LECTURE 4

• Fractional charge at the domain wall

Chiral symmetry:

$$\sigma_z H(k)\sigma_z = -H(k)$$

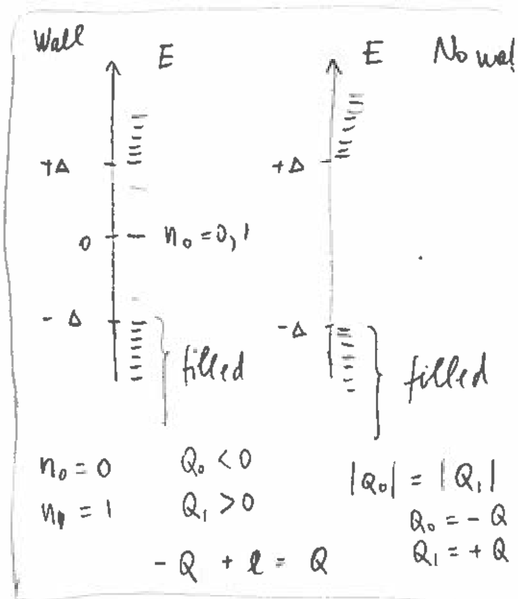
$$\sigma_z H_{eff}\sigma_z = -H_{eff}$$

$\Rightarrow$ For each eigenstate Ψ_E at energy E there exists eigenstate $\Psi_{-E} = \sigma_z \Psi_E$ at $-E$.

Proof: $\quad H \overset{\mathbb{1}=\sigma_z\sigma_z}{\,} \Psi_E = E\Psi_E \quad / \sigma_z$

$$\underbrace{(\sigma_z H \sigma_z)}_{-H}\underbrace{\sigma_z \Psi_E}_{\Psi_E} = E\underbrace{\sigma_z \Psi_E}_{\Psi_{-E}}$$

$$\boxed{H\Psi_{-E} = -E\Psi_{-E}} \quad \checkmark$$



$\Rightarrow Q = \frac{1}{2}e$

Fractional charge

Consider operator $\hat{O}$ (2x2 matrix) and its expectation value

$$\sum_E \Psi_E^+(x)\,\hat{O}\,\Psi_E(x)$$

$$= \sum_E \mathrm{Tr}\left[\Psi_E(x)\Psi_E^+(x)\hat{O}\right] = N\,\mathrm{Tr}[\hat{O}]$$

(N: total # of states)

If there is a zero mode

$$N\,\mathrm{Tr}[\hat{O}] = \Big(\sum_{E<0} + \sum_{E>0}\Big) \Psi_E^+(x)\, \hat{O}\, \Psi_E(x) \quad + \quad \Psi_0^+(x)\, \hat{O}\, \Psi_0(x)$$

$$\sum_{E>0} \Psi_E^+(x)\; \hat{O}\; \Psi_E(x) = \sum_{E>0} \Psi_{-E}^+(x)\, \sigma_z\, \hat{O}\, \sigma_z\, \Psi_{-E}(x)$$

~~assume~~ $$= \sum_{E<0} \Psi_E(x) \left(\sigma_z \hat{O} \sigma_z\right) \Psi_E(x)$$

Assume $\sigma_z \hat{O} \sigma_z = \hat{O}$

$$N\,\mathrm{Tr}[\hat{O}] = 2 \sum_{E<0} \Psi_E^+(x)\, \hat{O}\, \Psi_E(x) + \Psi_0^+(x)\, \hat{O}\, \Psi_0(x)$$

Expectation value of $\hat{O}$ over $E<0$ occupied states is

$$\boxed{O(x) \equiv \sum_{E<0} \Psi_E^+(x)\, \hat{O}\, \Psi_E(x) = \frac{1}{2}\left[N\,\mathrm{Tr}\,\hat{O} - \Psi_0^+(x)\, \hat{O}\, \Psi_0(x)\right]}$$

