# B.6.1 : SEMI-CLASSICAL METHODS

In the last chapter we already saw how there exist results that are unobtainable in any perturbative expansion about zero coupling. The idea of semiclassical theory is to make a quite different expansion, this time around the classical solution to the theory. Thus what we will end up doing is expanding in the strength of what are usually called the "quantum fluctuations" around the theory.

From a theoretical point of view this makes a lot of sense. In the enormously large "space of solutions" to any QFT, the classical limit is rather far away from the limit of zero coupling. Thus, any non-trivial non-linear solution to the classical field equations is completely inaccessible to a perturbation expansion (except under certain circumstances); and likewise tunneling phenomena (which, as we will see, can be studied using semiclassical theory) are likewise inaccessible.

As we shall see, the expansion around the classical solution is actually an expansion in powers of $\hbar$, and we have actually seen this in ordinary QM, as an expansion in fluctuations around the classical path. It also turns out that in diagrammatic terms, we can imagine that this expansion is an expansion in the number of loops in the relevant graphs - with the classical solution corresponding to a sum of "tree diagrams". However we need to be careful here - it turns out that certain infra-red diagrams containing loops can also contribute to the classical solution.

Of course the classical solutions are not always easy to find, particularly for the more complicated theories in which large varieties of topological and non-topological solitons may exist. For Yang-Mills theories, and for gravity, or for a system like superfluid $^3He$, we have only a very small sampling of the huge variety of different possible classical solutions. Moreover for certain systems it is not even clear what is the classical limit of the quantum theory. The simplest example of this is the non-relativistic system of spins in a solid, where the spins are localized on lattice points. Since there is no classical limit for a spin (if we let $\underline{S} = \hbar S \to 0$, we get nothing) this was for a long time a problem, until finally in the 1980's it was found how to write a path integral for spin.

In what follows we will develop the basic theory, and apply it to a few examples. However the emphasis here is on the methods — the number of examples is so large that there is no point in trying to give a general picture of these.

## B.6.1(a) THE CLASSICAL LIMIT

Obviously, before we can study fluctuations around the classical limit, we need to understand what this is. Anyone familiar with the enormous complexities of even the simplest classical mechanical system (eg., a set of 3 or more bodies

interacting gravitationally; or the simple vacuum of General Relativity) will appreciate that this may not be a simple matter. To make things more difficult, the limit of a quantum theory as $\hbar \to 0$ is a singular one, so it is by no means clear that the limit is unique, or that by taking it, we actually do end up with the correct classical theory. Here we look at some of these issues.

## (i) RELEVANT FEATURES OF CLASSICAL MECHANICS

Let us first recall some results in classical mechanics that inevitably play a role in this discussion. One deals typically with either the Lagrangian $L$ or the Hamiltonian $\mathcal{H}$; let us assume a set of generalized coordinates $\underline{Q} = (q_1 \ldots q_N)$, where $N$ may be infinite, and $\underline{P} = (p_1 \ldots p_N)$; here we are looking at a non-relativistic system to start with. Then we have the action

$$S = \int dt\, L[\underline{Q}, \dot{\underline{Q}}; t) = \int \underline{P} \cdot d\underline{Q} - \mathcal{H}(\underline{P}, \underline{Q})\, dt \tag{1}$$

with, if we minimize the action, the eqtns

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{Q}}} - \frac{\partial L}{\partial \underline{Q}} = 0 \tag{2}$$

and

$$\left.\begin{aligned} \dot{\underline{Q}} &= \partial \mathcal{H} / \partial \underline{P} \\ \dot{\underline{P}} &= -\partial \mathcal{H} / \partial \underline{Q} \end{aligned}\right\} \tag{3}$$

We also have the differential relations

$$\left.\begin{aligned} \partial S / \partial \underline{Q} &= \underline{P} \\ \frac{\partial S}{\partial t} &= -\mathcal{H}(\underline{P}, \underline{Q}; t) \end{aligned}\right\} \tag{4}$$

and, from (1), that

$$\frac{d}{dt} S = L(\underline{Q}, \dot{\underline{Q}}; t) \tag{5}$$

The 2nd eqtn of (4) is usually written as the Hamilton-Jacobi eqtn., viz.,

$$\frac{\partial S}{\partial t} + \mathcal{H}(\underline{Q}, \partial S / \partial \underline{Q}; t) = 0 \tag{6}$$

and if $\mathcal{H}$ is independent of $t$, so that $\mathcal{H}_0(\underline{Q}, \partial S / \partial \underline{Q}) = E_0$, we have

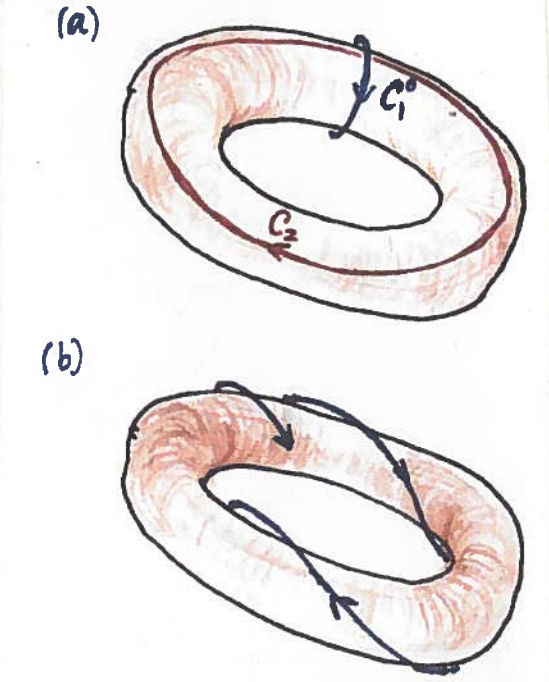
$$S(t) = S_0(\underline{Q}) - E_0 t \tag{7}$$

A key distinction must now be made between integrable and non-integrable systems. Thus we have the following 2 possibilities:

(a) INTEGRABLE SYSTEMS : If a system with $6N$ degrees of freedom (we imagine here a set of $N$ particles moving in 3d space) is completely integrable, it will then possess $3N$ constants of motion. What this means is that in the $6N$-dimensional phase space of the system, it will move on a "$3N$-torus", ie., a $3N$-dimensional hypertorus in the $6N$-dimensional space. There will then be $3N$ natural periods of motion of the system, with periods $T_j^\circ$, where $j = 1, 2, \ldots 3N$. The motion of the system can then be written in the form

$$\underline{Q}^\circ(t) = \prod_{j=1}^{3N} \sum_{n_j=-\infty}^{\infty} \underline{Q}^\circ_{n_1 n_2 \ldots n_{3N}} \, e^{i \sum_j n_j \omega_j^\circ t} \qquad (8)$$

where $\omega_j^\circ = 2\pi / T_j^\circ$. Each of these $3N$ independent periodic motions corresponds to motion around a closed circuit $C_j^\circ$ around the hypertorus; we show this in the figure for a 2-torus.




(a) TWO INDEPENDENT CLOSED CURVES $C_1^\circ$ AND $C_2^\circ$ ON THE 2-TORUS.

(b) A PATH MADE FROM $C_1^\circ$ AND $C_2^\circ$, WHICH IS BI-PERIODIC, BUT NOT CLOSED

A simple invariant torus, showing independent orbits, and also a trajectory involving 2 periods that are incommensurate.

It is convenient to rewrite things here in terms of the action-angle variables introduced by Delaunay in 1846. One makes the canonical transformation

$$(\underline{P}, \underline{Q}) \equiv (p_j, q_j) \rightarrow (J_j^\circ, \theta_j^\circ(t)) \qquad (9)$$

where the $\{J_j\}$ are the action accumulated around a circuit $C_j^\circ$, ie.,

$$J_j^\circ = \frac{1}{2\pi} \oint_{C_j^\circ} \underline{P} \cdot d\underline{Q} \qquad (10)$$

and the $\theta_j^\circ$ measure the angle (or "hyperangle") traced out during part of a circuit. Since the canonical relations $p_j = \partial S_0 / \partial q_j$ are invariant under the canonical transformation, we also have $\theta_j^\circ(t) = \partial S_0 / \partial J_j^\circ$, and Hamilton's eqns become

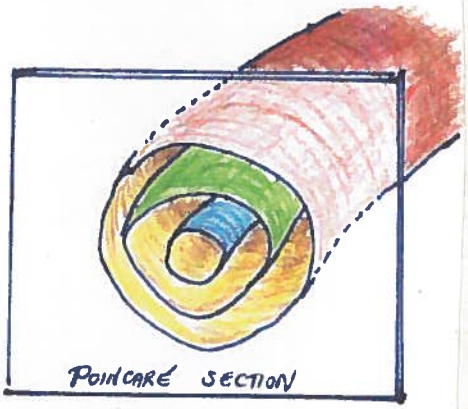
$$\left.\begin{aligned} \dot{J}_j(t) &= -\partial \mathcal{H}_0 / \partial \theta_j^\circ = 0 \\ \dot{\theta}_j^\circ(t) &= \partial \mathcal{H}_0 / \partial J_j^\circ = \omega_j^\circ \end{aligned}\right\} \qquad (11)$$

and so we have

$$\left.\begin{aligned} \omega_j^\circ &= \partial \mathcal{H}_0 / \partial J_j^\circ = \partial E_0 / \partial J_j^\circ \\ \theta_j^\circ(t) &= \omega_j^\circ t + \alpha_j \end{aligned}\right\} \qquad (12)$$

where the $\{\alpha_j\}$ are a set of integration constants. Thus the $\theta_j^\circ(t)$ change linearly with time - the system winds at a constant angular velocity around the circuits $C_j^\circ$ - and $\omega_j^\circ$ is a measure of the ratio between the total energy and the action $J_j^\circ$.

