# LECTURE 17

## Axion electrodynamics in 3D TIs

Electromagnetic action in vacuum:

$$S_{EM} = \frac{1}{8\pi}\int dt\, d^3x \left(\varepsilon_0 \vec{E}^2 - \frac{1}{\mu_0}\vec{B}^2\right)$$

$$= \frac{1}{8\pi}\int dt\, d^3x \;\; F^{\mu\nu}F_{\mu\nu}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EM field strength tensor

$\varepsilon_0 = \mu_0 = c = 1$

### Maxwell's Equations:

$$j^\mu = \frac{\delta S_{EM}}{\delta A^\mu} = \frac{1}{4\pi}\partial_\mu F^{\mu\nu} \qquad j^\mu = (\rho, \vec{j}\,)$$

4-current density

Dual field strength: $\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \Rightarrow \quad \begin{aligned}&\vec{\nabla}\cdot\vec{B} = 0\\ &\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0\end{aligned}}$$

$$\boxed{\begin{aligned}&\vec{\nabla}\cdot\vec{E} = 4\pi\rho\\ &\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}\end{aligned}}$$

EM response of an insulator

action describing the insulator

$$Z = \int \mathcal{D}[\bar{\Psi},\Psi] \int \mathcal{D}A_\mu \, e^{-iS_I[\bar{\Psi},\Psi;A_\mu] - iS_{EM}[A_\mu]} \qquad (\hbar = 1)$$

Effective EM action

$$e^{-iS^{\text{eff}}_{EM}[A_\mu]} = \int \mathcal{D}[\bar{\Psi},\Psi] \, e^{-iS_I[\bar{\Psi},\Psi;A_\mu] - iS_{EM}[A_\mu]}$$

"Ordinary insulators"

$$\boxed{S^{\text{eff}}_{EM} = \frac{1}{8\pi} \int dt\, d^3x \left( \varepsilon \vec{E}^2 - \frac{1}{\mu} \vec{B}^2 \right)}$$

$\varepsilon, \mu$ - describe dielectric, paramagnetic response of the insulator.

The Coulomb field of a test charge Q is

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon r}$$

$$= \frac{\theta e^2}{8\pi h} \int dt\, d^3x \, \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$$

"Axion" term:

$$\boxed{S_\theta = \frac{\theta e^2}{2\pi h} \int dt\, d^3x \, \vec{E} \cdot \vec{B}}$$

- in principle allowed
- studied in high energy physics "strong CP violation"

but $\theta$ consistent with [illegible]

Symmetries: How do $\vec{E}, \vec{B}$ transform under $\mathcal{T}, \mathcal{P}$?

Cont. equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$

Expect $\rho$ is even under both $\mathcal{T}$ & $\mathcal{P}$.

$\Rightarrow$ $\vec{j}$ is odd under both $\mathcal{T}$ & $\mathcal{P}$.

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ :

$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$.

$\mathcal{T}$: $\vec{E} \to \vec{E}$, $\mathcal{P}$: $\vec{E} \to -\vec{E}$

$\mathcal{T}$: $\vec{B} \to -\vec{B}$, $\mathcal{P}$: $\vec{B} \to \vec{B}$

$\Rightarrow$ $\vec{E}^2, \vec{B}^2$ respect $\mathcal{T}$ & $\mathcal{P}$

$\vec{E} \cdot \vec{B}$ break both $\mathcal{T}$ and $\mathcal{P}$.

- It would thus appear that $\theta = 0$ in a system that is $\mathcal{T}$ or $\mathcal{P}$ invariant.

However: It turns out that the amplitude

$e^{iS_\theta}$ is invariant under $\theta \to \theta + 2\pi$

because integral $\frac{e^2}{2\pi\hbar}\int dt\, d^3x\, \vec{E} \cdot \vec{B} = n \in \mathbb{Z}$

on any compact space-time manifold.

Proof: Nakahara: "Geometry, topology and physics" (Bristol, 1990)

Vazifeh & Franz, PRB $\underline{82}$, 233103 (2010) ← elementary proof relies on:

$$S_\theta = \frac{\theta e^2}{4\pi h} \int dt\, d^3x\, \varepsilon^{\alpha\beta\mu\nu} \partial_\alpha \left[A_\beta \partial_\mu A_\nu\right]$$

↑ surface term.

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Under $\mathcal{T}$ or $\mathcal{P}$, $\theta \to -\theta$. Because of $\theta \to \theta + 2\pi$ invariance

$$\Longrightarrow \boxed{\theta = 0, \pi}$$

are allowed in $\mathcal{T}$ or $\mathcal{P}$ invariant system.

$$\theta = \begin{cases} 0, & \text{trivial insulator} \\ \pi, & \text{strong topological insulator } (\nu_0 = 1) \end{cases}$$

$$\boxed{\theta = \pi \nu_0} \qquad \nu_0 = \text{strong } \mathbb{Z}_2 \text{ index}$$

- Qi, Hughes, Zhang PRB $\underline{78}$, 195424 (2008), see also Bernevig textbook $\underline{\text{ch 14}}$
- Essin, Moore & Vanderbilt, PRL $\underline{102}$, 146805 (2009).

