LECTURE 7

$\mathcal{P}, \mathcal{T}$ for the low-energy theory

$$\mathcal{T}: \quad \mathcal{H}^*(\vec{k}) = \mathcal{H}(-\vec{k})$$

$$\mathcal{P}: \quad \sigma_x \mathcal{H}(\vec{k}) \sigma_x = \mathcal{H}(-\vec{k})$$

$$\vec{k} = \vec{K} + \vec{q} \qquad \text{(valley 1)}$$

$$\vec{k} = -\vec{K} + \vec{q} \qquad \text{(valley 2)}$$

$$\mathcal{H}_{eff}(\vec{q}) = v_F (\tau_z \sigma_x q_x + \sigma_y q_y)$$

$$\mathcal{T}, \mathcal{P}: \quad \vec{k} \to -\vec{k} = -\vec{K} - \vec{q} \qquad \text{(valley 1)}$$

$$= +\vec{K} - \vec{q} \qquad \text{(valley 2)}$$

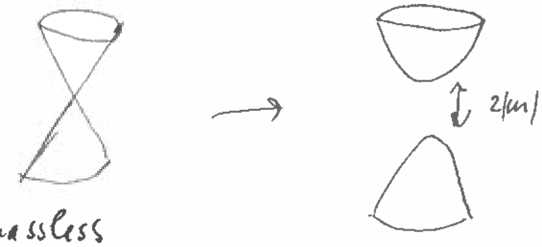
$\mathcal{T}, \mathcal{P}$ interchange the valleys: implement by $\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In the low-energy theory

$$\left[\begin{array}{l} \mathcal{T}: \quad \tau_x \mathcal{H}^*_{eff}(\vec{q}) \tau_x = \mathcal{H}_{eff}(-\vec{q}) \\ \mathcal{P}: \quad \tau_x \sigma_x \mathcal{H}_{eff}(\vec{q}) \sigma_x \tau_x = \mathcal{H}_{eff}(-\vec{q}) \end{array}\right]$$

Check!

Mass terms in graphene



massless → massive Dirac fermion.

① P-breaking "Semenoff" mass

[G. Semenoff, PRL 53, 2449 (1984)]

$$\delta \mathcal{H}_s = \sigma_z m_s$$

$\mathcal{T}$: $\quad \tau_x \, \delta \mathcal{H}_s^* \, \tau_x = \tau_x (\sigma_z m_s)^* \tau_x = \sigma_z m_s = \delta \mathcal{H}_s \quad \checkmark$ $\mathcal{T}$-invariant

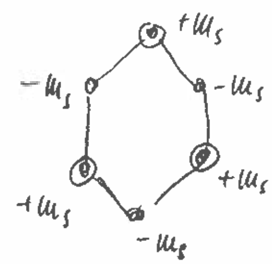
$\mathcal{P}$: $\quad \tau_x \sigma_x \, \delta \mathcal{H}_s \, \sigma_x \tau_x = \tau_x \sigma_x (\sigma_z m_s) \sigma_x \tau_x = -\delta \mathcal{H}_s \quad \times$ $\mathcal{P}$-broken

- Spectrum E_k

$$\mathcal{H}_{eff} = v_F (\tau_z \sigma_x q_x + \sigma_y q_y) + \sigma_z m_s$$

$$\mathcal{H}_{eff}^2 = v_F^2 (q_x^2 + q_y^2) + m_s^2 \qquad \text{(all terms anticommute!)}$$

$$E(\vec{q}) = \pm \sqrt{v_F^2 q^2 + m_s^2}$$



- Realization on the lattice

  - staggered on-site potential
  - realized in hexagonal boron-nitride (BN)

② $\mathcal{T}$-breaking "Haldane" mass

[D. Haldane, PRL 61, 2015 (1988)]

$$\delta \mathcal{H}_H = \tau_z \sigma_z m_H$$

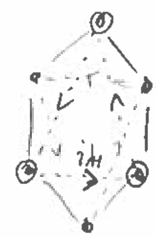
$\mathcal{T}$: $\quad \delta \mathcal{H}_H \to -\delta \mathcal{H}_H \quad \times$

$\mathcal{P}$: $\quad \delta \mathcal{H}_H \to \delta \mathcal{H}_H \quad \checkmark$

