1. (15 points) Axion angle calculation.

The axion angle θ for a 3D topological insulator can be evaluated using a non-Abelian gauge connection but this technique is generally quite messy and except for very special cases must be carried out numerically. However, for a special class of Bloch Hamiltonians which can be expressed as

$$\mathcal{H}(\mathbf{k}) = \sum_{\alpha=0}^{3} d_\alpha(\mathbf{k})\gamma_\alpha, \tag{1}$$

where $d_\alpha(\mathbf{k})$ are real functions of $\mathbf{k}$ and γ_α are mutually anticommuting 4×4 hermitian matrices with unit square ($\gamma_\alpha^2 = 1$), one can show that the axion angle can be written as

$$\theta = -\frac{1}{2\pi}\int_{\mathrm{BZ}} d^3k\,\epsilon^{\alpha\beta\mu\nu}\frac{d_\alpha(\partial_1 d_\beta)(\partial_2 d_\mu)(\partial_3 d_\nu)}{d^4}, \tag{2}$$

where $\epsilon^{\alpha\beta\mu\nu}$ represents the totally antisymmetric tensor with $\epsilon^{0123} = 1$.

a) Consider a system described by Hamiltonian (1) at half filling with

$$d(\mathbf{k}) = (M_\mathbf{k}, \sin k_x, \sin k_y, \sin k_z),$$

and $M_\mathbf{k} = \epsilon - 2t\sum_i \cos k_i$. Find the spectrum of $\mathcal{H}(\mathbf{k})$ and for a constant positive t identify the possible phase transitions between the insulating phases as a function of ϵ. Note that there is no need to assume a specific representation for the γ matrices: the spectrum only depends on their anticommutation property $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and can be obtained by the usual procedure of squaring the Hamiltonian.

b) When t and ϵ are small one can argue that the integral in Eq. (2) is dominated by regions close to the 8 TRIM. Here, the Hamiltonian can be linearized in $\mathbf{k}$ and approximated by a simple Dirac form. Show that in this approximation each TRIM contributes

$$\theta^\ell = -\frac{\pi}{2}\mathrm{sgn}(v_x^\ell v_y^\ell v_z^\ell m_\ell)$$

to the total axion angle $\theta = \sum_{\ell=1}^{8}\theta^\ell \mod 2\pi$, where v_i^ℓ are the corresponding Dirac velocities and m_ℓ the Dirac mass.

c) Use the result of part (b) to assign θ to the insulating phases identified in part (a).

2. (15 points) Majorana chain.

In a tight-binding chain consisting of N sites with spinless fermions the Hilbert space has dimension of 2^N since each site can be empty or occupied. In that sense we can say that the system has 2 states per site. In this problem we consider the analogous question for Majorana fermions. Consider the 'Majorana chain' described by the following 1D Hamiltonian

$$H = -t \sum_{j=1}^{N} (i\gamma_j \gamma_{j+1} + \text{h.c.}), \tag{3}$$

with γ_j the Majorana fermion operators satisfying

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \qquad (\gamma_j)^\dagger = \gamma_j.$$

The Hilbert space of the Majorana chain forms a representation of the above anti-commuting algebra.

a) Consider now a chain with N even and antiperiodic boundary conditions $\gamma_{j+N} = -\gamma_j$. Define the momentum-space Majorana operators $\gamma(k) = N^{-1/2} \sum_j e^{-\imath kj} \gamma_j$ and find their commutation relations as well as the relation between $\gamma^\dagger(k)$ and $\gamma(k)$. What are the allowed values of momentum k?

b) Show that Hamiltonian (3) takes the form

$$H = 4t \sum_{k>0} \sin k \; \gamma^\dagger(k)\gamma(k) + E_g.$$

Sketch the spectrum.

c) On the basis of results in parts (a) and (b) construct the Hilbert space of the Majorana chain. Show that it has dimension of $2^{N/2}$, corresponding to $\sqrt{2}$ states per site! Discuss this result in view of what we know about Majorana fermions.

3. Bonus problem (5 points) Interacting Majorana fermions.

Consider a system with 4 Majorana fermions γ_j where $j = 1, 2, 3, 4$. The system is described by the Hamiltonian

$$H = J\gamma_1\gamma_2\gamma_3\gamma_4$$

with J a positive constant. Find the energy of the ground state and its degeneracy.