Since $J_j^\circ$ is just a measure of the area of the orbit $C_j^\circ$ in phase space, we see that $\omega_j^\circ$ measures how rapidly $E_\circ$ varies as we change this area. Thus we can imagine doing what is shown in the figure, viz., expanding the tori by increasing the energy of the system. As we vary the energy, we get a family of tori, enclosing each other. As Poincaré (the first to seriously investigate all of this) noted, one can analyze the motion over time by making a "Poincaré section" through the hypertorus, and then plotting the points where the system passes through this section on repeated passes (this section is of course a hyperplane for large dimensions). Since the different periods $\{\omega_j^\circ\}$ are not in general commensurable, over time the system trajectory will cover the entire torus (if they are commensurable, it will simply repeat the same trajectory on each pass). Thus on the Poincaré section, for a given fixed energy, we expect to see the trajectory intersections to slowly build a "hypercircle" where the "invariant torus" crosses through the section.




INVARIANT TORI OF DIFFERENT ENERGY, INTERSECTING A POINCARÉ SECTION (FLAT HYPERPLANE).

Now all of this is for an integrable system, with Hamiltonian $\mathcal{H}_\circ$. We can also imagine a system to be partially integrable - this is the more usual case (there will always be some constants of motion, at least for a closed system). But - what happens in this case?

(b) NON-INTEGRABLE SYSTEMS: The famous question first posed over 200 yrs ago by Laplace, partially answered over 100 yrs ago by Poincaré, and given a much more complete answer over 60 yrs ago by "KAM" (Kolmogorov, 1954; Arnold, and Moser, 1962), can be summarized as follows:

Suppose we write the Hamiltonian $\mathcal{H}(\underline{P},\underline{Q}) = \mathcal{H}(\underline{J},\underline{\Theta})$ (where $\underline{\Theta}(t) = \{\Theta_j(t)\}$, and $\underline{J} = \{J_j\}$), in the form

$$\mathcal{H}(\underline{J},\underline{\Theta};t) = \mathcal{H}_\circ(\underline{J}^\circ,\underline{\theta}^\circ) + V(\underline{J},\underline{\Theta};t) \qquad (13)$$

We want to do perturbation theory in $V(\underline{J},\underline{\Theta})$; so let's expand it in terms of the original modes of the system, in the form

$$\left.\begin{aligned} V(\underline{J}_\circ^\circ,\underline{\theta}^\circ;t) &= \prod_{j=1}^{3N} \sum_{n_j} V_{n_1,\ldots n_{3N}}(J_1^\circ, J_2^\circ, \ldots J_{3N}^\circ)\, e^{-i\sum_{n_j} n_j \omega_j^\circ t} \\ &\equiv \sum_{\underline{n}} V_{\underline{n}}(\underline{J}_\circ^\circ)\, e^{-i\underline{n}\cdot\underline{\omega}^\circ t} \end{aligned}\right\} \qquad (14)$$

so that, integrating over time, we have a change in the action of the form

$$\Delta S = i \sum_{\underline{n}}{}' \frac{V_{\underline{n}}(\underline{J}^0)}{\underline{n}\cdot\underline{\omega}^0} e^{-i\underline{n}\cdot\underline{\omega}^0 t} \tag{15}$$

where $\sum'$ indicates we omit the $\underline{n} = 0 \equiv (0,0,\dots)$ term. The change in the solution $\underline{Q}(t)$ is then, in terms of the unperturbed solution $\underline{Q}^0(t)$

$$\Delta \underline{Q}(t) = \int^t dt' \, V(\underline{Q}, t') \, \underline{Q}^0(t') + O(V^2) \tag{16}$$

or, in Fourier components

$$\begin{aligned} \Delta Q(t) &\sim \int^t dt' \sum_{\underline{n}} \sum_{\underline{m}} V_{\underline{n}} Q^0_{\underline{m}} e^{-i(\underline{n}+\underline{m})\cdot\underline{\omega}^0 t'} \\ &\sim i \sum_{\underline{n}}{}' \sum_{\underline{m}}{}' \frac{V_{\underline{n}} Q^0_{\underline{m}}}{(\underline{n}+\underline{m})\cdot\underline{\omega}^0} e^{-i(\underline{n}+\underline{m})\cdot\underline{\omega}^0 t} \\ &\equiv i \prod_{j=1}^{3N} \prod_{k=1}^{3N} \sum_{n_j}{}' \sum_{m_k}{}' \frac{V_{n_1 n_2 \dots n_{3N}} Q^0_{m_1 m_2 \dots m_{3N}}}{n_j \omega_j^0 + m_k \omega_k^0} e^{-i\left(\sum_{n_j} n_j \omega_j^0 + \sum_{m_k} m_k \omega_k^0\right) t} \end{aligned} \tag{17}$$

To see the significance of this result, let us imagine we are dealing with a very simple case in which only 2 frequencies $\omega_1^0$ and $\omega_2^0$ are involved, so that (17) becomes

$$\Delta Q(t) \sim \sum_{n_1 n_2}{}' \sum_{m_1 m_2}{}' \frac{V_{n_1 n_2} Q_{m_1 m_2}}{(n_1+m_1)\omega_1^0 + (n_2+m_2)\omega_2^0} e^{-i[(n_1+m_1)\omega_1^0 + (n_2+m_2)\omega_2^0]t} \tag{18}$$

where now the $\Sigma'$ excludes vanishing energy denominators – this formula of course breaks down for very long times, when the terms $\sim O(V^2)$ must come in.

Now in general $\omega_1^0$ and $\omega_2^0$ are not commensurate – however we can approximate their ratio $\omega_1^0/\omega_2^0$ arbitrarily closely by taking the ratio $r_1/r_2$ with sufficiently large $r_1, r_2$, where $r_1$ and $r_2$ are integers. Indeed, it is known from number theory that the best approximant $a_{12}(M) = r_1/r_2$, for $\mathrm{Max}(r_1, r_2) = M$, has an error

$$|a_{12}(M)| = \left|\frac{\omega_1^0}{\omega_2^0} - a_{12}(M)\right| \sim O\left(\frac{1}{M^2}\right) \tag{19}$$

so that the error goes down rapidly with increasing $M$.

This problem of "vanishing energy denominators" was emphasized by Laplace, and Poincaré invented the field of topology in order to try and solve it. We see that as time $t$ increases, the effect of very small energy denominators can become very large – arbitrarily large for high frequency components of the perturbation. Going to higher order in $V$ obviously doesn't

help; we simply introduce chains of energy denominators, and the structure here is one you will recognize from ordinary scattering theory, either in QM (the Lippmann-Schwinger eqtn.) or QFT (the Bethe-Salpeter eqtn.), and can indeed be represented diagrammatically in a very familiar way.

This problem had great interest at that time, in the context of solar system dynamics. Consider, eg., the coupled dynamics of Jupiter and Saturn around the sun. Currently, the periods of these 2 planets are $T_J = 11.8618$ yrs and $T_S = 29.4571$ yrs. The ratio of their orbital frequencies is thus $\omega_S/\omega_J \approx 0.40268$ not so far from 2/5. Their interaction is attractive, and so if they continually meet at the same points in their orbits, there is a strong tendency for this weak attraction to pull them into resonance. In reality $\omega_S/\omega_J$ fluctuates in time, and Laplace realized that a dynamical system could not only move between different resonant frequency ratios; it could also become unstable. Poincaré uncovered an almost infinitely complex structure in phase space, one that we would now describe as "fractal".

