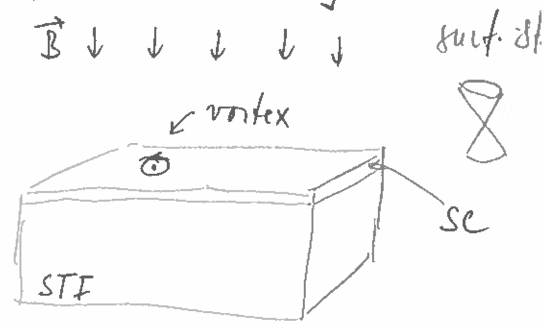
LECTURE 21

Vortices in the Fu-Kane model and their non-Abelian exchange statistics

- Fu-Kane model [Fu & Kane, PRL 100, 096407 (2008)]



$$\mathcal{H} = \int d^2r \; \hat{\Psi}_r^{+} \, H_{FK}(\vec{r}) \, \hat{\Psi}_r$$

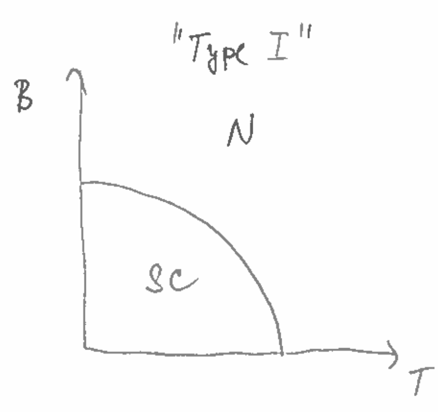
$$\hat{\Psi}_r = \begin{pmatrix} c_{\uparrow r} \\ c_{\downarrow r} \\ c^{+}_{\downarrow r} \\ -c^{+}_{\uparrow r} \end{pmatrix}, \qquad H_{FK}(\vec{r}) = \begin{pmatrix} v\vec{p}\cdot\vec{\sigma} - \mu & \Delta(\vec{r}) \\ \Delta^*(\vec{r}) & -v\vec{p}\cdot\vec{\sigma} + \mu \end{pmatrix}$$

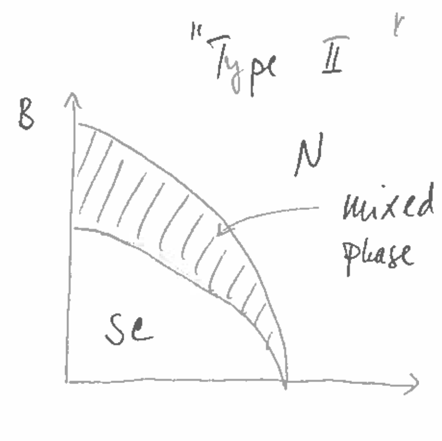
or $\quad H_{FK}(\vec{r}) = \tau^z(v\vec{p}\cdot\vec{\sigma} - \mu) + \tau_x \Delta_1 + \tau_y \Delta_2 , \qquad \Delta(\vec{r}) = \Delta_1(\vec{r}) + i\Delta_2(\vec{r})$

Spectrum: (for constant Δ)

$$E_k = \pm\sqrt{(vp \pm \mu)^2 + \Delta^2} \quad \leftarrow \text{gapped superconductor}$$

Vortices in SC:

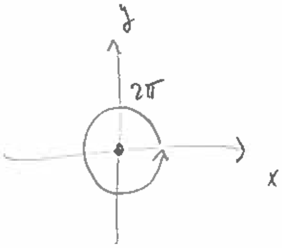




Mag. field penetrates in quantized increments $\Phi_0^* = \frac{hc}{2e}$, one vortex

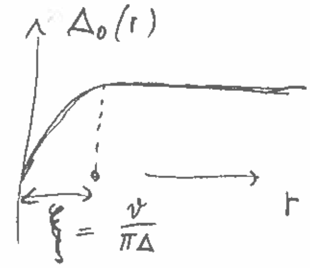
• **Vortex** – topological defect in a SC generated by applied magnetic field or by thermal fluctuation

$$\boxed{\Delta(\vec{r}) = \Delta_0(r)\, e^{i\varphi}} \qquad \vec{r} = r(\cos\varphi, \sin\varphi)$$



SC phase φ winds by 2π around the center

$\Delta_0(r) \longrightarrow 0$ as $r \to 0$.



"SC coherence length"

• **Want to solve**

$$H_{FK}\, \Phi(\vec{r}) = 0$$

in the presence of a vortex at the origin.

$$v\vec{p}\cdot\vec{\sigma} = v\begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} \equiv v\begin{pmatrix} 0 & p_- \\ p_+ & 0 \end{pmatrix}, \qquad p_\pm = p_x \pm i p_y$$
$$= -i(\partial_x \pm i\partial_y)$$

– go to cylindrical coordinates

$$p_\pm = e^{\pm i\varphi}\left(-i\partial_r \pm \frac{1}{r}\partial_\varphi\right)$$

