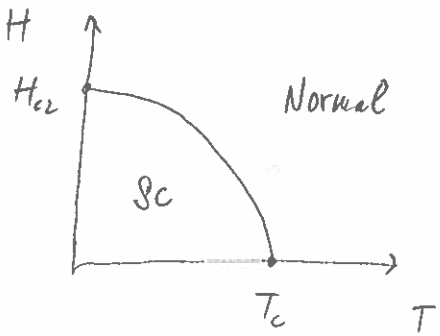
Lecture 18

TOPOLOGICAL SUPERCONDUCTORS, MAJORANA FERMIONS

Elliott & Franz, Rev. Mod. Phys. 87, 137 (2015)

I. Superconductivity: brief review



SC:
- zero resistance
- Meissner effect

$T_c \sim mK - 100K$

$H_{c2} \sim mT - 100T$

BCS Theory

- formation of Cooper pairs due to attractive interaction between electrons (phonons in low-T_c SC, unknown in high-T_c)
- BCS ground state wf:

$$|\Psi_{BCS}\rangle = \prod_{\vec{k}} \left(u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |0\rangle$$

u_k, v_k - BCS coherence factors

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{\xi_k^2}{E_k^2}} \qquad E_k^2 = \xi_k^2 + \Delta_k^2$$

$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$ - normal state energy dispersion

Δ_k - BCS gap function

E_k - BCS excitation spectrum

Bogolubov - de Gennes theory

- generalization of BCS to $T \neq 0$ and non-uniform situations.

$$H = \int d^3r \left\{ \sum_\sigma c_\sigma^+(\vec r) \overbrace{\left(-\frac{\hbar^2 \nabla^2}{2m} - \mu \right)}^{\hat h_0} c_\sigma(\vec r) - V n_\uparrow(\vec r) n_\downarrow(\vec r) \right\}$$

$n_\sigma(\vec r) = c_\sigma^+(\vec r) c_\sigma(\vec r)$ # operator

$\uparrow$ Hubbard interaction term.

$V > 0$ attractive interaction

- Bogolubov MF decoupling:

$$-V n_\uparrow n_\downarrow = -V c_\uparrow^+ c_\uparrow c_\downarrow^+ c_\downarrow = +V c_\uparrow^+ c_\downarrow^+ c_\uparrow c_\downarrow$$

$$\simeq +V \left[\langle c_\uparrow^+ c_\downarrow^+ \rangle c_\uparrow c_\downarrow + c_\uparrow^+ c_\downarrow^+ \langle c_\uparrow c_\downarrow \rangle - \langle \ \rangle \langle \ \rangle \right]$$

Define SC order parameter:

$$\Delta(\vec r) = V \langle c_\uparrow(\vec r) c_\downarrow(\vec r) \rangle$$

$$\boxed{H_{MF} = \int d^3r \left\{ \sum_\sigma c_\sigma^+(\vec r) \hat h_0 c_\sigma(\vec r) + \left[\Delta(\vec r) c_\uparrow^+(\vec r) c_\downarrow^+(\vec r) + h.c. \right] - \frac{1}{V} |\Delta(\vec r)|^2 \right\}}$$

To diagonalize H_{MF} employ Bogolubov transformation

$$c_\uparrow(\vec r) = \sum_n \left[u_n(\vec r) a_{n\uparrow} - v_n^*(\vec r) a_{n\downarrow}^+ \right]$$

$$c_\downarrow(\vec r) = \sum_n \left[u_n(\vec r) a_{n\downarrow} + v_n^*(\vec r) a_{n\uparrow}^+ \right]$$

$$\{ a_{n\sigma}, a_{m\lambda}^+ \} = \delta_{nm} \delta_{\sigma\lambda}$$

One can show that

$$\left[H_{MF} = \underset{\substack{\uparrow \\ \text{ground state energy}}}{E_g} + \sum_{n\sigma} \underset{\substack{\nwarrow \\ \text{excitations}}}{E_n a^+_{n\sigma} a_{n\sigma}} \right]$$

(BdG)

and u_n, v_n satisfy the Bogoliubov-de Gennes eqs:

$$\underbrace{\begin{pmatrix} h_0 & \Delta(\vec{r}) \\ \Delta^*(\vec{r}) & -h_0^* \end{pmatrix}}_{\text{BdG Hamiltonian}} \begin{pmatrix} u_n(\vec{r}) \\ v_n(\vec{r}) \end{pmatrix} = E_n \begin{pmatrix} u_n(\vec{r}) \\ v_n(\vec{r}) \end{pmatrix}$$

- Uniform case: $\Delta(\vec{r}) = \Delta_0 \in \mathbb{R}$

$\rightarrow$ Fourier transf $\quad u(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} u_k$

$$\begin{pmatrix} h_k & \Delta \\ \Delta & -h_k^* \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$$h_k = \frac{\hbar^2 k^2}{2m} - \mu \qquad h^*_{-k} = h_k$$

$$\begin{pmatrix} h_k & \Delta \\ \Delta & -h_k \end{pmatrix} \Rightarrow E_k = \pm\sqrt{h_k^2 + \Delta^2} \qquad \text{BCS spectrum}$$

