P525b | **Problem set # 2** | **Due: Jan. 17, 12:30pm**

1. (20 points) Berry's phase and curvature in a two-level system.

Consider a two-level system described by the Hamiltonian

$$H = \mathbf{d} \cdot \boldsymbol{\sigma}$$

where $\mathbf{d}$ is a real vector and $\boldsymbol{\sigma}$ the vector of Pauli matrices.

a) Using the spherical representation $\mathbf{d} = d(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ write the Hamiltonian H and show that its normalized, orthogonal eigenstates corresponding to eigenvalues $\pm d$ can be written as

$$|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix}.$$

b) Find the components of the Berry connection $\mathcal{A}_d$, $\mathcal{A}_\theta$ and $\mathcal{A}_\varphi$ for the negative-energy state. Show that the only non-zero component of the Berry curvature is $\mathcal{F}_{\theta\varphi} = \frac{1}{2}\sin\theta$.

c) Repeat the calculation in part (b) in a different gauge, obtained by the transformation $|\pm\rangle \to e^{i\varphi}|\pm\rangle$. Show that although $\mathcal{A}$ changes, $\mathcal{F}$ remains the same.

d) Now calculate the Berry curvature directly from the Hamiltonian making use of the formula $\mathcal{F}_{\theta\varphi} = \frac{1}{2}\hat{\mathbf{d}} \cdot (\partial_\theta \hat{\mathbf{d}} \times \partial_\varphi \hat{\mathbf{d}})$.

e) On the basis of the above results show that the Berry phase acquired by the system when vector $\hat{\mathbf{d}}$ sweeps a closed contour C on the unit sphere is equal to $\frac{1}{2}\Omega$, where Ω is the corresponding solid angle.

2. (10 points) Thouless charge pump.

Consider the Su-Schrieffer-Heeger (SSH) model discussed in class. Propose a time-dependent generalization of the SSH Hamiltonian $H(k, \tau)$, where τ is the time parameter, such that the system realizes the Thouless charge pump.

Hints: This problem requires some careful thinking but almost no calculations. The simplest solution involves modifying the SSH Hamiltonian by addition of terms periodic in τ, such as $\cos\tau$ and $\sin\tau$. What you want to achieve is that the unit vector $\hat{\mathbf{d}}(k, \tau)$ covers the entire surface of the unit sphere as k and τ vary over one period.