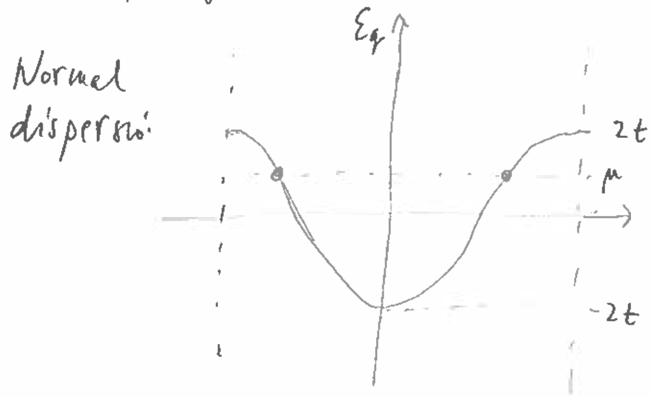
LECTURE 20

Kitaev chain cont'd

$$H = \sum_q \underbrace{(-2t\cos q - \mu)}_{\varepsilon_q} c_q^\dagger c_q + \Delta \sum_q (i \sin q \, c_q c_{-q} + h.c.)$$

Topological phase: when $|2t| > |\mu|$



there exists a Fermi point in the right half of the BZ.

Topological invariant for 1D SC chain [Kitaev]

- Introduce "Majorana polarization" and relate it to the bulk properties of the chain.

For Hamiltonian

$$H = \frac{i}{4} \sum_{l,m} \sum_{\alpha\beta} B_{\alpha\beta}(l-m)\, \gamma_{l\alpha} \gamma_{m\beta}$$

the invariant, Majorana number $\mathcal{M}$ is given by

$$\mathcal{M} = \operatorname{sgn}\left\{ \operatorname{Pf}[\tilde{B}(0)] \cdot \operatorname{Pf}[\tilde{B}(\pi)] \right\} = \pm 1$$

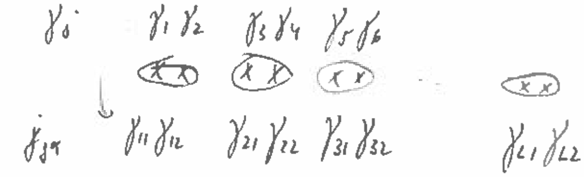
where $\tilde{B}(q)$ is the Fourier transform of matrix $B(j)$.
$\mathcal{M} = -1$ signals the topological phase with MF at ends.

- In the limit of weak SC pairing, i.e. $|\Delta| \ll |t|$,

$$\underline{\mathcal{M} = (-1)^{\nu}}, \qquad \nu = \#\text{ of Fermi points in the right half of the BZ.}$$

(in the normal state)

- Calculate $\mathcal{M}$ for Kitaev chain



$$\cancel{H = \frac{i}{2}\sum_j \left[-\mu\, \gamma_{2j-1}\gamma_{2j} + (t+\Delta)\gamma_{2j}\gamma_{2j+1} + (-t+\Delta)\gamma_{2j-1}\gamma_{2j+2}\right]}$$

$$H = \frac{i}{2}\sum_j \left[-\mu\, \gamma_{j1}\gamma_{j2} + (t+\Delta)\gamma_{j2}\gamma_{j+1,1} + (-t+\Delta)\gamma_{j1}\gamma_{j+1,2}\right]$$

FT: $$\gamma_{j\alpha} = \frac{1}{\sqrt{2N}}\sum_q e^{iqj}\gamma_{q\alpha}$$

$$H = \frac{i}{4}\sum_q (\gamma_{q1}, \gamma_{q2}) \underbrace{\begin{pmatrix} 0 & D_q \\ -D_q^* & 0 \end{pmatrix}}_{\tilde{B}(q)} \begin{pmatrix} \gamma_{-q1} \\ \gamma_{-q2} \end{pmatrix}$$

$$D_q = -\mu - 2t\cos q - 2i\Delta \sin q$$

Check!

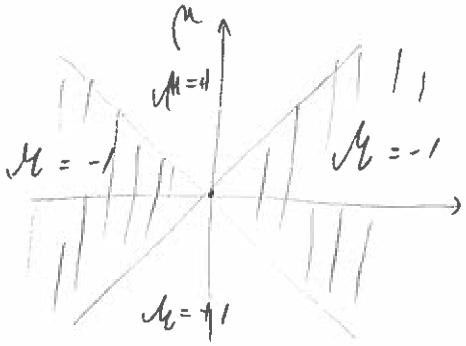
$$\mathrm{Pf}\,[\tilde{B}(0)] = D_0 = -\mu - 2t$$

$$\mathrm{Pf}\,[\tilde{B}(\pi)] = D_\pi = -\mu + 2t$$

Majorana # :

