1. (10 points) In class, we have used $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}$ to efficiently find the eigenvalues of 2×2 matrix Hamiltonians. We now generalize this to 4×4 Hamiltonians with matrices Γ_i.

a) Using the knowledge of 2×2 Pauli matrices construct *all* independent anticomuting Hermitian 4×4 matrices Γ_i. In this construction it is useful to employ the tensor product notation in which a 4×4 matrix can be given as $\tau_i \otimes \sigma_j$ where τ_i and σ_j are Pauli matrices. For example, one may write

$$\tau_1 \otimes \sigma_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

b) Find the energy spectrum of a 4×4 Bloch Hamiltonian $H = \sum_j \Gamma_j d_j(\mathbf{k})$ where $d_j(\mathbf{k})$ are real functions of the crystal momentum $\mathbf{k}$.

c) Using the method developed in part (a) construct all independent anticommuting Hermitian 8×8 matrices. Based on this how many such matrices of dimension $2^n \times 2^n$ are there?

2. (20 points) Flux state on the square lattice.

The effect of the external magnetic field on the tight binding electrons can be taken into account by means of Peierls substitution,

$$t_{ij} \to t_{ij} e^{i\phi_{ij}}, \quad \phi_{ij} = \frac{e}{\hbar c} \int_{\mathbf{R}_i}^{\mathbf{R}_j} \mathbf{A} \cdot dl$$

where $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and the integral is taken along the straight line connecting points $\mathbf{R}_i$ and $\mathbf{R}_j$. Consider tight binding electrons with nearest neighbor hopping t and on-site energy $E_0 = 0$ on a 2D square lattice with uniform magnetic field B applied perpendicular to the lattice plane.

a) Consider the field strength

$$B = \frac{p}{q} \frac{\Phi_0}{a^2}$$

with p, q integer, $\Phi_0 = hc/e$ the flux quantum and a the lattice constant. Show that in this case the smallest primitive unit cell contains q lattice sites. Construct explicitly an example of such unit cell. What does this imply for the band structure? *Hint:* It is easiest to work in the gauge $\mathbf{A} = B(0, x, 0)$. Also note that $\oint_C \mathbf{A} \cdot dl = \Phi$, the flux enclosed by contour C.

b) Calculate the energy spectrum for a special case of $p = 1$, $q = 2$, i.e. half flux quantum per elementary plaquette. Show that in this case the spectrum has two Dirac points per Brillouin zone, just like graphene. Sketch the first BZ and mark the position of the Dirac points.

c) Construct the representation of $\mathcal{P}$ and $\mathcal{T}$ operators for this system and show that the Hamiltonian is $\mathcal{P}$- and $\mathcal{T}$-invariant. Explain why the $\mathcal{T}$-invariance holds although the system is in external magnetic field. Would this be the case for an arbitrary $p,\ q$?

d) In analogy with our analysis of graphene construct the physical perturbations to the system that represent the $\mathcal{P}$- and $\mathcal{T}$-breaking masses for the low-energy Dirac Hamiltonian. Describe clearly what these perturbations are in the original lattice model.

e) Argue that with the $\mathcal{T}$-breaking mass the flux state on the square lattice realizes the Chern insulator. What is its Chern number n?