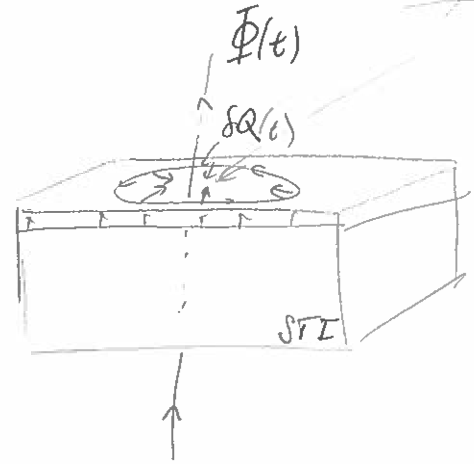
- Flux tube insertion thought experiment, the "wormhole effect"

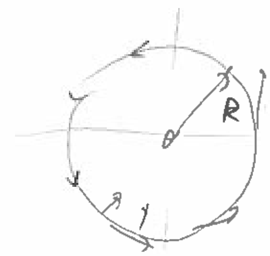
[Rosenberg, Guo, Franz, PRB 82, 041104R (2010)]



Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \int d\vec{S}$$

$$\boxed{\text{Hall current response} \quad \vec{j} = \sigma_{xy}\,(\vec{E} \times \hat{z})}$$



$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial \Phi}{\partial t} \Rightarrow$$

$$= 2\pi R E = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

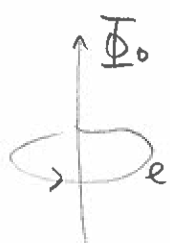
$$j = \sigma_{xy}\frac{1}{2\pi R}\left(-\frac{1}{c}\frac{\partial \Phi}{\partial t}\right)$$

$$\sigma_{xy} = \frac{e^2}{2h}, \quad \Phi_0 = \frac{hc}{e}$$

$$\boxed{\delta Q = \int_0^T dt\; 2\pi R\, j = \sigma_{xy}\frac{\delta\Phi}{c}}$$

Chose $\delta\Phi = \Phi_0$

$$\boxed{\delta Q = \frac{e^2}{2h}\,\frac{hc}{ec} = \frac{1}{2}e}$$



Phase 2π

$e^{i2\pi} = 1$

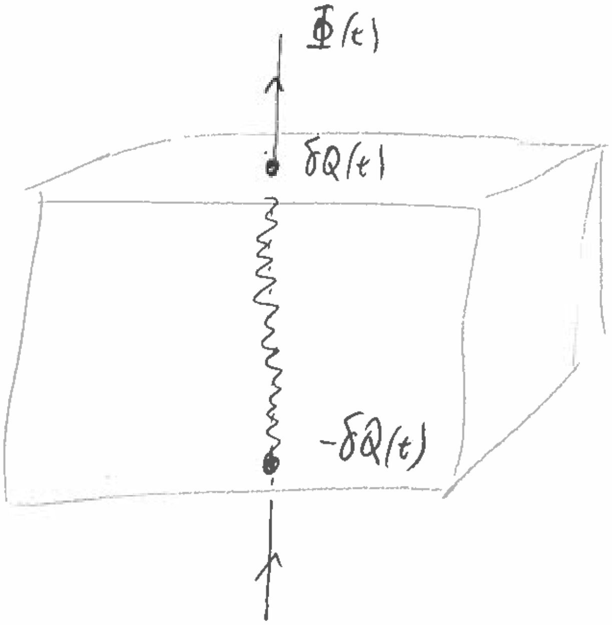
However, a flux tube with Φ_0 flux can be removed form the Hamiltonian by a gauge transf.

This leads to a paradox:

: a state with charge $\pm \frac{1}{2} e$ should be an eigenstate of a non-interacting Hamiltonian.