$$\mathcal{M} = \mathrm{sgn}\left\{\mathrm{Pf}[\tilde{B}(0)]\,\mathrm{Pf}[\tilde{B}(\pi)]\right\} = \mathrm{sgn}\left[(\mu-2t)(\mu+2t)\right]$$

$$= \mathrm{sgn}\left(\mu^2 - 4t^2\right)$$



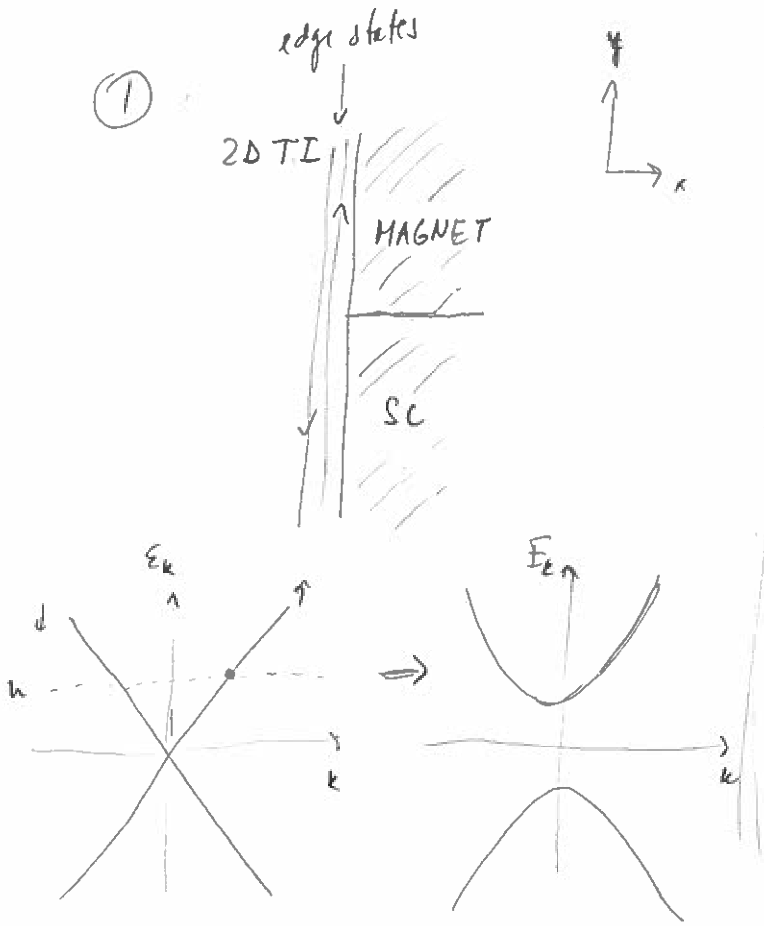
- odd # of Fermi points ✓

PHYSICAL REALIZATIONS OF KITAEV'S PARADIGM

1) Edge state of a 2D TI with SC and magnetic domain walls. [Fu & Kane, PRB 79, 161408 R (2009)]

2) 3D TI nanowire proximity coupled to a SC [Cook & Franz, PRB 84 201105R (2011)]

3) Semiconductor nanowire with strong spin-orbit coupling coupled to a SC
[Alicea, Rep. Prog. Phys. 75, 076501 (2012)]
[Exp: Mourik et. al, Science 336, 1003 (2012)]



$$H_0 = v s_y k_y \longrightarrow v s_y(-i\partial_y)$$

With mag. order

mag. order in x-direction

$$H_1 = v s_y k + m s_x \longrightarrow v s_y(-i\partial_z) + m(z) s_x$$

With mag. order + SC

$\Delta \in \mathbb{R}$

$$H_{BdG} = \begin{pmatrix} H_k & \Delta \\ \Delta^* & -s_y H^*_{-k} s_y \end{pmatrix} = \begin{pmatrix} v s_y k + m s_x & \Delta \\ \Delta^* & -v s_y k + m s_x \end{pmatrix}$$

$$= vk\, s_y \tau_z + m s_x + \Delta \tau_x \longrightarrow v(-i\partial_y) s_y \tau_z + m(y) s_x + \Delta(y)\tau_x$$

