1. (20 points) Topological classification of Bi_2Se_3 via parity eigenvalues.

The effective theory of the archetypal 3D topological insulator Bi_2Se_3 is given by the following Bloch Hamiltonian on the simple cubic lattice,

$$\mathcal{H}(\mathbf{k}) = \lambda\sigma_z(s_x \sin k_y - s_y \sin k_x) + \lambda_z\sigma_y \sin k_z + \sigma_x M_{\mathbf{k}}.$$

Here $M_{\mathbf{k}} = \epsilon - 2t(\cos k_x + \cos k_y) - 2t_z \cos k_z$ and $\mathbf{s}$, $\boldsymbol{\sigma}$ represent the Pauli matrices in spin and orbital space, respectively. λ, λ_z, ϵ and t, t_z are model parameters, reflecting the layered structure of Bi_2Se_3. Hereafter, we take $\lambda = 1$ and measure the remaining parameters in units of λ.

a) Find the spectrum of $\mathcal{H}(\mathbf{k})$. Show that $\mathcal{H}(\mathbf{k})$ respects both $\mathcal{T}$ and $\mathcal{P}$. (The inversion operation here involves exchange of the two orbitals on the same site implemented by σ_x).

b) Use the Fu-Kane parity criterion to classify the topological phases of $\mathcal{H}(\mathbf{k})$ when the two negative-energy bands are filled. Consider the case when $t = t_z > 0$ and $\lambda_z = \lambda$. Sketch the phase diagram as a function of ϵ. Please label clearly the phase transitions and assign the Z_2 indices $(\nu_0; \nu_1\nu_2\nu_3)$ to the phases as appropriate.

c) Sketch the representative Fermi contours for the $z = 0$ surface in the above phases.

d) Now relax the condition $t = t_z$. For what values of ϵ and t_z will the system be in the $(1; 110)$ phase? Sketch the representative Fermi contours for the $x = 0$ surface in this phase.

2. (15 points) Bi_2Se_3 surface states.

Consider the above Hamiltonian with parameters appropriate for the (1;000) topological phase but close to the transition to the trivial (0;000) phase. To keep things simple focus again on the case when $t = t_z > 0$ and $\lambda_z = \lambda$.

a) Formulate the effective low-energy theory for the electrons *in the bulk* for this situation.

b) Show explicitly that this theory implies a single gapless surface state on the boundary between the TI and the trivial phase. Find the low-energy Hamiltonian for this surface state appropriate for the boundary in the $z = 0$ plane and in the $x = 0$ plane.

c) Sketch the spectrum of the surface states for the two planes and indicate the spin orientation.

3. (Bonus problem, 5 points) $\mathcal{T}$-breaking in Bi_2Se_3.

Show that by allowing terms in the above-defined $\mathcal{H}(\mathbf{k})$ that break time reversal the distinction between various topological phases disappears. *Hint:* This is easiest done by finding a $\mathcal{T}$-breaking term that anticommutes with $\mathcal{H}(\mathbf{k})$.