They also showed

$$\theta = \frac{1}{2\pi} \int_{BZ} d^3k \; \varepsilon_{ijk} \, \mathrm{Tr}\left[ \mathcal{A}_i \partial_j \mathcal{A}_k - \frac{2i}{3} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k \right] \bmod 2\pi$$

$$\mathcal{A}_j^{\mu\nu} = i \langle u_\mu | \frac{\partial}{\partial k_j} | u_\nu \rangle , \quad \mu, \nu \in \text{occupied bands}$$

"Non-Abelian gauge connection" for a 3D band insulator

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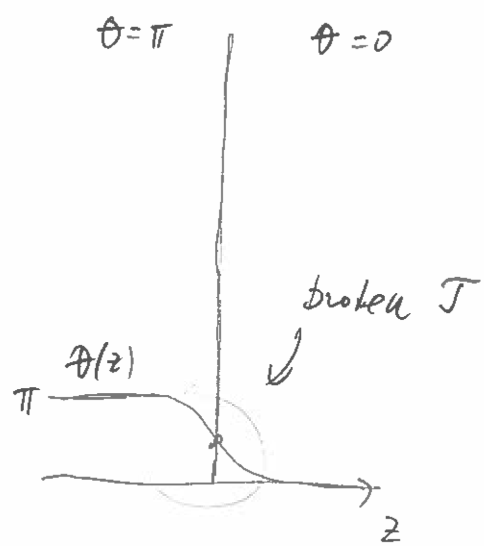
Physical consequences

- Consider boundary between TI ($\theta = \pi$) and vacuum ($\theta = 0$)

$$S_\theta = \frac{e^2}{4\pi h} \int dt \, d^3x \; \theta(z) \, \varepsilon^{\alpha\beta\mu\nu} \partial_\alpha A_\beta \partial_\mu A_\nu$$

$$= -\frac{e^2}{4\pi h} \varepsilon^{3\beta\mu\nu} \int dt \, dx dy dz \; \partial_z \theta(z) \; A_\beta \partial_\mu A_\nu \qquad \alpha = 3$$




Choose the gauge such that $A_\mu$ indep. of $z$

$$\boxed{S_\theta = \frac{e^2}{4h} \varepsilon^{3\beta\mu\nu} \int dt \, dx dy \; A_\beta \partial_\mu A_\nu}$$

"Chern-Simmons action describing a 2D integer Quantum Hall state with $\sigma_{xy} = \frac{e^2}{2h}$

$$j^\mu = \frac{\delta S^{\text{eff}}}{\delta A^\mu}$$

Axion-modified Maxwell eqs.:

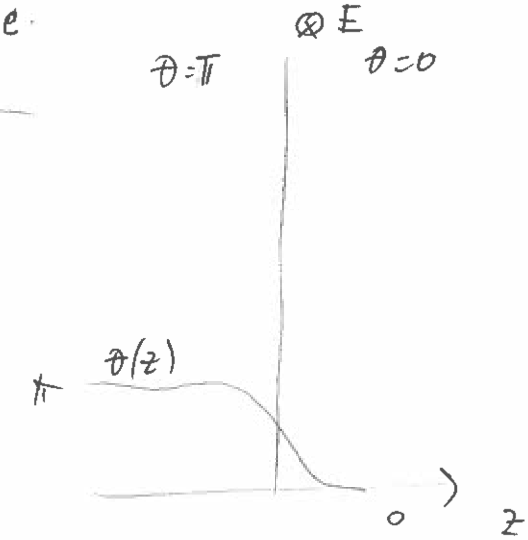
$$\left[\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho - \underline{\frac{e^2}{2\pi h}\vec{\nabla}\theta\cdot\vec{B}} \\ \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= \vec{j} + \underline{\frac{e^2}{2\pi h}\left(\vec{\nabla}\theta\times\vec{E} + \frac{\partial\theta}{\partial t}\vec{B}\right)} \end{aligned}\right]$$

- Boundary between TI and vac.

$$\vec{j} = -\frac{e^2}{2\pi h}\vec{\nabla}\theta\times\vec{E}$$

$\vec{j}$ – in plane

$\vec{\nabla}\theta = \hat{z}\frac{\partial\theta}{\partial z}$




$$\vec{j}_{2D} = \int_{-\infty}^{\infty} dz\,\vec{j}$$

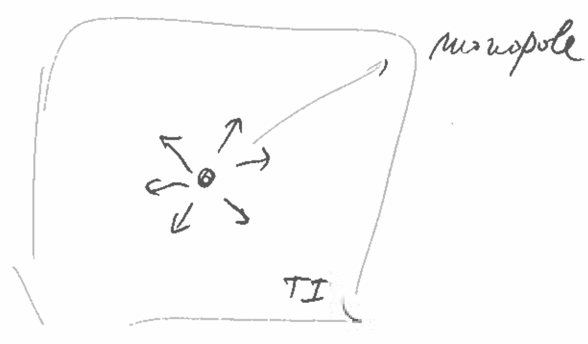
$$= -\frac{e^2}{2\pi h}\underbrace{\int dz\frac{\partial\theta}{\partial z}}_{-\pi}\hat{z}\times\vec{E}$$

$$\boxed{\vec{j}_{2D} = \frac{e^2}{2h}\left(\hat{z}\times\vec{E}\right)}$$

$\leftarrow$ QHE in the surface with $\sigma_{xy} = \frac{e^2}{2h}$ ✓

- Witten effect [Phys. Lett. B 86, 283 (1979)]

$$\vec{\nabla}\cdot\vec{B} = \Phi_0\,\delta(\vec{r})$$




Consider $\theta(t)$: $0 \to \pi$

$$\vec{\nabla}\theta = \ldots = 0$$

$$\vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} = -\frac{e^2}{2\pi h} \frac{\partial \theta}{\partial t} \vec{\nabla} \cdot \vec{B} \qquad / \int dt$$

$$\vec{\nabla} \cdot \vec{E} = -\frac{e^2}{2\pi h} (\pi) \, \Phi_0 \, \delta(\vec{r}) = -\frac{e^2}{2h} \frac{h}{e} \delta(\vec{r})$$

$$\boxed{\vec{\nabla} \cdot \vec{E} = -\frac{e}{2} \delta(\vec{r})}$$

$\Rightarrow$ Charge $Q = -\frac{e}{2}$ bound to the monopole!