The solution to the problem given by KAM can be stated as

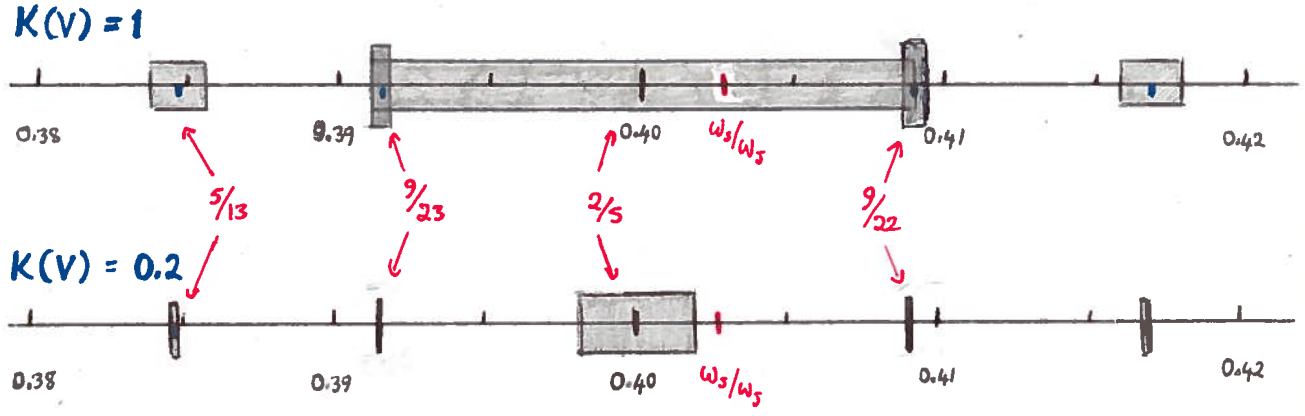
"If $|V|$ is sufficiently small, then for almost all irrational $\underline{\omega}^\circ$ (ie., sets of frequencies $\{\omega_j^\circ\} = (\omega_1^\circ, \omega_2^\circ, \ldots \omega_{3N}^\circ)$ with irrational ratios $\omega_j^\circ/\omega_1^\circ$) there exists an invariant torus $T_V(\underline{\omega}^V)$ of the perturbed system such that $T_V(\underline{\omega}^V)$ is close to the unperturbed torus $T_0(\underline{\omega}^\circ)$."

Moreover, KAM showed that, quite generically, there exists a function of $V$, usually called $K(V)$, which goes to zero regularly with $V$, such that the above result is valid provided (here we assume $\omega_2^\circ > \omega_1^\circ$):

$$\left| \frac{\omega_1^\circ}{\omega_2^\circ} - \alpha_{12}(r_2) \right| \equiv \left| \frac{\omega_1^\circ}{\omega_2^\circ} - \frac{r_1}{r_2} \right| > \frac{K(V)}{r_2^{5/2}} \qquad (\forall\ r_2) \qquad (20)$$

in which case we say that the invariant torus is "preserved". It is easy to see that this result leaves most parts of the interval $0 < \alpha_{12} < 1$ preserved, even if $V$ is not that small.

To see how this works out for the Jupiter/Saturn pair, we plot results for (20) for 2 different values of $K(V)$, near the actual ratio $\omega_S/\omega_J$.




The region around $\omega_S/\omega_J = 0.40268$ on the line. Regions of size $K(V)/q^{5/2}$ are excluded around any fraction $p/q$, as shown in shaded rectangles.

We see from this how the KAM theorem works. That only a small fraction of the interval is rendered unstable for small $K$ is seen by making the crude estimate for the measure of the real line that is excised when (20) is not valid; if the measure is $M(K)$, we have

$$M(K) < \sum_{r=1}^{\infty} r \frac{K}{r^{3/2}} = K \sum_{r=1}^{\infty} \frac{1}{r^{3/2}} \sim O(K) \qquad (21)$$

where $M(K)$ is considerably less than the sum on the R.H.S., because this sum double-counts many fractions (thus, eg., $1/2$ is also counted as $2/4, 3/6, 4/8$, etc.).

What happens when the inequality (20) is violated? One possible result is chaotic behaviour, particularly if $V$ is repulsive. Another is the formation of resonantly coupled frequency pairs. We see how complicated is the possible behaviour for just 2 degrees of freedom - for $3N$ coupled degrees of freedom, it is clear that an almost infinitely bewildering variety is possible in the dynamics (including instability).

(c) INTERACTING FIELDS : It is very tempting, although incorrect, to think of the different modes discussed above as oscillators, so that by adding a perturbation we are simply coupling these oscillators. Actually the KAM discussion is much more general - it applies, eg., to orbiting planets, which do not in general behave like oscillators under perturbations.

However a simple interacting bosonic field can be thought of as a set of coupled oscillators. This is clear both for a relativistic scalar field, whose action, when Fourier transformed, reads in the case of $\phi^4$ theory as

$$S = -\left\{ \sum_q \tfrac{1}{2} \phi_q (q^2 + m^2) \phi_q + \frac{g}{4!} \sum_{kk'q} \phi_k \phi_{k'} \phi_{k'+q} \phi_{k-q} \right\} \qquad (22)$$

and for a set of interacting phonons (where the coupling is more complicated; see section B.3, eqtn (82)) but of the same general form. Thus one can clearly learn something about coupled fields by studying the simple non-relativistic problem of coupled oscillators, with action

$$S = \int dt \left[ \tfrac{1}{2} \sum_q m_q (\dot{x}_q^2 - \omega_q^2 x_q^2) - \sum_{n \geq 3} V_n(q_1 \cdots q_n) x_{q_1} x_{q_2} \cdots x_{q_n} \right] \qquad (23)$$

as an example. The distribution of frequencies $\{\omega_q\}$ is clearly important, but in a typical interacting field, we can assume a quasi-continuous distribution with a density of states $N(\omega)$; in a solid there may exist gaps between different oscillator "bands".

The first serious investigation of a problem like this required a computer (as does much of modern work on non-linear dynamics); it was done by Fermi, Pasta, and Ulam in , in an attempt to understand how a set of oscillators would thermally relax once interactions were introduced. To their great surprise, they found that new modes appeared which did not relax -

we now think of these modes as a kind of "non-topological soliton" which hardly interacts with the other modes.

The situation now is, if anything, more confused. We are far more aware of the enormous variety of possible behaviour in non-linear dynamical systems. Moreover, as we will discuss below, the existence of constraints in the dynamics of a classical system can have unforeseen consequences - this has become clear in the study of solutions to Einstein's eqtns in General Relativity, and is in a sense obvious in subjects like fluid dynamics, and can have important consequences for the quantum theory. However it has to said immediately that there is still very little known about how to give a general theoretical treatment here. As we will see, most work in QFT has focussed on soliton excitations and on their imaginary time counterpart, the "instanton". It is perfectly obvious that this is only scratching the surface of a very large problem.

We will return in detail to all of this later. But it is worthwhile giving a very brief look at one simple example, which is understood in great detail, both in the classical and quantum theories. This is the Sine-Gordon model, which is used both in relativistic and non-relativistic QFT. It takes the following form:

- Relativistic Sine-Gordon Model: This describes a scalar field in $1+1$ dimensions, with action

$$S = \int d^2x \left\{ \tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^4}{g} \left[ \cos \frac{g^{1/2}}{m} \phi - 1 \right] \right\} \tag{24}$$

$$S = \int d^2x \left\{ \tfrac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) + \frac{g}{4!} \phi^4 - \frac{1}{6!} \frac{g^2}{m^2} \phi^6 + \cdots \right\} \tag{25}$$

so that for $g/m^2 \ll 1$ this looks like $\phi^4$ theory in 2d. The eqtn of motion for $\phi(x)$ is then

$$\partial^2 \phi + \frac{m^3}{g^{1/2}} \sin \frac{g^{1/2}}{m} \phi = 0 \tag{26}$$

which for $g/m^2$ looks like a free field equation.