LECTURE 18 cont'd

In the presence of SOC the BdG Hamiltonian takes the form

$$\mathcal{H}_{BdG} = \int d^d r = \left[\hat{\Psi}^{+}(\vec{r})\, H_{BdG}\, \hat{\Psi}(\vec{r}) - \frac{1}{V} |\Delta(\vec{r})|^2 \right]$$

$$H_{BdG} = \begin{pmatrix} \hat{h}(\vec{r}) & \hat{\Delta}(\vec{r}) \\ \hat{\Delta}^*(\vec{r}) & -\sigma^y \hat{h}^*(\vec{r}) \sigma^y \end{pmatrix}, \qquad \hat{\Psi}(\vec{r}) = \begin{pmatrix} c_\uparrow(\vec{r}) \\ c_\downarrow(\vec{r}) \\ c_\downarrow^+(\vec{r}) \\ -c_\uparrow^+(\vec{r}) \end{pmatrix} \equiv \begin{pmatrix} \hat{\psi}(\vec{r}) \\ i\sigma^y \hat{\psi}^+(\vec{r}) \end{pmatrix}$$

$\hat{h}(\vec{r})$ – 2×2 matrix in spin space

$\hat{\Delta}(\vec{r})$ – $\begin{pmatrix} \Delta(\vec{r}) & 0 \\ 0 & \Delta(\vec{r}) \end{pmatrix}$

H_{BdG} – 4×4 matrix in combined spin & Nambu space (particle-hole)

Important: in constructing $\mathcal{H}_{BdG}$ we have DOUBLED the total number of degrees of freedom by going from (2×2) to (4×4) matrices.

- Once again, this problem is solved by finding eigenstates $\Phi_n(\vec{r})$ of H_{BdG}:

$$H_{BdG}\, \Phi_n(\vec{r}) = E_n\, \Phi_n(\vec{r}), \qquad \Phi_n(\vec{r}) = \begin{pmatrix} u_{n\uparrow}(\vec{r}) \\ u_{n\downarrow}(\vec{r}) \\ v_{n\uparrow}(\vec{r}) \\ v_{n\downarrow}(\vec{r}) \end{pmatrix}$$

This brings $\mathcal{H}_{BdG}$ into diagonal form

$$\mathcal{H}_{BdG} = \sum_n^{(1)} E_n a_n^+ a_n + E_g \qquad \leftarrow \text{sum over } E_n > 0 \text{ modes}$$

$$a_n = \int d^d r \; \Phi_n^+(\vec r)\, \hat\Psi(\vec r)$$

$$= \int d^d r \left[u_{n\uparrow}^*(\vec r)\, c_\uparrow(\vec r) + u_{n\downarrow}^* c_\downarrow + v_{n\uparrow}^* c_\downarrow^+ - v_{n\downarrow}^* c_\uparrow^+ \right]$$

a_n — quasiparticle excitation operators.

- Particle-hole (Charge conjugation) symmetry

(H_{BdG} is invariant under $\mathcal{C} H_{BdG} \mathcal{C}^{-1} = -H_{BdG}$ with $\mathcal{C} = \tau_y \sigma_y K$ (Check!))

$$\mathcal{C} = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} K \equiv CK \qquad C = \tau_y \sigma_y \quad \text{Charge conjug. matrix}$$

$$\boxed{C H_{BdG}^* C = - H_{BdG}}$$

$$\boxed{\mathcal{C}^2 = +1 \quad \begin{matrix} \text{BDI} \\ \text{D} \\ \text{DIII} \end{matrix}}$$

classes in periodic table

- Consequently, for each eigenstate Φ at energy E there exists an eigenstate $\tilde\Phi = C\Phi^*$ at energy $-E$.

PROOF:
$$H_{BdG} \overset{\mathcal{C}^{-1}\mathcal{C}}{\,} \Phi = E\Phi \quad / \mathcal{C}$$

$$\left.\underbrace{(\mathcal{C} H_{BdG} \mathcal{C}^{-1})}_{-H_{BdG}} \mathcal{C}\Phi = E \mathcal{C}\Phi \right\} \Rightarrow H_{BdG} \mathcal{C}\Phi = -E\,\mathcal{C}\Phi$$

- MAJORANA ZERO MODES

An interesting situation arises if there exists a single, isolated zero-energy solution,

$$H_{BdG}\, \Phi_0 = 0 .$$

According to the preceding discussion, only $\frac{1}{2}$ of this solution is physical! Also, it must be self-conjugate under $\mathcal{C}$:

$$\Phi_0 = \tau_y \sigma_y \Phi_0^* \quad \Rightarrow \quad \begin{pmatrix} u_{0\uparrow} \\ u_{0\downarrow} \\ v_{0\uparrow} \\ v_{0\downarrow} \end{pmatrix} = \begin{pmatrix} -v_{0\downarrow}^* \\ v_{0\uparrow}^* \\ u_{0\downarrow}^* \\ -u_{0\uparrow}^* \end{pmatrix}$$

$\Phi_0(\vec{r})$ has only 2 independent (complex) components, say $(u_{0\uparrow}, u_{0\downarrow})$

- Consider the zero mode operator:

$$a_0 = \int d^d r\, \Phi_0^\dagger(\vec{r})\, \hat{\Psi}(\vec{r}) =$$

$$= \int d^d r \left[u_{0\uparrow}^* c_\uparrow + u_{0\downarrow}^* c_\downarrow + u_{0\downarrow} c_\downarrow^\dagger + u_{0\uparrow} c_\uparrow^\dagger \right]$$

This means: $\boxed{a_0^\dagger = a_0}$ Particle = antiparticle

"Majorana fermion"

[E. Majorana, 1937]