$\mathcal{H}_k^{BdG}$ has double degrees of freedom compared to h_k ("BdG doubling") $\Rightarrow$ only half of them are physical

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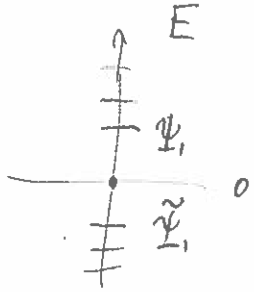
(1) + (2) $\Rightarrow$ take only positive-energy eigenstates.



Zero-modes are interesting

(i) if there is a single zero mode then it is protected, $\Psi_k = \tau_y \Psi_k^*$

(ii) Only "a half" of this mode is physical it is a <u>Majorana fermion</u>.

LECTURE 19

<u>Majorana Fermions</u>

For a general review see F. Wilczek, Nature Physics $\underline{5}$, 614 (200.

$$\{c_i, c_j^+\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^+, c_j^+\} = 0$$

Write

$$c_j = \tfrac{1}{2}(\gamma_{j1} + i\gamma_{j2}) \qquad \boxed{\gamma_{j\alpha}^+ = \gamma_{j\alpha}} \quad \text{(self-conjugate)}$$

$$\gamma_{j1} = (c_j + c_j^+)$$

$$\gamma_{j2} = \frac{1}{i}(c_j - c_j^+)$$

$$\{\gamma_{j1}, \gamma_{k1}\} = \{c_j + c_j^\dagger, c_k + c_k^\dagger\} = 2\delta_{jk}$$

$$\{\gamma_{j2}, \gamma_{k2}\} = 2\delta_{jk}$$

$$\{\gamma_{j1}, \gamma_{k2}\} = \frac{1}{i}\{c_j + c_j^\dagger, c_k - c_k^\dagger\} = 0$$

$$\gamma_{k\alpha}^2 = 1 \qquad \gamma_{k\alpha}^\dagger = \gamma_{k\alpha}$$

Can't form a # operator:

$$\gamma_{k\alpha}^\dagger \gamma_{k\alpha} = \gamma_{k\alpha}\gamma_{k\alpha} = 1$$

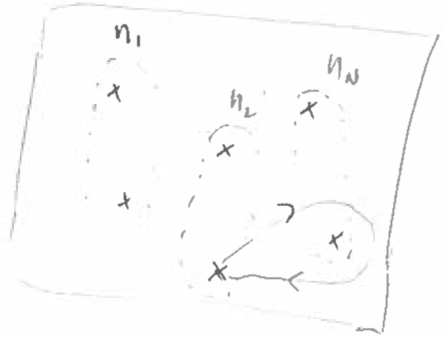
$$\begin{aligned} n_k = c_k^\dagger c_k &= \frac{1}{4}(\gamma_{k1} - i\gamma_{k2})(\gamma_{k1} + i\gamma_{k2}) \\ &= \frac{1}{4}\Big(\underbrace{\gamma_{k1}\gamma_{k1}}_{1} + i\gamma_{k1}\gamma_{k2} - i\gamma_{k2}\gamma_{k1} + \underbrace{\gamma_{k2}\gamma_{k2}}_{1}\Big) \\ &= \frac{1}{2}\left(1 + i\gamma_{k1}\gamma_{k2}\right) \end{aligned}$$

<u>Connection with quantum computation</u>: (1) encode a quantum bit (qubit) into an eigenvalue of $n_k = 0, 1$ with γ_{k1} and γ_{k2} spatially separated. To read the qubit one needs to coherently probe both Majoranas; environment cannot do this!

γ_{k1} $\qquad\qquad$ γ_{k2}

(2) For 2N Majoranas in 2D plane

the ground state has degeneracy 2^N



N pairs, $n_j = 0, 1$

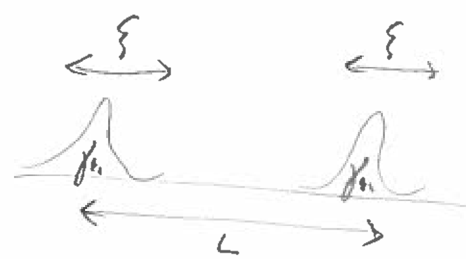
$$|\Psi_G\rangle = \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{pmatrix}$$

- One can perform unitary operations on $|\Psi_G\rangle$ by <u>braiding</u> MFs. $|\Psi_G\rangle \to \underset{N\times N \text{ matrix}}{U} |\Psi_G\rangle$

$\Rightarrow$ Quantum computer, topologically protected.