- Non-relativistic Sine-Gordon Model: Here we deal with a 1-dimensional system, which contains a periodic potential in the field amplitude, with action:

$$\left.\begin{aligned} S &= \int dt\, dx \left[ \tfrac{1}{2} \left( \frac{\partial \Theta(x,t)}{\partial t} \right)^2 - H(x,t) \right] \\ H(x,t) &= \int dt\, dx \left\{ \tfrac{1}{2} c_0^2 \left( \frac{\partial \Theta}{\partial x} \right)^2 + \omega_0^2 \left( 1 - \cos \Theta(x) \right) \right\} \end{aligned}\right\} \tag{27}$$

which describes a 1-d field in many different solid-state systems (as well as in polymers like polyacetylene chains). The eqtn. of motion is then

$$(\partial_t^2 - c_0^2 \partial_x^2)\,\theta(x,t) + \omega_0^2 \sin\theta(x,t) = 0 \qquad (28)$$

These 2 systems are identical, and have the same classical physics. It is common to rescale the relativistic form, using

$$\left.\begin{aligned} x &\to \bar{x} = mx \\ t &\to \bar{t} = mt \\ \phi &\to \bar{\phi} = (g^{1/2}/m)\,\phi \end{aligned}\right\} \qquad (29)$$

so that

$$S = \int d^2\bar{x}\ \frac{m^2}{g}\left[\tfrac{1}{2}\bar{\partial}_\mu\bar{\phi}\bar{\partial}^\mu\bar{\phi} + (\cos\bar{\phi} - 1)\right] \qquad (30)$$

with eqtn. of motion:

$$\left(\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2} - \frac{\partial^2\bar{\phi}}{\partial\bar{x}^2}\right) + \sin\bar{\phi}(\bar{x},\bar{t}) = 0 \qquad (31)$$

and conserved energy

$$E = \frac{m^3}{g}\int d\bar{x}\left\{\tfrac{1}{2}\left[\left(\frac{\partial\bar{\phi}}{\partial\bar{t}}\right)^2 + \left(\frac{\partial\bar{\phi}}{\partial\bar{x}}\right)^2\right] + (1-\cos\bar{\phi})\right\} \qquad (32)$$

Thus the rescaled relativistic theory is like the non-relativistic theory, with energy gap $\omega_0 = 1$, and light velocity $c_0 = 1$. They are both Lorentz invariant.

This system has 2 kinds of excitation. There are the small oscillation modes (phonons or photons), with

$$\left.\begin{aligned} \theta_q^{QP}(x,t) &\sim \theta_0 e^{i(qx-\omega_q t)} \\ \omega_q^2 &= \omega_0^2 + c_0^2 q^2 \end{aligned}\right\} \qquad (33)$$

where "QP" signifies "quasiparticle". However there are also "kink" and "anti-kink" solutions, given by

$$\psi_\pm^V(x,t) = 4\tan^{-1}\left[\exp\left(\pm\frac{\omega_0}{c_0}\frac{x-vt}{(1-v^2/c_0^2)^{1/2}}\right)\right] \qquad (34)$$

$$\bar{\Phi}_\pm^V(v;\bar{x},\bar{t}) = \pm 4\tan^{-1}\left[\exp\left(\frac{\bar{x}-v\bar{t}}{(1-v^2)^{1/2}}\right)\right] \qquad (35)$$

in the 2 cases. The existence of these solutions reflects the discrete symmetry present in the system, whereby

$$\left.\begin{aligned} \phi(\bar{x},\bar{t}) &\longleftrightarrow -\phi(\bar{x},\bar{t}) \\ \bar{\phi}(\bar{x},\bar{t}) &\longleftrightarrow \bar{\Phi}(\bar{x},\bar{t}) + 2n\pi \end{aligned}\right\} \qquad (36)$$

with the same for $\theta(x,t)$; we also notice that the Lorentz invariance is explicit in these solutions. A common way to think of these solutions is in terms of the physical model of an infinite chain of pendula hanging in a gravitational field, with nearest neighbours connected by springs. Then the quasiparticles correspond to small oscillations about the energy minimum, and the kinks to configurations in which the line rotates all the way around the upwards vertical, through an angle $2\pi$.

One can also construct exact multi-soliton solutions to this system. However we wish to focus on one simple point, with a view to future developments. Suppose we have a single soliton/kink which is, for simplicity, taken to be stationary ($v = 0$); and we wish to see how its presence affects the quasiparticle modes. This is easy to do; let's assume a solution

$$\theta_{\pm}(x,t) = \psi_{\pm}^{v}(x,t) + \varphi(x,t) \tag{37}$$

where $\varphi(x,t)$ is a very small perturbation around $\psi_{\pm}^{v}(x,t)$. Substituting this into the equation of motion, we get a new equation

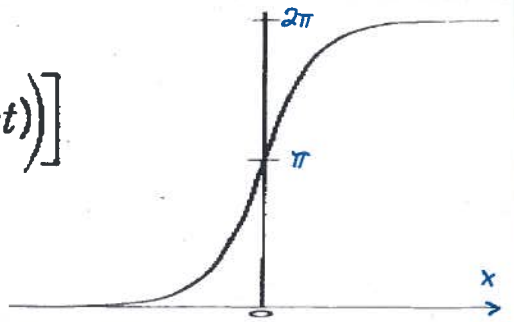
$$\left\{ (\partial_t^2 - c_0^2 \partial_x^2) + \omega_0^2 \left[ 1 - 2\,\mathrm{sech}^2\left(\frac{\omega_0}{c_0} x\right) \right] \right\} \varphi(x,t) = 0 \tag{38}$$

with the "kink potential" $V(x) = \omega_0^2 [1 - 2\,\mathrm{sech}^2(\omega_0 x / c_0)]$ acting on $\varphi(x,t)$.

$$\psi_{\pm}^{v}(x,t) = 4 \tan^{-1}\left[ \exp\left( \pm \frac{\omega_0}{c_0} \gamma (x - vt) \right) \right]$$

$$\gamma \equiv (1 - v^2/c_0^2)^{-1/2}, \quad |v| < c_0$$

The kink soltn for the 1-d Sine-Gordon model.




To solve for $\varphi(x,t)$, we assume the ansatz $\varphi(x,t) = f(x) e^{-i\omega t}$, and substitute in (38). Then we find the eqtn (here $q_0 = \omega_0/c_0$):

$$\left\{ c_0^2 \frac{d^2}{dx^2} + \left[ \omega^2 - \omega_0^2 (1 - 2\,\mathrm{sech}^2 q_0 x) \right] \right\} f(x) = 0 \tag{39}$$

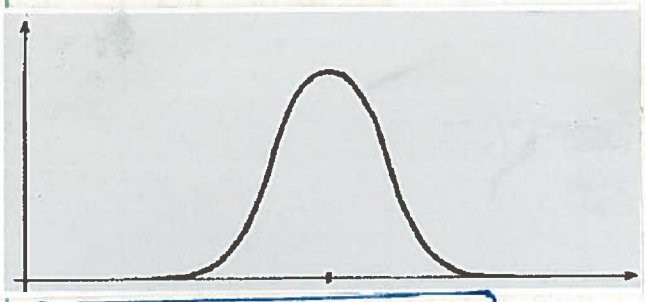
which is a Schrodinger eqtn., with solutions

$$\left.\begin{aligned} f_b(x) &= 2 q_0 \,\mathrm{sech}\, q_0 x \qquad (\text{with } \omega = 0) \\ f_q(x) &= \frac{1}{\sqrt{2\pi}} \left( \frac{c_0}{\omega_q} \right) e^{iqx} \left[ q_0 + i q_0 \tanh q_0 x \right] \qquad (\text{with } \omega_q^2 = \omega_0^2 + c_0^2 q^2) \end{aligned}\right\} \tag{40}$$

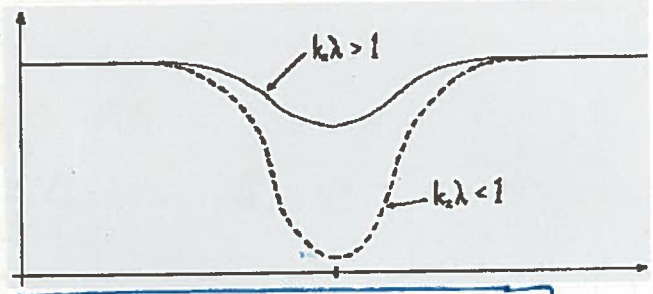
which describe 2 kinds of quasiparticle state. The first, sometimes called a "zero mode", has $\omega = 0$, and is stuck to the soliton as a localized bound state. The second has the form of a running wave, but with reduced amplitude in the region of the kink. This is not surprising - all these modes are orthonormal, so that

$$\left.\begin{aligned} \int dx\, f_q^*(x)\, f_{q'}(x) &= \delta(q-q') \\ \int dx\, f_q(x)\, f_b(x) &= 0 \end{aligned}\right\} \qquad (41)$$

and so we expect the delocalized quasiparticles to have reduced amplitude around the soliton, specially for long wavelengths. All this is illustrated in the figures



$|\psi_b(x)|^2 = f_b^2 = 4q_0^2 \operatorname{sech}^2 q_0 x$




$|\psi_q(x)|^2 = |f_q(x)|^2$ for quasiparticle solns.

One should not imagine that the interaction between the running modes and the kink, or the introduction of a new bound state, means that there is any scattering between the modes. Actually a quasiparticle traveling through the kink suffers a phase shift $\delta_q$ given by

$$\delta_q = \pi \operatorname{sign} q - 2\tan^{-1}(q/q_0) \qquad (42)$$

but there is no physical scattering.