Solution:

"Wormhole effect"

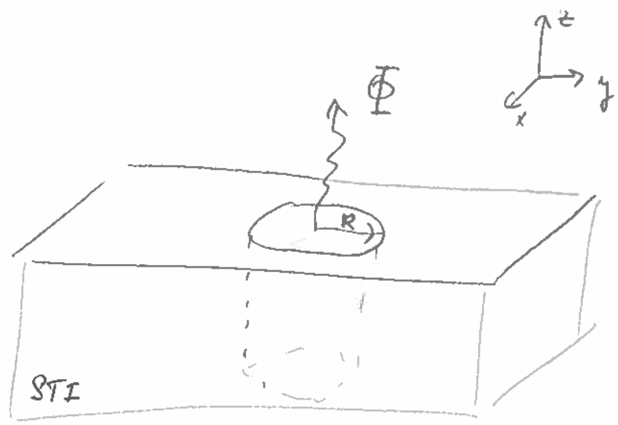


when $\Phi(t) = \frac{1}{2}\Phi_0$ there is a gapless mode formed along the tube inside STI.

The charge $\delta Q(t)$ escapes along the wormhole to the other side.

- Image monopole effect [Qi, Li, Zhang & Zhang, Science 323, 1184 (2009)]

Wormhole effect calculation



According to Ostrovsky, Gornyi & Mirlin [PRL 105, 036803 (2010), Supplement] surface states of a STI in a curved geometry are governed by

$$\mathcal{H} = \frac{i}{2}\left[\vec{s}\cdot\vec{p},\ \vec{s}\cdot\hat{n}\right] = \frac{\vec{\nabla}\cdot\hat{n}}{2} + \frac{1}{2}\left\{\hat{n}\cdot(\vec{p}\times\vec{s}) + (\vec{p}\times\vec{s})\cdot\hat{n}\right\}$$

where $\vec{p} = -i\vec{\nabla}$ and $\hat{n}$ is a surface normal unit vector.

- For cylindrical hole

$$\hat{n} = -\left(\frac{x}{R},\ \frac{y}{R},\ 0\right),$$

- $(\vec{p}\times\vec{s})\,\hat{n} = -(\vec{s}\times\vec{p})\cdot\hat{n} = -\vec{s}\cdot(\vec{p}\times\hat{n})$

$$\vec{\nabla}\times\hat{n} = 0, \quad \vec{\nabla}\cdot\hat{n} = -\frac{1}{R}$$

$$\boxed{\mathcal{H} = -\frac{1}{2R} + \vec{s}\cdot(\hat{n}\times\vec{p})}$$ For cylindrical hole

Minimal substitution to include mag. flux

$$\vec{p} \to \vec{\Pi} = \vec{p} - \frac{e}{c}\vec{A} \qquad \vec{A} = \frac{\hat{z}\times\vec{r}}{2\pi r}\,\gamma\Phi_0 = (-y, x, 0)\frac{\gamma\Phi_0}{2\pi r^2}$$

γ - flux in units of flux q.

Transform to cylindrical coordinates

$$x = r\cos\varphi \qquad y = r\sin\varphi \qquad z = z$$

$$\hat{n} = -(\cos\varphi, \sin\varphi, 0)$$

$$\hat{n} \times \hat{\Pi} = - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\varphi & \sin\varphi & 0 \\ \Pi_x & \Pi_y & \Pi_z \end{vmatrix} = (-\hat{x}\sin\varphi + \hat{y}\cos\varphi)\Pi_z - \hat{z}(\cos\varphi\,\Pi_y - \sin\varphi\,\Pi_x)$$

$$\left[\mathcal{H} = -\frac{1}{2R} + (-s_x\sin\varphi + s_y\cos\varphi)(-i\partial_z) - s_z\left[\cos\varphi(-i\partial_y - A_y) - \sin\varphi(-i\partial_x - A_x)\right] \right]$$

$$\partial_x = \cos\varphi\,\partial_r - \sin\varphi\,\frac{1}{r}\partial_\varphi$$

$$\partial_y = \sin\varphi\,\partial_r + \cos\varphi\,\frac{1}{r}\partial_\varphi$$

$$\frac{e}{c}A_x = -\frac{\sin\varphi}{r}\,\eta \qquad \frac{e}{c}A_y = \frac{\cos\varphi}{r}\,\eta$$