$$\Delta(\vec{r}) = \Delta_0(r)\, e^{i(\varphi + \alpha)}$$

α – constant phase due to other distant vortices

Zero mode wavefunction has a simple form for $\mu = 0$

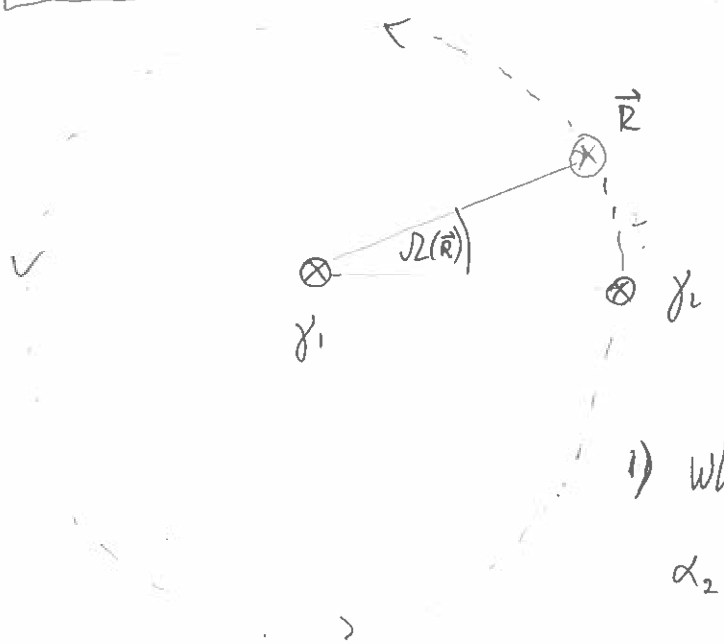
$$\Phi_0 = \frac{1}{\sqrt{2}} f_0(r) \begin{pmatrix} e^{i(\alpha/2 - \pi/4)} \\ 0 \\ 0 \\ -e^{-i(\alpha/2 - \pi/4)} \end{pmatrix}, \qquad f_0(r) = A\, e^{-\frac{1}{v}\int_0^r dr' \Delta_0(r')}$$

Zero mode operator:

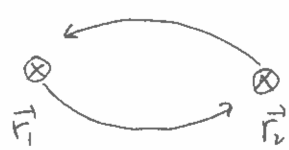
$$\gamma = \frac{1}{\sqrt{2}} \int d^2 r \left[e^{i(\alpha/2 - \pi/4)} c_{r\uparrow} + e^{-i(\alpha/2 - \pi/4)} c^+_{r\uparrow} \right] f_0(r)$$

$\gamma^+ = \gamma$ – Majorana

- Vortex exchange statistics



generally:



$$\Psi(\vec{r}_1, \vec{r}_2) \longrightarrow \pm\, \Psi(\vec{r}_2, \vec{r}_1)$$

bosons / fermions

$$[a_i^+, a_j]_\pm = \delta_{ij}$$

1) When γ_2 encircles γ_1 the phase α_2 in wavefunction Φ_2 advances by 2π

Thus $\Phi_2 \to -\Phi_2$ or $\gamma_2 \to -\gamma_2$

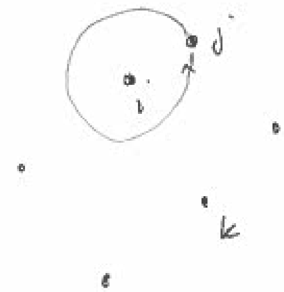
Also, phase α_1 in Φ_1 advances by 2π

Thus $\Phi_1 \to -\Phi_1$ or $\gamma_1 \to -\gamma_1$

To summarize, under encircling

$$\gamma_1 \mapsto -\gamma_1, \quad \gamma_2 \mapsto -\gamma_2$$

This can be implemented by a unitary operator



$$\boxed{\gamma_k \longmapsto U_{ij}\,\gamma_k\, U_{ij}^{\dagger}, \qquad U_{ij} = \gamma_i \gamma_j}$$

2) An exchange of γ_i and γ_j can be thought of as $\frac{1}{2}$ of the encircling operation. (Two consecutive exchanges give encircling.)

$\Rightarrow$ Exchange is implemented by

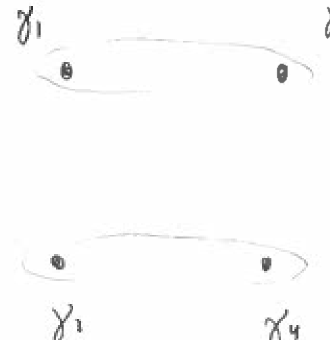
$$T_{ij} = (U_{ij})^{1/2} = \frac{1}{\sqrt{2}}\left(1 + \gamma_i \gamma_j\right)$$

$$\gamma_k \longmapsto T_{ij}\,\gamma_k\, T_{ij}^{\dagger}$$

$$\boxed{\gamma_i \longmapsto -\gamma_j, \quad \gamma_j \longmapsto \gamma_i, \quad \gamma_k \longmapsto \gamma_k \qquad (k \neq i, j)}$$

non-Abelian exchange statistics

- *Physical consequences, use in quantum computation*



$$c_a = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$c_b = \frac{1}{2}(\gamma_3 + i\gamma_4)$$

\# operators

$$n_a = c_a^+ c_a = \frac{1}{2}(1 - i\gamma_1\gamma_2) = 0,1$$

$$n_b = c_b^+ c_b = \frac{1}{2}(1 - i\gamma_3\gamma_4) = 0,1$$

- Encode quantum information into a state $|n_a, n_b\rangle$ (4 states: $|00\rangle$ $|01\rangle$ $|10\rangle$ $|11\rangle$)

  by forming linear superpositions

$$|\Psi\rangle = \sum_{n_a, n_b} c_{n_a n_b} |n_a, n_b\rangle$$

- Manipulate the state vector $|\Psi\rangle$ in a coherent fashion by performing adiabatic exchanges between γ_i ("quantum gates")

For instance:

$$T_{12}\, |n_a, n_b\rangle = e^{i\frac{\pi}{4}(1-2n_a)} |n_a, n_b\rangle$$

$$T_{31}\, |n_a, n_b\rangle = \frac{1}{\sqrt{2}}\left[|n_a, n_b\rangle + (-1)^{n_a} |\bar{n}_a, \bar{n}_b\rangle \right]$$

$$T_{31} T_{31}\, |n_a, n_b\rangle = (-1)^{n_a} |\bar{n}_a, \bar{n}_b\rangle$$

[Unfortunately, this does not give a universal quantum computer.]