## (ii) CLASSICAL LIMIT FOR SPIN

: In some cases it is not clear what is the correct classical limit for a quantum system. For any path-integral based method this is a serious problem, because one sums over all paths for a classical action, written in terms of classical variables. This problem emerges most notoriously in the case of spin (a problem which famously defeated Feynman), because in the classical limit $\hbar \to 0$, the spin $\underline{S} = \hbar \underline{\sigma}$, with $|\sigma| = \pm\frac{1}{2}, 1, \ldots$, disappears completely. It has its counterpart in field theories.

In old-style treatments of the problem, one treated spin as though it was an angular momentum. A classical angular momentum $\underline{L}$ has a rotational energy given by

$$T = \tfrac{1}{2}\underline{L}\cdot\underline{\Omega} = \tfrac{1}{2}\Omega_i I_{ij} \Omega_j = \tfrac{1}{2} L_i I_{ij}^{-1} L_j \qquad (43)$$

where $\underline{\Omega} = d\underline{\theta}/dt$ is the angular velocity, and $I_{ij}$ the moment of inertia of the body.

Then the Lagrangian of the system is

$$\mathcal{L} = T - U(\underline{\theta}) \tag{44}$$

where $U(\underline{\theta})$ is a potential energy a function of the orientation angle $\underline{\theta}(t)$ of the object. The Hamiltonian is then $\mathcal{H} = T + U(\underline{\theta})$; and $\underline{L}$ and $\underline{\Omega}$ are related as usual by

$$L_i = I_{ij}\Omega_j \tag{45}$$

Note there is no difficulty in quantizing this and taking the classical limit; if we let $|\underline{L}| = \hbar$, then the classical limit involves letting $\hbar \to 0$ and $\ell \to \infty$ at the same time.

If we try this with spin, we run into a problem. Suppose we try to write the spin as $\underline{S} = \hbar S\,\underline{n}(t)$, where $\underline{n}(t)$ is a unit vector and $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Can we then write a Lagrangian

$$\left.\begin{aligned} L(n,\dot{n}) &= \tfrac{1}{2}\dot{n}_i I_{ij}\dot{n}_j - U(\underline{S}) \\ &\equiv \tfrac{1}{2}\underline{S}\cdot\dot{\underline{n}} - U(\underline{S}) \end{aligned}\right\} \; ? \tag{46}$$

for the spin? If so, we apparently must write $S_i = I_{ij}\dot{n}_j$, but what does this mean? There is no obvious significance to $I_{ij}$. If we forget about this problem for the moment, one can adopt the following tactic - assume a Hamiltonian $\mathcal{H}(\underline{S})$ for the system, and an equation of motion

$$\dot{\underline{S}}(t) = -\gamma\left(\underline{S}(t) \times \frac{\partial \mathcal{H}(\underline{S})}{\partial \underline{S}}\right) \tag{47}$$

where $\gamma = g\mu_B = \hbar g e/2m$; and then write a Lagrangian

$$L(\underline{S},\dot{\underline{S}}) = P\dot{Q} - H(\underline{S}) \tag{48}$$

with canonical variables

$$\left.\begin{aligned} P(t) &= \hbar S\cos\theta(t) \\ Q(t) &= \varphi(t) \end{aligned}\right\} \tag{49}$$

where the unit vector $\underline{n}(t) = (\theta(t), \varphi(t))$, in polar coordinates. Hamilton's eqtns then give us back (47). While this procedure works, it is also unsatisfactory, because it assumes a special role for the $\hat{z}$-axis, which may not exist in the Hamiltonian.

This problem was solved in a way which is discussed in more detail in part A. One introduces a set of coherent states $|\underline{n}(t)\rangle$, such that

$$\langle \underline{n}|\hat{\underline{S}}|\underline{n}\rangle = \hbar S\,\underline{n} \tag{50}$$

so that, e.g., $\langle \underline{n} | \hat{S}_z | \underline{n} \rangle = \hbar S \cos\theta$, and $\langle \underline{n} | \hat{S}^{\pm} | \underline{n} \rangle = \hbar S \sin\theta\, e^{\pm i\varphi}$. These states thus pick out a specific direction in spin space, or the Bloch sphere, but they are not $\delta$-functions of angle, and two states $|\underline{n}_1\rangle$ and $|\underline{n}_2\rangle$, oriented in different directions, still have a finite overlap. Thus the state $|\underline{n}_z\rangle \equiv |\hat{\underline{z}}\rangle$, oriented along the north pole, we have

$$|\underline{n}_z\rangle \equiv |\hat{\underline{z}}\rangle = |S, m = S\rangle \tag{51}$$

ie., the state with total spin $S$ and projection $S$ along the $\hat{z}$-axis; and for some arbitrary direction $\underline{\theta}$, we have

$$|\underline{n}_{\underline{\theta}}\rangle = |\theta,\varphi\rangle \equiv \hat{R}(\theta,\varphi)\,|\underline{n}_z\rangle \tag{52}$$

where $\hat{R}(\theta,\varphi)$ is just the rotation operator for the vector $\underline{n}$; in terms of the eigenstates $|S,m\rangle$ of $\hat{S}_z$, we have

$$\left.\begin{aligned} |\underline{n}_{\underline{\theta}}\rangle &= \left(\cos\tfrac{\theta}{2}\right)^{2S} \exp\left[\tan\tfrac{\theta}{2}\, e^{i\varphi}\, \hat{S}_-\right] |S,S\rangle \\ &\equiv \sum_{m=-S}^{S} \left(\frac{2S!}{(S-m)!\,(S+m)!}\right)^{1/2} \left(\cos\tfrac{\theta}{2}\right)^{S+m} \left(\sin\tfrac{\theta}{2}\right)^{S-m} e^{im\varphi}\, |S,S\rangle \end{aligned}\right\} \tag{53}$$

and, although these states are complete, ie.,

$$\left.\begin{aligned} (2S+1)\int d\underline{n}\, |\underline{n}\rangle\langle\underline{n}| &\equiv (2S+1)\int \frac{d\theta\, d\varphi}{4\pi} \sin\theta\, |n\rangle\langle n| \\ &= 1 \end{aligned}\right\} \tag{54}$$

we still have the overlap

$$\left.\begin{aligned} \langle \underline{n}_1 | \underline{n}_2 \rangle &= \left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\, e^{-i(\varphi_1-\varphi_2)}\right]^{2S} \\ &\equiv \left(\frac{1+\underline{n}_1\cdot\underline{n}_2}{2}\right)^{S} e^{iS\beta_{12}} \end{aligned}\right\} \tag{55}$$

where

$$\tan\frac{\beta_{12}}{2} = \tan\left(\frac{\varphi_1-\varphi_2}{2}\right) \frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} \tag{56}$$

Now we can do path integrals - for it turns out - ie., it can be proved that* - the propagator for a single spin between states $|n_1\rangle$ and $|n_2\rangle$ takes the form

$$\langle \underline{n}_2 | \hat{G}(t_2,t_1) | \underline{n}_1 \rangle = \int_{\underline{n}_1(t_1)}^{\underline{n}_2(t_2)} \mathcal{D}n(t)\; e^{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\; L(\underline{n},\dot{\underline{n}};t)} \tag{57}$$

---

* Write $G(2,1) = \lim_{N\to\infty} (2S+1)^N \prod_{j=1}^{N-1} \int d\underline{n}_j \langle \underline{n}_{j+1} | e^{-\frac{i}{\hbar}H(t_j)\,dt} | \underline{n}_j \rangle$, and for infinitesimal $dt$, rewrite $\langle \underline{n}_{j+1} | e^{-\frac{i}{\hbar}H(t_j)\,dt} | \underline{n}_j \rangle = \langle \underline{n}_{j+1} | \underline{n}_j \rangle \langle \underline{n}_j | e^{-\frac{i}{\hbar}H(t_j)\,dt} | \underline{n}_j \rangle$, and then use (55).

where the Lagrangian is now for a classical vector $\underline{n}(t)$, taking the form

$$L(\underline{n},\dot{\underline{n}};t) = \hbar S\, \underline{A}\cdot\dot{\underline{n}}(t) - \mathcal{H}(S\underline{n};t) \tag{58}$$

with a fake vector potential $\underline{A}$ which is designed to describe the field of a unit monopole situated at the centre of the Bloch sphere, ie., it satisfies

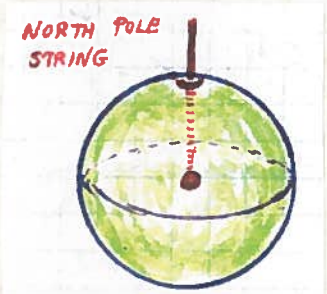
$$\underline{n}\cdot(\underline{\nabla}\times\underline{A}) \equiv n_\alpha \epsilon^{\alpha\beta\gamma}\frac{\partial A^\beta}{\partial n^\gamma} = 1 \tag{59}$$

with the vector operations taken in spin space; in the same way we have

$$\oint_C d\underline{n}\cdot\underline{A} = \omega \tag{60}$$

where the circuit $C$ is taken on the Bloch sphere, and $\omega$ is the solid angle enclosed by this circuit.