Consider charge operator $\hat{\rho} = e\mathbb{1}$

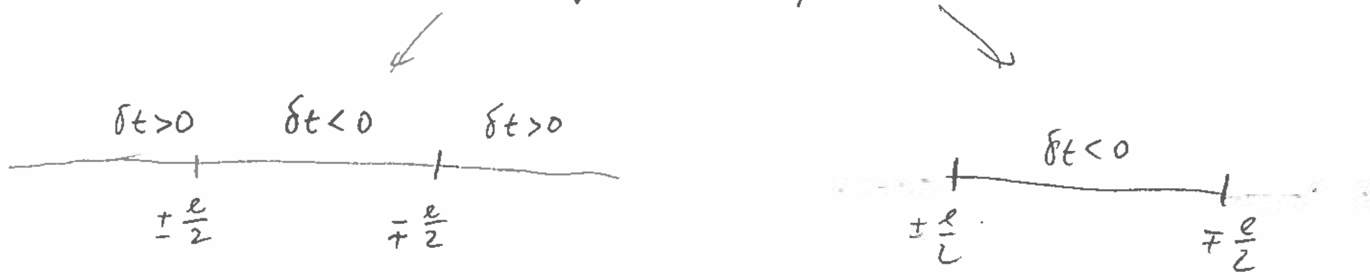
$$\delta\rho_{\mathrm{WALL}}(x) \equiv \rho_{\mathrm{WALL}}(x) - \rho_0 = \frac{e}{2}\left[N\,\mathrm{Tr}\,\mathbb{1} - \Psi_0^+(x)\,\mathbb{1}\,\Psi_0(x) - \cancel{N\,\mathrm{Tr}\,\mathbb{1}}\right]$$

$$= -\frac{e}{2}\,|\Psi_0(x)|^2$$

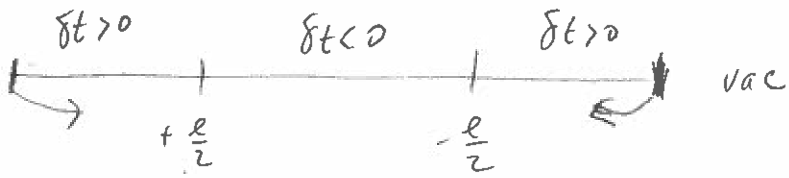
$$\boxed{\delta Q_{\mathrm{WALL}} = \int_{-\infty}^{\infty} dx\; \delta\rho_{\mathrm{WALL}}(x) = -\frac{e}{2}}$$

fractional charge at the domain wall

• Domain wall charge vs. polarization



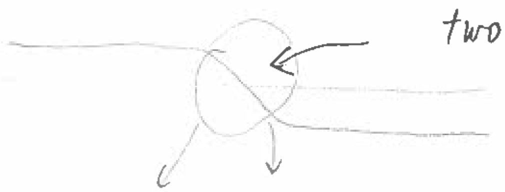
$\Rightarrow$ topologically trivial ($\gamma_c = 0$) phase is equivalent to vacuum



shrink the trivial region to zero, fractional charge remains.

Real polyacetylene

• Have to include electron spin $\uparrow, \downarrow$



two zero modes, spin $\uparrow$ and $\downarrow$

(neglect spin-orbit coupling)

$n_{0\uparrow}$	$n_{0\downarrow}$	Q	S_z
0	0	-1	0
1	0	0	$+\frac{1}{2}$
0	1	0	$-\frac{1}{2}$
1	1	$+1$	0

- simple example of spin-charge separation

See "Solitons in conducting polymers" by Heeger, Kivelson, Schrieffer, Su

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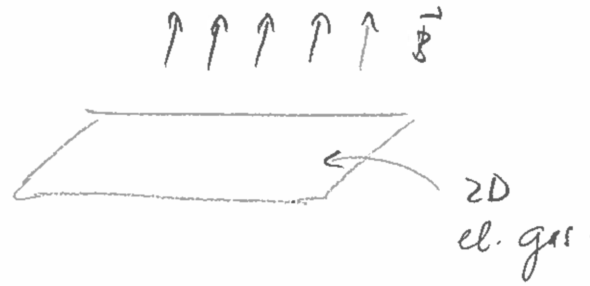
Topology in 2D systems