<u>But not universal.</u>

- $n_k = 0, 1$ states are degenerate when γ_{k1}, γ_{k2} are sufficiently far apart



$$H = \varepsilon n_k = \frac{\varepsilon}{2}(1 + i\gamma_{k1}\gamma_{k2})$$

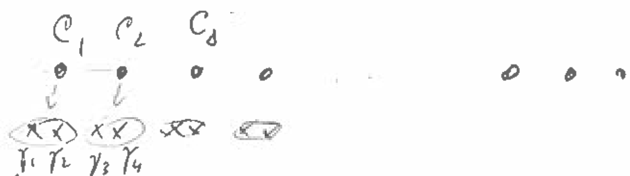
ε - overlap between MF wavefunctions

$\varepsilon \sim e^{-L/\xi} \to 0$ for large $L \gg \xi$

The Kitaev chain

[A. Kitaev, Usp. Fiz. Nauk Suppl. **171**, 131 (2001)]

1D chain with spinless fermions c_j



$$H = \sum_j \left[-t\,(c_j^\dagger c_{j+1} + h.c.) - \mu \left(c_j^\dagger c_j - \tfrac{1}{2}\right) + (\Delta c_j c_{j+1} + h.c.)\right]$$

hopping — chem pot. — "p-wave" SC

Consider $\Delta \in \mathbb{R}$ and L number of sites,

Write c_j in terms of MF

$$\gamma_{j1} = c_j + c_j^\dagger \qquad\Rightarrow\qquad c_j = \tfrac{1}{2}(\gamma_{j1} + i\gamma_{j2})$$

$$\gamma_{j2} = -ic_j + ic_j^\dagger \qquad\qquad c_j^\dagger = \tfrac{1}{2}(\gamma_{j1} - i\gamma_{j2})$$

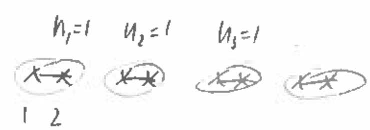
$$\gamma_k^\dagger = \gamma_k$$

Check!

$$H = \frac{i}{2}\sum_j \left[-\mu\, \gamma_{j1}\gamma_{j2} + (t+\Delta)\,\gamma_{j2}\gamma_{j+1,1} + (-t+\Delta)\,\gamma_{j1}\gamma_{j+1,2}\right]$$

- Study 2 special cases

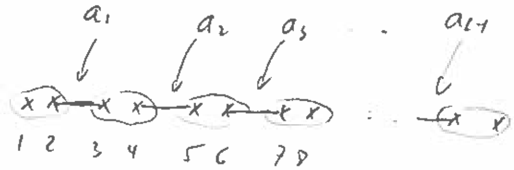
a) $\Delta = t = 0, \quad \mu < 0$



$$H = \frac{i}{2}(-\mu)\sum_j \gamma_{j1}\gamma_{j2} = -\mu \sum_j \left(c_j^\dagger c_j - \tfrac{1}{2}\right)$$

- trivial phase with original fermions localized on the individual sites of the lattice.

b) $\Delta = t > 0$, $\mu = 0$



$$H = it \sum_{j=1}^{L-1} \gamma_{j2}\gamma_{j+1,1}$$

$\boxed{\gamma_1 \text{ and } \gamma_{2N}}$ do not enter H $\Rightarrow$ they represent a zero-energy, spatially separated Majorana fermions.

Solve by defining new fermion operators:

$$a_j = \frac{1}{2}\left(\gamma_{j2} + i\gamma_{j+1,1}\right)$$

$$H = 2t \sum_{j=1}^{L-1} \left(a_j^+ a_j - \tfrac{1}{2}\right)$$

GAPPED TOPOLOGICAL PHASE supporting isolated Majorana end-modes.