At this point we have a gauge freedom in choosing $\underline{A}(\theta,\varphi)$, defined as a vector field on the surface of the Bloch sphere. Actually, by the 2-d application of Poincare's conjecture (ie., that every vector field on a sphere has to have a singular point) we know that a "Dirac string" can be used to define this vector field; and in fact the choice of the point where the string emerges is equivalent to a choice of quantization axis for the spin. Let us choose this quantization axis to be the north-directed $\hat{z}$-axis; then one popular choice for $\underline{A}(\theta,\phi)$ is




The "Dirac string" showing the point on the Bloch sphere where $\underline{A}(\underline{n})$ is singular.

$$\underline{A}(\theta,\varphi) = -\hat{\varphi}\cot\theta/2. \tag{61}$$

and another (produced from the first by a non-singular gauge transformation) is

$$\underline{A}(\theta,\varphi) = -\hat{\varphi}\,\frac{1+\cos\theta}{\sin\theta} \tag{62}$$

Thus we have the amusing result that a spin $\underline{S}$, moving in some potential which is a function of the angle $\underline{\theta} = (\theta,\varphi)$, can be described in path integral terms as though it was a particle of charge $\hbar S$, moving on the Bloch sphere surface in the same field, but now with the unit monopole at the centre of the sphere (so that there is a Lorentz force acting on the particle velocity $\dot{\underline{n}}$). This then gives us the correct eqtn of motion (47) for the classical dynamics. However, as will discuss below, it also has profound implications for the quantum dynamics.

This treatment for a single spin is easily adapted to the case of a lattice of spins, to give us a lattice field theory. In all cases the prescription is the same — for a quantum Hamiltonian $\hat{\mathcal{H}}(\hat{\underline{S}})$, written in terms of spin operators, we simply substitute $\hat{\underline{S}} \to \hbar S\underline{n}$, converting the Hamiltonian to a

classical one for the vector $S\underline{n}$. In other words, for a typical exchange + anisotropy Hamiltonian, we have the correspondence

$$\sum_j \hat{\mathcal{H}}_j(\hat{\underline{S}}_j) + \sum_{i\neq j}\sum_j \hat{\mathcal{H}}_{ij}(\hat{\underline{S}}_i, \hat{\underline{S}}_j) \quad \Longleftrightarrow \quad \sum_j H_j(\theta_j,\varphi_j) + \sum_{i\neq j}\sum H_{ij}(\theta_i,\varphi_i;\theta_j,\varphi_j) \tag{63}$$

As an example, consider the Hamiltonian

$$\hat{\mathcal{H}} = \sum_j\left(\frac{1}{S}\left[K_2^{\parallel}\hat{S}_{jz}^2 + K_2^{\perp}S_{jx}^2\right] + \frac{1}{S^3}K_4^{\parallel}S_{jz}^4\right) + \sum_{i\neq j}\sum\frac{1}{S}J_{ij}^{\parallel}\hat{S}_{iz}\hat{S}_{jz} \tag{64}$$

This then has the classical counterpart

$$\left.\begin{aligned} H = {} & \hbar S\sum_j\left(\left[K_2^{\parallel}\cos^2\theta_j + K_2^{\perp}\sin^2\theta_j\cos^2\varphi_j\right] + K_4^{\parallel}\cos^4\theta_j\right) \\ & + \hbar S\sum_{(i\neq j)} J_{ij}\cos\theta_i\cos\theta_j \end{aligned}\right\} \tag{65}$$

with obvious generalizations to other forms of spin Hamiltonian.

If we insist on defining everything in a specific gauge, such as the one employed here in which the Dirac string goes through the north pole, then we can easily recover the results as written in the form (48) and (49). We first need to go to a phase space formulation of the spin path integral, according to which

$$\left.\begin{aligned} G(2,1) = {} & \int_{P_1(t_1)}^{P_2(t_2)}\mathcal{D}P\int_{Q_1(t_1)}^{Q_2(t_2)}\mathcal{D}Q\, e^{\frac{i}{\hbar}S[P,Q]} \\ S[P,q] = {} & \int_{t_1}^{t_2}dt\,\left(P\dot{Q} - H[P,Q]\right) \end{aligned}\right\} \tag{66}$$

However, we do not pursue this here.

## B.6.1 (6) FLUCTUATIONS & THE LOOP EXPANSION

Now we return to the quantum theory, and look at expansions around the classical result in the functional integral formulation. This kind of expansion has its roots in simple WKB theory, which we begin by briefly recalling here. Suppose we want to solve a differential eqtn containing a very small parameter, such as the Schrodinger eqtn, written as

$$[\hbar^2 - f(x)]\psi(x) = 0 \tag{67}$$

where, for a 1-particle system, $f(x) = 2m(V(x) - E)$. The basic idea of WKB theory is to write the solution in the form

$$\psi(x) = A(x)\,e^{\Phi(x)/\hbar} \equiv e^{\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n\varphi_n(x)} \tag{68}$$

where $A(x)$ is assumed to very slowly, compared to $\Phi(x)$, in the limit where

$\hbar \ll 1$. Note that this limit $\hbar \to 0$ is singular, as we can see by looking at any simple QM problem; for example, take the "particle in a box", for a specific value of $2m(V-E) = -2mE$ (ie., for specific finite energy). As $\hbar \to 0$, we go to ever higher energy levels for the same energy, and the solution becomes more and more rapidly oscillatory - the limit $\hbar = 0$ is discontinuous, because for $\hbar = 0$, the solution to (67) is $\psi(x) = 0$. Thus (68) is obviously an asymptotic series. Substituting (68) into (67) gives the hierarchy

$$\left.\begin{aligned} (\varphi_0'(x))^2 &= f(x) \\ 2\varphi_0'(x)\varphi_1'(x) + \varphi_0''(x) &= 0 \\ 2\varphi_0'(x)\varphi_n'(x) + \varphi_{n-1}''(x) + \sum_{r=1}^{n-1} \varphi_r' \varphi_{n-r}'(x) &= 0 \qquad n \geq 2 \end{aligned}\right\} \quad (69)$$

a result which we will use several times in this chapter. The solutions for the $\varphi_n(x)$ then have the following results

- The lowest term $\varphi_0(x)$ is just the classical action, ie., we have the general result

$$\varphi_0(x) \equiv S_{cl}(x) = \pm \int^x dx' \, f^{1/2}(x) \quad (70)$$

where in the case $f(x) < 0$, this becomes imaginary.

- The next term is the leading contribution to the prefactor $A(x)$ in the usual QM formula, ie., we have

$$\varphi_1(x) \equiv A_0(x) = -1/4 \ln f(x) \quad (71)$$

corresponding in path integral theory to the 1st fluctuation determinant (see below).

- The next set of terms has the following form, and corresponds, in the path integral theory of QM, to the higher fluctuations about the classical path:

$$\varphi_2(x) = \pm \int dx' \left[ \frac{1}{8} \frac{f''(x')}{f^{3/2}(x')} - \frac{5}{32} \frac{(f'(x'))^2}{f^{5/2}(x')} \right] \quad (72)$$

$$\varphi_3(x) = \left[ \frac{5}{64} \frac{(f'(x))^2}{f^3(x)} - \frac{1}{16} \frac{f''(x)}{f^2(x)} \right] \quad (73)$$

and so on; for a more detailed discussion one should go to part A.

These results are reflected in the path integral formulation, although to get them requires the calculation of higher fluctuation determinants - all of this is

part of our discussion of path integrals in part A.

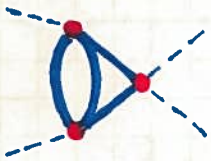
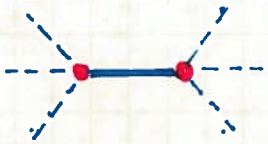
One other thing should be noted here, viz., the connection between this expansion in powers of $\hbar$ and a "gradient expansion", ie., an expansion in higher derivatives of both the function $f(x)$ and the solutions $\varphi_n(x)$ which feed into $\psi(x)$. We see that higher powers in $\hbar$ are associated with higher derivatives (or powers of derivatives) of $f(x)$.

## (i) LOOP EXPANSION for SCALAR FIELDS :

We now look at the connection between the asymptotic expansion in powers of $\hbar$ and the "loop expansion", and the usual scalar field theory provides a nice example of this. The basic point here is quite simple, and we shall look at it in 2 different ways: first, by looking at things purely diagrammatically, and then by looking at things as a fluctuation expansion about a classical configuration.