I. Quantum Hall states



- Landau levels
  - free electrons in $\vec{B}$ field

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \qquad \vec{A} = (0, Bx, 0) \quad \text{(Landau gauge)}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{z} B$$

$$= \frac{1}{2m}\left[p_x^2 + \left(p_y - \frac{eB}{c}x\right)^2\right]$$

Translational invariance in y-direction $\qquad \vec{p} = -i\hbar\vec{\nabla}$

$$\psi(x,y) = e^{iky}\,\Phi(x)$$

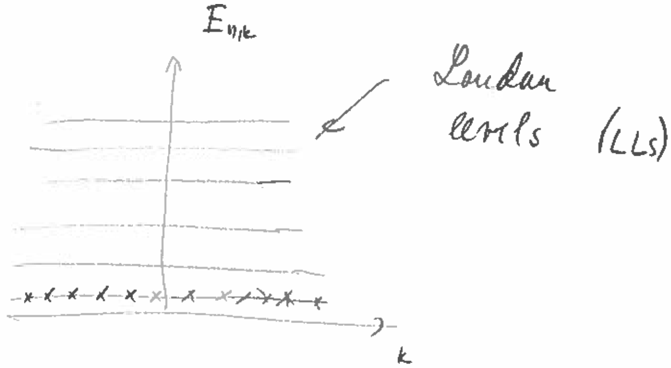
$$H\psi = E\psi \quad \Rightarrow \quad \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial x^2} + \left(\frac{eB}{c}\right)^2\Big(x - \underbrace{\frac{\hbar c}{eB}k}_{x_k}\Big)^2\right]\Phi = E\Phi$$

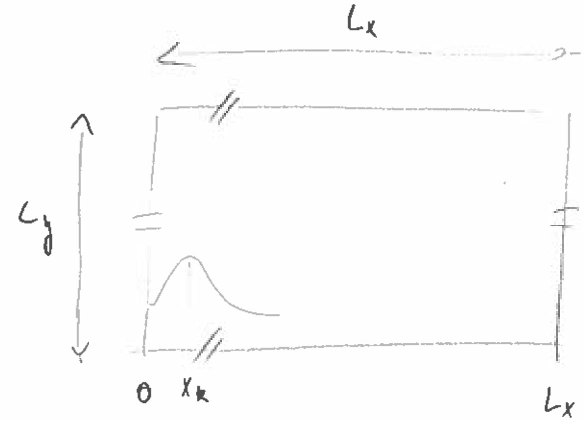
$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_c^2(x - x_k)^2\right]\phi = E\phi \qquad \omega_c = \frac{eB}{mc}$$

shifted harmonic oscillator $\qquad$ "cyclotron freq."

$$\psi_{nk}(x,y) = e^{iky}\,\Phi_n(x - x_k), \qquad x_k = \frac{\hbar c}{eB}k$$

$$E_{nk} = \hbar\omega_c\left(n + \tfrac{1}{2}\right), \qquad n = 0, 1, \ldots, \qquad \text{indep. of } k$$





LL degeneracy (# of states in a LL)

Consider p.b.c $\quad \psi(x, y+L_y) = \psi(x,y)$

$$k = \frac{2\pi}{L_y} m \qquad m = 0, 1, 2 \ldots$$

$$x_k = \frac{\hbar c}{eB} k \qquad 0 < x_0 < L_x$$

Φ total mag. flux

$$\Rightarrow \quad 0 \le m \le \frac{eBL_xL_y}{2\pi\hbar c} = \frac{(BL_xL_y)}{\Phi_0} \qquad \Phi_0 = \frac{hc}{e} \text{ flux quanta}$$

= # of flux quanta in the system.

LL degeneracy: $\quad m_{MAX} = \dfrac{\Phi}{\Phi_0} = \dfrac{BL_xL_y}{\Phi_0} = \dfrac{L_xL_y}{2\pi \ell_B^2}$

where $\ell_B^2 = \dfrac{\hbar c}{eB}$ "magnetic length"