LECTURE 1

- **Exact quantization in solids**

- most quantities measurable in solids are non-quantized, e.g. resistivity, susceptibility, heat capacity, ...
depend on exact composition, T, P, impurity content etc.

Examples of EXACTLY QUANTIZED measurable quantities

1) Flux quantization in superconductors
2) Hall conductivity quantization in 2D Quantum Hall insulators.

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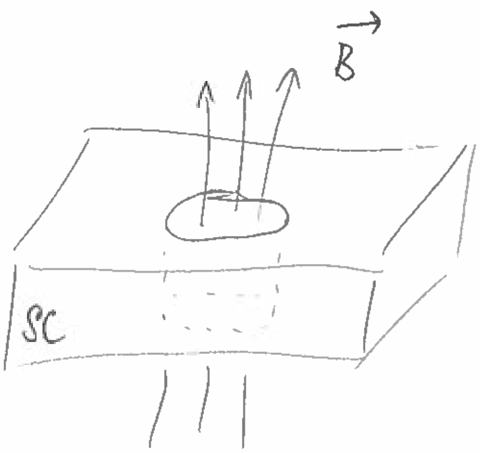
1 Flux quantization

below T_c

[$\rho = 0$, resistivity]

Meissner effect

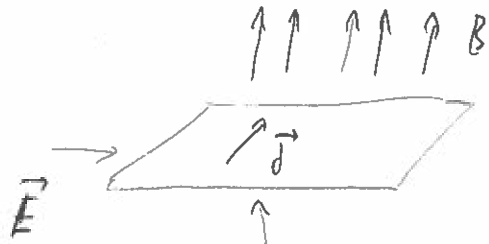


$$\Phi \equiv \int \vec{B} \cdot d\vec{S} = n \Phi_0$$

$$\Phi_0 = \frac{hc}{2e}, \quad n \in \mathbb{Z}$$

SC flux quantum

(2) Integer Quantum Hall effect (IQHE)

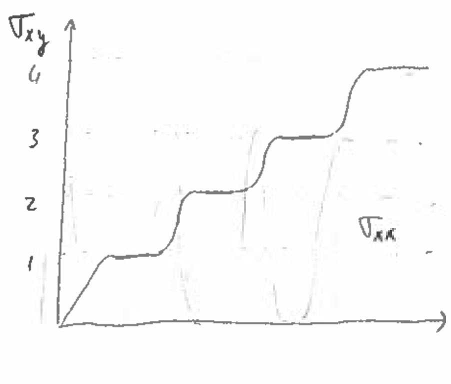


2D electron gas (2DEG)

$$\sigma_{xy} = \frac{j_y}{E_x} = n\frac{e^2}{h}, \quad n \in \mathbb{Z}$$

$$\sigma_{xx} = \frac{j_x}{E_x} = 0$$

$\frac{e^2}{h}$ – "quantum of conductance"



[Also: Fractional Quantum Hall effect, FQHE

$$\sigma_{xy} = p\frac{e^2}{h} \qquad p = \frac{n}{m} \qquad n, m \in \mathbb{Z}$$

e-e interactions play a key role]

- The most accurate known determination of fundamental physical constants (h and e) come from flux quantization and IQHE measurements

$$\frac{hc}{2e}, \quad \frac{e^2}{h}, \quad c$$

The root cause of exact quantization:

$\boxed{\text{TOPOLOGICAL ORDER}}$

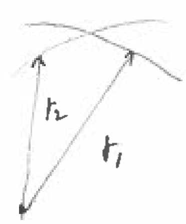
Def: "When certain physical properties of a system depend on global topology - and not on local details, such as disorder - then the system is said to realize a <u>topological phase</u>."

SC and IQHE are examples where a physical observable is proportional to an integer-valued TOPOLOGICAL INVARIANT.

- Geometrical example of a topological invariant.

<u>Gauss - Bonnet theorem</u>:

$$\int_M \kappa \, dA = 4\pi(1-g) \qquad \kappa = \frac{1}{r_1 r_2} \qquad \text{"Gaussian curvature"}$$



sphere
$g=0$

$$\int \frac{1}{R^2} dA = 4\pi R^2/R^2 = 4\pi \checkmark$$

Torus
$g=1$

$$\int \kappa \, dA = 0$$

g = # of holes (genus)

$\Rightarrow$ stable against arbitrary <u>smooth</u> deformations

$\frac{1}{r_1}$, $\frac{1}{r_2}$ "principal curvatures"

$r_1 = \min(r)$

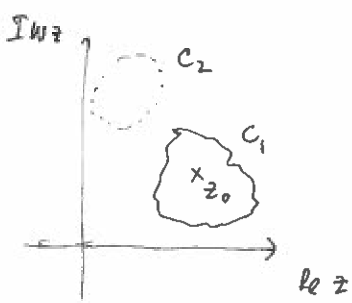
$r_2 = \max(r)$

• Example from complex analysis:

Residue theorem: $\oint_C f(z)\,dz = 2\pi i \sum_j \mathrm{Res}[f(z)]_{z=z_j}$, z_j - simple poles

Consider $f(z) = \frac{1}{z - z_0}$

$$N_C = \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0} = \begin{cases} 1, & \text{if } z_0 \text{ inside } C \\ 0, & \text{otherwise} \end{cases}$$



$N_{C_1} = 1$

$N_{C_2} = 0$ - independent of the shape of the contour

$\boxed{N_C \text{ is a "topological invariant"}}$

TOPOLOGICAL BAND THEORY

• Band theory of solids: electrons in a periodic potential, no interactions

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}) \qquad V(\vec{r}+\vec{R}) = V(\vec{r})$$

$\vec{R} \in$ Bravais lattice

Bloch's theorem: Eigenstates of $H(\vec{r})$ can be chosen as

$$|\psi_n(\vec{r},\vec{k})\rangle = e^{i\vec{k}\cdot\vec{r}}\,|u_n(\vec{r},\vec{k})\rangle$$ "Bloch state"

where $|u_n(\vec{r}+\vec{R},\vec{k})\rangle = |u_n(\vec{r},\vec{k})\rangle$

$\vec{k}$ - "crystal momentum" $\vec{k} \in$ 1st Brillouin zone

n - band index

Define: $H(\vec{k}) = e^{i\vec{k}\cdot\vec{r}}\,H\,e^{-i\vec{k}\cdot\vec{r}}$

"Bloch Equation"

$$\Rightarrow \quad H(\vec{k})\,|u_n(\vec{k})\rangle = E_n(\vec{k})\,|u_n(\vec{k})\rangle \qquad (1)$$

$E_n(\vec{k})$ defines the "band structure" of the solid

Lattice translational symmetry implies $H(\vec{k}+\vec{G}) = H(\vec{k})$ for all reciprocal lattice vectors $\vec{G}$, $[\vec{R}\cdot\vec{G} = 2\pi m,\ m \in \mathbb{Z}]$

$\Rightarrow$ $\vec{k}$ and $\vec{k}+\vec{G}$ are equivalent, and the domain of the crystal momentum $\vec{k}$ in d-dimensions is T^d (torus).