### DIAGRAMMATIC ARGUMENT :

Let's consider how the $\hbar$ factors enter the diagrams in a theory like $\phi^4$ theory. Let's begin with a table showing how all this works for the lowest-order graphs for the irreducible vertex $\Gamma_n(x_1 \cdots x_n)$ (ie., having no external legs). We work in 4-momentum space, and find:

| GRAPH | EXPRESSION | $O(\hbar)$ | $L$ |
|---|---|---|---|
|  | $\left(\frac{-ig}{\hbar}\right)^3 \sum_{kQ} \sum (i\hbar)^4 \Delta_F(Q)\Delta_F(Q+k-p) \times \Delta_F(k)\Delta_F(p+p'-k)$ | $\hbar$ | 2 |
|  | $\left(\frac{-ig}{\hbar}\right)^2 i\hbar\,\Delta_F(k)$ | $\hbar^{-1}$ | 0 |
|  | $\frac{-ig}{\hbar} \sum_k i\hbar\,\Delta_F(k)$ | $\hbar^0$ | 1 |
|  | $\left(\frac{-ig}{\hbar}\right)^2 \sum_{kq} \sum (i\hbar)^3 \Delta_F(k+q)\Delta_F(k)\Delta_F(p-q)$ | $\hbar$ | 2 |
|  | $\left(\frac{-ig}{\hbar}\right)^3 \sum_{kk'q} \sum (i\hbar)^5 \mathcal{F}(k,k',p,q)$ | $\hbar^2$ | 3 |

with $\mathcal{F}(k,k'pq) = \Delta_F(p-q)\Delta_F(k+q)\Delta_F(k) \times \Delta_F(k'+q)\Delta_F(k')$

where as usual we have the rules given in section B.3 (eqtns (40) and (41) in the Figures) for the vertex and the propagator.

The result is clear already from looking at these graphs; we have, in any graph, a number of loops given by eqtn. (43) of section B.3, ie., that $L = I + 1 - V$, where there are $I$ internal lines and $V$ vertices; and the graph will obey

$$\Gamma_n(k_1..k_n) = \hbar^{L-1} f_n(k_1..k_n) \qquad (74)$$

where the $\{k_n\}$ are the external leg arguments, and $f_n(k_1..k_n)$ is independent of $\hbar$. To prove this result is simple; each internal line gives a factor $\hbar$, while each vertex gives a factor $\hbar^{-1}$; thus we get a contribution $\hbar^{I-V}$ from any graph, ie., $\hbar^{L-1}$.

From this result we see that by summing all "tree graphs" (containing no loops at all) for the vertex generating functional, we will get the complete classical result for the classical generating functional $\Gamma_{cl}[\bar\Phi]$ in terms of the classical solution $\bar\Phi(x)$ of the field theory. We have already seen this result in section B3; cf eqtn (37). There we showed it by expanding $\Gamma_{cl}[\bar\Phi]$ in a graphical expansion, and noting that we got tree graphs; here we have proved it by a general analysis of graphs for the theory.

Actually this proof is not watertight - it was discovered in 2004 that in the case where there are massless fields in the theory (eg., in QED) it is possible for the classical and quantum contributions to mix together at the same order in a loop expansion.* This arises when there are non-analytic terms in the internal lines which yield square-root infrared singularities. - which happens when there are two or more massless propagators in the diagram. This topic is outside the scope of these lectures.

FUNCTIONAL INTEGRAL ARGUMENT: In the path integral formulation of QM, we have seen in part A how one can extract the leading correction to the classical contribution to the 1-particle propagator, in the form of a prefactor or "fluctuation determinant. What we wish to do here is see how this works for a theory like $\phi^4$ theory; we also want to see the structure of the whole asymptotic expansion, not just the first correction.

As a warm-up, let's do the problem for ordinary QM, but now going beyond the 1st correction. Thus, we consider the amplitude

$$\left.\begin{aligned} G_0(2,1) &= \int_1^2 \mathcal{D}q(t)\, e^{\frac{i}{\hbar}S[q]} \\ &= \int_1^2 \mathcal{D}q(t)\, e^{\frac{i}{\hbar}\int_1^2 dt\,[\frac{1}{2}m(\dot q^2 - \omega^2 q^2) - V(q)]} \end{aligned}\right\} \qquad (75)$$

where $V(q)$ includes the non-quadratic part of the potential, and the coordinate $q$ is 1-d.

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* See, eg., B.R. Holstein, J.F. Donoghue, Phys. Rev. Lett. 93, 201602 (2004); and S.J. Brodsky, P. Hoyer, Phys. Rev. D83, 045026 (2011).

Then, in a simple generalization of the usual procedure for extracting the fluctuation determinant, we write

$$q(t) = Q_0(t) + x(t) \tag{76}$$

where $Q_0(t)$ is the classical trajectory, and $x(t)$ the deviation from this trajectory; We then have

$$G(2,1) = A(2,1)\, e^{\frac{i}{\hbar} S_{cl}[Q_0]} \tag{77}$$

where $S_{cl}[Q_0]$ is the classical action, and the prefactor $A(2,1)$ is

$$A(2,1) = \int_{\bar{x}(t_1)=0}^{\bar{x}(t_2)=0} \mathcal{D}\bar{x}(t)\, \exp\left\{-i \int_{t_1}^{t_2} dt \left[\frac{m}{2}\bar{x}(t)\left(\frac{d^2}{dt^2} + \omega_0^2\right)\bar{x}(t) + \sum_{n=3}^{\infty} \frac{\alpha_n}{n!}\, \bar{x}^n(t)\right]\right\} \tag{78}$$

where the coefficient $\alpha_n$ is

$$\alpha_n = \hbar^{\frac{n}{2}-1} \left.\frac{d^n V(\bar{x})}{d\bar{x}^n}\right|_{\bar{x}=0} \tag{79}$$

so that we have connected the higher terms with higher powers of $\hbar$; since only terms of even order contribute, this becomes an expansion in integer powers of $\hbar$ (NB: rescaled $x(t)$ to $\bar{x}(t) = x(t)/\hbar^{1/2}$).

If we ignore the higher terms in $\hbar$, including only the leading correction to the classical result, then we find

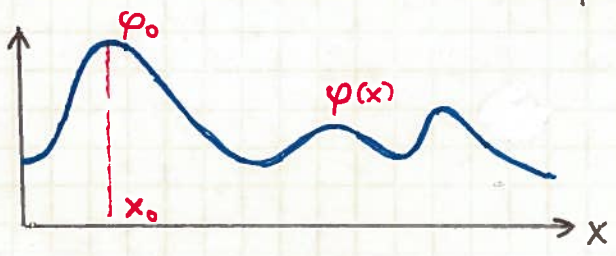
$$\left.\begin{aligned} A(2,1) &= e^{-\frac{1}{2} \ln \det \left| m\left(\frac{d^2}{dt^2} + \omega_0^2\right)\right|} \\ &= \left(\det \left| m\left(\frac{d^2}{dt^2} + \omega_0^2\right)\right|\right)^{-1/2}. \end{aligned}\right\} \tag{80}$$

where as usual the determinant refers to the product of eigenvalues of the differential operator (later on we will rewrite this determinant in terms of a generalized zeta function).

In what follows I will not calculate in full the form of the higher corrections, but simply indicate their form. This can be understood by considering the asymptotic expansion of the integral

$$I(\hbar) = \int_{-\infty}^{\infty} dx\, e^{-\varphi(x)/\hbar} \tag{81}$$

where we assume that the function $\varphi(x)$ has an absolute maximum at a value $x = x_0$, given by $\varphi(x_0) = \varphi_0$. Writing $z = x - x_0$, we then make the expansion




$$\varphi(z) = \varphi_0 + \sum_{n=2}^{\infty} \varphi_n z^n \tag{82}$$

using Laplace's method ( which is just a special case of the method of steepest descents); this would be produced from (78) by rotating to Euclidean space, and specializing to a 1-d integral). Then standard methods give us the result

$$\left.\begin{aligned} I(\hbar) &= e^{-\varphi_0/\hbar} \int_{-\infty}^{\infty} dz\, e^{-\frac{\varphi_2}{2\hbar} z^2}\, e^{-\frac{1}{\hbar}\sum_{n=3}^{\infty} \frac{\varphi_n}{n!} z^n} \\ &= \left(\frac{2\pi\hbar}{\varphi_2}\right)^{1/2} e^{-\varphi_0/\hbar} \int_{-\infty}^{\infty} \frac{d\bar{z}}{\sqrt{2\pi}}\, e^{-\bar{z}^2/2}\, e^{-\sum_{n=3}^{\infty} \frac{1}{n!} \frac{\varphi_n}{(\varphi_2)^{n/2}} \hbar^{\frac{n}{2}-1} \bar{z}^n} \end{aligned}\right\} \quad (83)$$

which when expanded out gives

$$\ln I(\hbar) = -\left\{ \varphi_0/\hbar + \tfrac{1}{2} \ln\left(\frac{\varphi_2}{2\pi\hbar}\right) + \frac{\hbar}{\varphi_2}\left[\frac{1}{8}\frac{\varphi_4}{\varphi_2} - \frac{5}{24}\left(\frac{\varphi_3}{\varphi_2}\right)^2\right] + O(\hbar^2) \right\} \quad (84)$$

which for a path integral is generalized to infinite dimensions.

