Phys 511: HOMEWORK ASSIGNMENT No (2)

January 16th 2007

DUE DATE: Wednesday 28th February 2007.
(Please note that late assignments may not receive a full mark.)

QUESTION (1): SPIN WAVE EXPANSIONS

(i) Consider the simple Hamiltonian describing a ferromagnet on a 3-d cubic lattice, in a magnetic field, given by

$$\mathcal{H} = -g\mu_B H_o \sum_j S_j^z \; + \; \frac{1}{2}\sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \tag{1}$$

Show that this system can be rewritten using a spin wave expansion in terms of Holstein-Primakoff boson operators, in the form

$$\mathcal{H} \; = \; E_o \; + \; \sum_{\mathbf{q}} \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}} \; + \; \frac{1}{2N} \sum_{\mathbf{k},\mathbf{q}_1,\mathbf{q}_2} \Gamma^{(4)}_{\mathbf{q}_1\mathbf{q}_2}(\mathbf{k}) \; b^\dagger_{\mathbf{q}_1+\mathbf{k}} b^\dagger_{\mathbf{q}_2-\mathbf{k}} b_{\mathbf{q}_1} b_{\mathbf{q}_2} \;\; + \;\; O(b^6) \tag{2}$$

and derive expressions for the energies E_o, ω_q, and $\Gamma^{(4)}_{\mathbf{q}_1\mathbf{q}_2}(\mathbf{k})$.

(ii) Now consider the simple 3-d AFM system with single ion easy-plane anisotropy and exchange anisotropy, and with Hamiltonian

$$\mathcal{H} \; = \; \frac{1}{2}\sum_{<ij>} \left[J_z S_i^z S_j^z + J_\perp (S_i^x S_j^x + S_i^y S_j^y) \right] \; + \; K_z \sum_j (S_j^z)^2 \tag{3}$$

where the notation $< ij >$ refers to a sum over all pairs of nearest-neighbour lattice sites on the cube. Find the Holstein-Primakoff expansion for the spin wave spectrum for this system up to 2nd order in magnon operators, and give an expression for the energy of the magnons.

QUESTION (2): SCHRIEFFER-WOLFF TRANSFORMATION

Suppose an impurity can be described by the simple Anderson model of form $\mathcal{H} = H_o + V_o$, where

$$H_o \; = \; E_d d_\sigma^\dagger d_\sigma \; + \; \sum_{\mathbf{k}\sigma} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k}\sigma} \; + \; \frac{1}{2} U_o n_\sigma^d n_{-\sigma}^d \tag{4}$$

$$V_o \; = \; \sum_{\mathbf{k},\sigma} [v_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} d_\sigma \; + \; H.c.] \tag{5}$$

where μ is the chemical potential, and we consider a single d-orbital only on the impurity site.

Suppose we now make a canonical transformation of form $\mathcal{H} \rightarrow \tilde{\mathcal{H}} = e^{iS}\mathcal{H}e^{-iS}$, where the operator $\hat{S}$ satisfies

$$\hat{V}_o + i[\hat{S}, \hat{H}_o] \; = \; 0 \tag{6}$$

Derive a suitable form for $\hat{S}$ and show that it satisfies this last equation, and then derive all the terms for the new effective Hamiltonian up to $\sim O(v_{\mathbf{k}}^2)$, giving their expression in terms of the parameters in the original Hamiltonian.

QUESTION (3): PERTURBATION THEORY FOR SPINS

Suppose we have a spin of magnitude S in a combined easy-axis anisotropy field and a transverse term, so that the Hamiltonian is given by

$$\mathcal{H}_o = -K_o \hat{S}_z^2 + E_o \hat{S}_x^2 \tag{7}$$

where $K_o, E_o > 0$

(i) Find the energy levels of the problem when $E = 0$

(ii) Suppose now that $E_o > 0$, but $E_o/K_o \ll 1$. Show that if S has an integer value, then pairs of levels which have projection $S_z = \pm M$ are split by the transverse term, with a splitting given by perturbation theory in the parameter E_o/K_o as

$$\Delta_M^S \; = \; A_M^S K_o \left(\frac{E_o}{16 K_o} \right)^M \quad ; \qquad A_M^S = \frac{8}{[(M-1)!]^2} \frac{(S+M)!}{(S-M)!} \tag{8}$$

and find an approximate expression for the ground-state splitting Δ_S^S when $S \gg 1$.

(iii) The Hamiltonian in (ii) can be diagonalised numerically very easily. Do this for $S = 20$, and plot $ln \Delta_S^S$ as a function E_o/K_o. Compare with the answer in (ii) to see how accurate the perturbative expansion is.

END OF ASSIGNMENT (2)