Looking for zero modes

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{pmatrix} 0 & m-ik & \Delta & 0 \\ m+ik & 0 & 0 & \Delta \\ \Delta & 0 & 0 & m+ik \\ 0 & \Delta & m-ik & 0 \end{pmatrix} \underbrace{\begin{pmatrix} u_\uparrow \\ u_\downarrow \\ v_\uparrow \\ v_\downarrow \end{pmatrix}}_{\Phi(\vec{r})} = 0 \qquad \cancel{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 0 & m_+ & \Delta & 0 \\ m_- & 0 & 0 & \Delta \\ \Delta & 0 & 0 & m_+ \\ 0 & \Delta & m_- & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =}$$

$$\cancel{U^+ H U = \begin{pmatrix} 0 & \begin{matrix} \Delta & m+ik \\ m+ik & \Delta \end{matrix} \\ & 0 \end{pmatrix} \quad \Delta + \sigma_x (m+ik)}$$

$$\begin{pmatrix} m-ik & \Delta \\ \Delta & m-ik \end{pmatrix} \underbrace{\begin{pmatrix} u_\downarrow \\ v_\uparrow \end{pmatrix}}_{\chi} = 0 \qquad \begin{pmatrix} m+ik & \Delta \\ \Delta & m+ik \end{pmatrix} \underbrace{\begin{pmatrix} u_\uparrow \\ v_\downarrow \end{pmatrix}}_{\chi} = 0$$

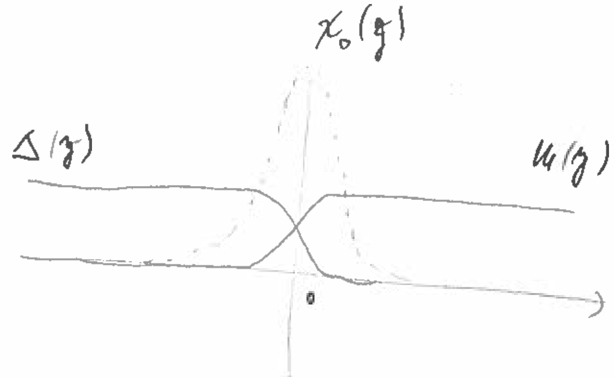
$$\Psi = e^{i\frac{\pi}{4}\sigma_y} \chi$$

$$\left[(m+ik)\mathbb{1} + \sigma_x \Delta\right] \chi = 0 \qquad e^{i\frac{\pi}{4}\sigma_y} \sigma_x e^{-i\frac{\pi}{4}\sigma_y} = \sigma_z$$

$$\to \left[(m+ik)\mathbb{1} + \sigma_z \Delta\right] \chi = 0 \qquad e^{i\frac{\pi}{4}\sigma_y} = \cos\frac{\pi}{4} + i\sigma_y \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}(1 + i\sigma_y) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \cdot & \cdot \end{pmatrix}$$

$$ik \to i(-i\partial_y) = \partial_y$$



$$[\partial_y + m(y) + \Delta(y)]\,\tilde{\chi}_1 = 0$$

$$[\partial_y + m(y) - \Delta(y)]\,\tilde{\chi}_2 = 0$$

$$\boxed{\tilde{\chi}_2(y) = \chi_0(y) \equiv A\, e^{-\int_0^y dy' [m(y') - \Delta(y')]}}$$

– only normalizable solution!

- Zero-mode w.f.

$$\Psi(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_0(y) \\ 0 \\ 0 \\ -\chi_0(y) \end{pmatrix} = \frac{\chi_0(z)}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

- Field operator for the zero mode

$$\hat{\psi}_0 = \int dz\, (c_\uparrow, c_\downarrow, c_\downarrow^{+}, -c_\uparrow^{+}) \cdot \Psi(z)$$

$$= \frac{1}{\sqrt{2}} \int dz\, \chi_0(z) \left[c_\uparrow(z) + c_\uparrow^{+}(z) \right]$$

$$\hat{\psi}_0^{+} = \psi_0$$ Majorana fermion.

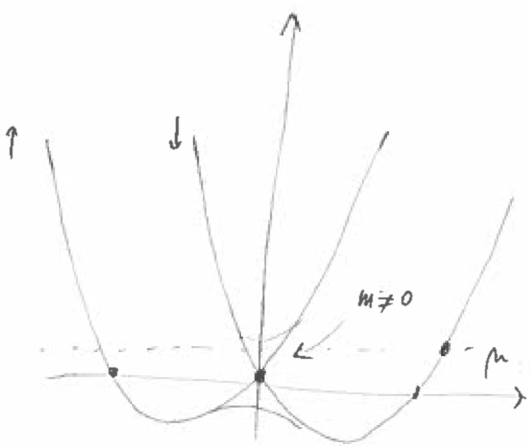
(3) Semicond. nanowires with SOC. (InSb)

$$H = \frac{\hbar^2 k^2}{2m} + \alpha k S_x + m S_z$$

kin. energy — SOC — Zeeman mag. field. $m = \mu_B B_z$

$\underline{m = 0}$

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} \pm \alpha |k|$$



- odd # of Fermi points when μ is inside the Zeeman gap
- expect topological SC.

Show Delft pictures.