\sigma_{xy} = \frac{e^2}{h}}$$

If we have $N_{R,L}$ right/left moving modes, we get by the same calculation:

$$\sigma_{xy} = (N_R - N_L)\frac{e^2}{h} = \Delta n \frac{e^2}{h}$$

$$\boxed{N_R - N_L = \Delta n}$$

Bulk - boundary correspondence for Chern insulators.