- Spectrum E_k

$$E_k = \pm \sqrt{v_F^2 q^2 + m_H^2}$$

- Lattice realization



- imaginary-valued second neighbor hopping

$$d_z^H = \lambda_H \sum_{p<p'=1}^{3} S_{pp'} \sin \vec{k}\cdot(\vec{\delta}_p - \vec{\delta}_{p'}) \qquad , \qquad m_H = 3\sqrt{3}\,\lambda_H$$

$$S_{pp'} = \pm 1 \quad \text{for} \quad p' = (p \pm 1) \bmod 3$$

Chern # in the gapped phases of graphene

$$n = \frac{1}{2\pi} \int_{BZ} \mathcal{F}_{xy} d^2k \quad , \qquad \mathcal{F}_{ij} = \frac{1}{2} \hat{d} \cdot (\partial_i \hat{d} \times \partial_j \hat{d}) \qquad i,j = 1,2 \; (x,y)$$

- hard to evaluate for the lattice model
- try for the low-e Dirac Hamiltonian

$$\mathcal{H}_{eff}(\vec{q}) = \vec{d}(\vec{q}) \cdot \vec{\sigma}$$

$$\vec{d}(\vec{q}) = (v_x q_x, v_y q_y, m)$$

$$d \equiv |d| = \sqrt{v_x^2 q_x^2 + v_y^2 q_y^2 + m^2} \qquad \hat{d} = \frac{\vec{d}}{d}$$

$$\mathcal{F}_{ij} = \frac{1}{2} \frac{\vec{d}}{d} \cdot \left(\partial_i \frac{\vec{d}}{d} \times \partial_j \frac{\vec{d}}{d} \right) =$$

$$= \frac{1}{2} \frac{\vec{d}}{d} \cdot \left(\frac{d\, \partial_i \vec{d} - \vec{d}\, \partial_i d}{d^2} \times \frac{d\, \partial_j \vec{d} - \vec{d}\, \partial_j d}{d^2} \right)$$

$$= \frac{1}{2} \frac{\vec{d}}{d^3} \cdot (\partial_i \vec{d} \times \partial_j \vec{d})$$

$$\partial_x \vec{d} = (v_x, 0, 0)$$

$$\partial_y \vec{d} = (0, v_y, 0)$$

$$\mathcal{F}_{xy} = \frac{1}{2d^3} \begin{vmatrix} v_x q_x & v_y q_y & m \\ v_x & 0 & 0 \\ 0 & v_y & 0 \end{vmatrix} = + \frac{1}{2d^3} v_y v_x m$$

$$n = \frac{1}{2\pi} \int dq_x\, dq_y \, \frac{+ v_x v_y m}{2\left(v_x^2 q_x^2 + v_y^2 q_y^2 + m^2\right)^{3/2}}$$

$$x = |v_x| q_x$$
$$y = |v_y| q_y$$

$$= \frac{+\,\mathrm{sgn}(v_x v_y)}{4\pi} \int_{-\infty}^{\infty} dx\, dy \, \frac{m}{(x^2 + y^2 + m^2)^{3/2}}$$

$$r = \sqrt{x^2 + y^2}$$

$$= +\tfrac{1}{2}\,\mathrm{sgn}(v_x v_y) \underbrace{\int_0^{\infty} dr \, \frac{m r}{(r^2 + m^2)^{3/2}}}_{\mathrm{sgn}(m)}$$

$$= +\tfrac{1}{2}\,\mathrm{sgn}(v_x v_y m)$$

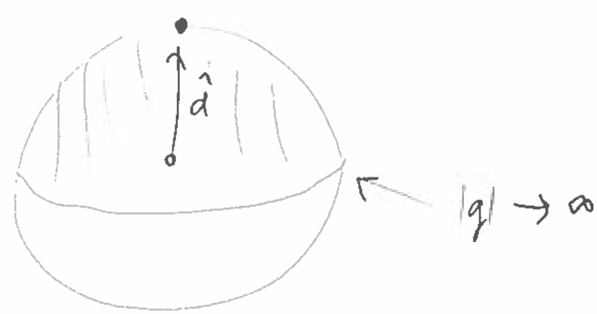
$$\rho = r/m$$

$$\tfrac{1}{2}\,\mathrm{sgn}(v_x v_y)\,\mathrm{sgn}(m) \underbrace{\int_0^{\Lambda/m} d\rho \, \frac{\rho}{(\rho^2 + 1)^{3/2}}}_{\to 1 \text{ as } \frac{\Lambda}{m} \to \infty}$$

Simple geometric argument

$$n = \frac{1}{2\pi} \int \mathcal{F}_{xy}\, d^2 q = \frac{1}{2\pi}\left(\tfrac{1}{2}\Omega\right)$$

solid angle swept by $\hat{d}(\vec{q})$



$$\Omega = 2\pi$$

$$n = \frac{1}{2\pi}\left(\tfrac{1}{2}\, 2\pi\right) = \tfrac{1}{2} \ \checkmark$$