- Assume plane wave solutions along z: $-i\partial_z \to k$

$$\mathcal{H}_k = -\frac{1}{2R} + \begin{pmatrix} \frac{1}{R}(i\partial_\varphi + \eta) & -ik\,e^{-i\varphi} \\ ik\,e^{i\varphi} & -\frac{1}{R}(i\partial_\varphi + \eta) \end{pmatrix}$$

$$\mathcal{H}_k \Psi_k(\varphi) = E_k \Psi_k(\varphi)$$

Assume solution

$$\Psi_k(\varphi) = e^{i\varphi\ell}\begin{pmatrix} f_k \\ e^{i\varphi} g_k \end{pmatrix} = e^{i\varphi\ell}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}\begin{pmatrix} f_k \\ g_k \end{pmatrix} \qquad \ell = 0, \pm 1, \pm 2

$$\tilde{\mathcal{H}}_k \tilde{\Psi}_k = E_k \tilde{\Psi}_k \qquad \tilde{\Psi}_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix}$$

$$\boxed{\tilde{\mathcal{H}}_k = s_y k - s_z \frac{1}{R}\left(l + \frac{1}{2} - \varepsilon\right)}$$

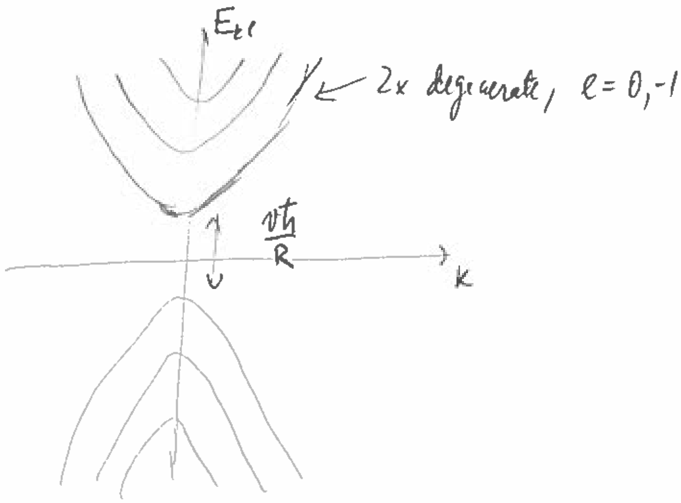
$l = 0, \pm 1, \pm 2 \ldots$

$\varepsilon = \frac{\Phi}{\Phi_0}$

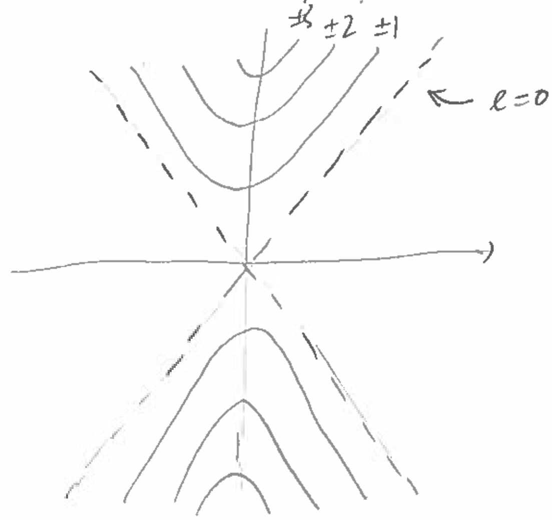
$$\tilde{\mathcal{H}}_{kl} = U_l^{-1} \mathcal{H}_k U_l$$

$$E_{kl} = \pm v\hbar \sqrt{k^2 + \frac{1}{R^2}\left(l + \frac{1}{2} - \varepsilon\right)^2} \qquad l = 0, \pm 1, \pm 2 \ldots$$

$\varepsilon = 0$ (zero flux)



$\varepsilon = \frac{1}{2}$ (half flux q.)



$$\boxed{\varepsilon = \frac{1}{2} : \quad E_{k0} = \pm v\hbar |k| \quad \text{indep. of } R}$$

$\to$ "Wormhole effect"

Back to the Flux insertion exp.:

slope consistent with $\sigma_{xy} = \frac{e^2}{2h}$