GS: $n_j = 0$ $\forall j = 1 \ldots L-1$

Excitations $n_j = 1$ for some j, cost energy $2t$

- Away from special case (b) Majorana wavefunctions extend into the bulk:

$$\xi \approx \frac{t}{\Delta} \qquad \text{for } \Delta \ll t$$

- Bulk Hamiltonian (infinite chain, or finite ring)

$$c_q = \frac{1}{\sqrt{L}} \sum_j e^{iqj} c_j \,, \qquad q \in (-\pi, \pi)$$

$$H = \sum_q (-2t\cos q - \mu)\, c_q^+ c_q + \Delta \sum_q \left(i \sin q \, c_q c_{-q} + h.c.\right)$$

$$H = (c_q^+, c_{-q}) \begin{pmatrix} h_q & \Delta_q^* \\ \Delta_q & -h_{-q} \end{pmatrix} \begin{pmatrix} c_q \\ c_{-q}^+ \end{pmatrix}$$

$\Delta_q = 2i\Delta \sin q$

$h_q = (-2t\cos q - \mu)$

$$E(q) = \pm\sqrt{(2t\cos q + \mu)^2 + 4\Delta^2 \sin^2 q}$$

↑ spectrum of excitations.

Phase diagram:

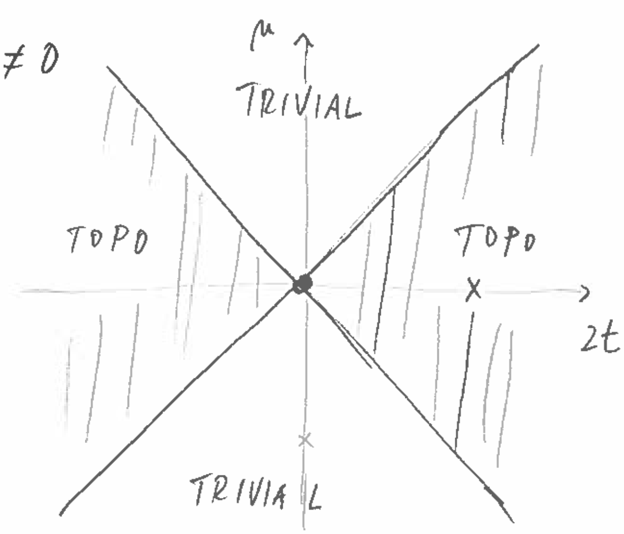
Topological: $\Delta = t > 0$, $\mu = 0$

Trivial: $\Delta = t = 0$, $\mu < 0$

Phase transition: when the bulk gap closes, $E(q) = 0$ for some q.

$$E(q) = 0 \quad \text{when} \quad \begin{cases} \sin^2 q = 0 & \text{i.e.} \quad q = 0, \pi \\ (2t\cos q + \mu) = 0, & \text{i.e.} \quad 2t = \pm\mu \end{cases}$$

$|\Delta| \neq 0$



Topological phase in the Kitaev chain occurs when $|\Delta| > 0$ and $|2t| > |\mu|$

$$\sum_j c_j c_{j+1} = \frac{1}{L} \sum_{q,q'} c_q c_{q'} \sum_j e^{-iqj} e^{-iq'(j+1)}$$

$$= \sum_{q,q'} c_q c_{q'} e^{-iq'} \delta_{q,-q'} = \sum_q c_q c_{-q} e^{iq}$$

$$= \frac{1}{2} \sum_q \left(c_q c_{-q} e^{iq} + c_{-q} c_q e^{-iq} \right)$$

$$= \frac{1}{2} \sum_q c_q c_{-q} \left(e^{iq} - e^{-iq} \right)$$

$$= i \sum_q c_q c_{-q} \sin q$$

$$\frac{i}{2}(-\mu) \sum_j \gamma_{j1} \gamma_{j2} = \frac{i}{2}(-\mu) \frac{1}{2N} \sum_{q,q'} \gamma_{q1} \gamma_{q'2} \sum_j e^{i(q+q')j}$$

$$= \frac{i}{4}(-\mu) \sum_q \gamma_{q1} \gamma_{-q2} \quad \checkmark$$