Let us now return to $\phi^4$ theory. The classical solution was already discussed in section B.3 ; again making a Euclidean rotation, we have the classical solution $\varphi_0(x)$ satisfying the eqtn. of motion

$$(\partial^2 - m^2)\,\Phi_0(x) - \frac{g}{3!}\Phi_0^3(x) + J(x) = 0 \quad (85)$$

(cf. section B.3, eqtn (29) ; we write $\Phi_0(x)$ here in place of the $\bar{\phi}(x)$ used there).

Now making the expansion in the deviation $\psi(x) = \phi(x) - \Phi_0(x)$, we have

$$\left.\begin{aligned} S[\phi, J] = S_{cl}[\Phi_0, J] + \int d^4x \Big\{ &\tfrac{1}{2}\left[(\partial\psi)^2 + \left(m^2 + \frac{g\Phi_0}{2}\right)\psi^2\right] \\ &+ \frac{g}{3!}\Phi_0\,\psi^3 + \frac{g}{4!}\psi^4 \Big\} \end{aligned}\right\} \quad (86)$$

where the terms linear in $\psi(x)$ vanish because of the eqtn of motion (85). We now expand the action in powers of $\hbar$, in exactly the same way as we have done above. The result can be written as

$$e^{-S[\phi,J]/\hbar} = N^{-1} e^{-S_{cl}[\Phi_0,J]} \int \mathcal{D}\psi\, e^{-\Delta S[\psi]} \quad (87)$$

where the classical action is

$$S_{cl}[\Phi_0, J] = \int d^4x \left[ J(x)\Phi_0(x) - \tfrac{1}{2}\left((\partial\Phi_0)^2 + m^2\Phi_0^2\right) - \frac{g}{4!}\Phi_0^4 \right] \quad (88)$$

in terms of the solution $\Phi_0(x)$ to the classical eqtn of motion (85), with the non-zero source field $J(x)$ included; where the normalization factor is

$$N = \int \mathcal{D}\psi \; e^{-\frac{1}{\hbar}\int d^4x \left[ \frac{1}{2}\left((\partial\psi)^2 + m^2\psi^2\right) + \frac{g}{4!}\psi^4 \right]} \tag{89}$$

and the shift in the action is, from (86)

$$\Delta S[\psi] = \frac{1}{\hbar}\int d^4x \left\{ \frac{1}{2}\left[(\partial\psi)^2 + \left(m^2 + g\frac{\Phi_0}{2}\right)\psi^2\right] + \frac{g}{3!}\Phi_0\psi^3 + \frac{g}{4!}\psi^4 \right\} \tag{90}$$

Making the same rescaling, viz., $\bar{\psi} = \hbar^{1/2}\psi$, the expansion in powers of $\hbar$ becomes explicit; we get

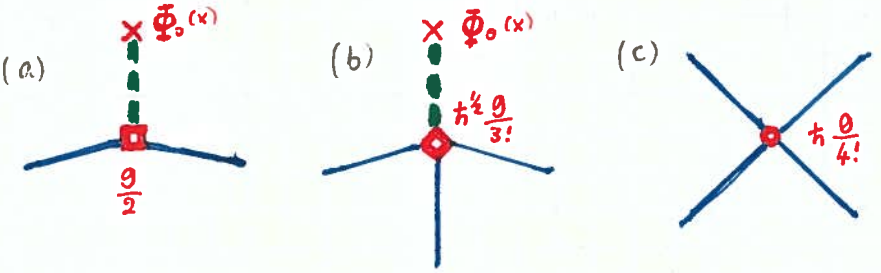
$$\left.\begin{aligned} \Delta S[\psi] &= \int d^4x \left\{ \frac{1}{2}\left[(\partial\psi)^2 + \left(m^2 + g\frac{\Phi_0}{2}\right)\psi^2\right] + \hbar^{1/2}\frac{g}{3!}\Phi_0\psi^3 + \hbar\frac{g}{4!}\psi^4 \right\} \\ N &= \int \mathcal{D}\psi \; \exp\left\{ -\int d^4x \left[ \frac{1}{2}\left((\partial\psi)^2 + m^2\psi^2\right) + \hbar\frac{g}{4!}\psi^4 \right] \right\} \end{aligned}\right\} \tag{91}$$

The 1st correction to the classical action is obtained by what we now familiar methods, and we get the result we already know (eqtns (107) and (108) of section B.3), by doing the Gaussian integral. Thus we get

$$S[\phi, J] = S_{cl}[\Phi_0, J] + \frac{\hbar}{2}\ln\left[\frac{\det\left(\partial^2 + m^2 + \frac{1}{2}g\Phi_0^2\right)}{\det\left(\partial^2 + m^2\right)}\right] + O(\hbar^2) \tag{92}$$

so that the role of the "frozen field" $V_0(x)$ in section B.3 is now played by the classical field $\Phi_0(x)$; otherwise everything we did from eqtns (109) - (114) in section B.3 can be applied directly here. Note in particular the x-dependent effective mass, so that $m^2 \to m^2 + \frac{1}{2}g\Phi_0(x)$.

There are however some key differences, notably in the interaction terms in $\Delta S[\psi]$; we have a new coupling $\propto \psi^3$, and in fact the theory boasts 3 different interaction vertices, as shown in the figure.




(93)

The rules for the number of loops and vertices are different here; if $V_3$ is the number of vertices involving $\psi^3$, and $V_4$ those involving $\psi^4$, we find that the number of loops is

$$L = 1 + \frac{1}{2}V_3 + V_4 \tag{94}$$

Now all of this is only as good as our classical solution $\Phi_0(x)$, since this enters as the "background field". If we know $\Phi_0(x)$, then the

problem becomes one of a self-interacting "quantum fluctuation field" $\psi(x)$, coupled also to a dynamical perturbation $\phi_0(x)$, which affects its mass and adds a new vertex $\propto \psi^3$.

Unfortunately it is hard to find $\Phi_0(x)$; its eqtn of motion (85) is non-linear, and we have seen how difficult such problems are, even for a few interacting particles. Suppose, however, we could find it - what then? Then the fluctuations $\psi(x)$ can, at first approximation, by dropping the interaction terms, since they are higher-order in $\hbar$. Then, with only the vertex (3) in (93) left, we can try solving for the problem - the results being encapsulated in (92).

More generally, we can ask - how good is this semiclassical starting point? This is a hard question to answer in relativistic QFT, since we can't easily modify the fields or the couplings. However one can study toy models, and/or use numerical approaches.

In condensed matter systems it is a quite different story. One can modify the couplings (using external fields, pressure, etc.). And we can even modify the fields: for example, in the FQHE (Fractional Quantum Hall Effect) one modifies the underlying quantum fields by changing the external magnetic field, and in spin systems, one can just change the spin. The key message that emerges is that in many condensed matter systems, the real behaviour can be very far from classical. In other words, the quantum field is very different from the classical field.

# B.6.1 (c) QUANTUM SPIN FLUCTUATIONS

Spin systems teach us many interesting lessons — we shall see some here, and more later on. One can write down a non-relativistic field theory for various spin systems — some of these turn out to be closely related to important relativistic QFT's. Remarkably, given that spin has no classical limit, some of these theories are very well described by the classical limit (eg., ferromagnetic systems); others not at all (eg., 1-dimensional antiferromagnetic systems).

Here we will look at both of these cases, and also at how the theory relates to both the real experimental world and to relativistic QFT. The functional formulation turns out to be very powerful, but other methods are also useful. This is a huge subject, and we will only scratch the surface here.

**(i) GENERAL REMARKS :** Before beginning, it is useful to say a little more about the functional formulation of spin dynamics, and how one can represent fluctuations about the classical state.

**SINGLE SPIN DYNAMICS :** We start with a single spin, where as we saw in eqtn. (57), the spin propagator can be written as a path integral; between 2 arbitrary states we have

$$\langle \psi_2 | \hat{G}(t_2,t_1) | \psi_1 \rangle = \int d\underline{n}_1 \int d\underline{n}_2 \langle \psi_2 | \underline{n}_2 \rangle \langle \underline{n}_2 | \hat{G}(t_2,t_1) | \underline{n}_1 \rangle \langle \underline{n}_1 | \psi_1 \rangle \qquad (95)$$

with $\langle \underline{n}_2 | \hat{G}(t_2,t_1) | \underline{n}_1 \rangle$ given in (57). So then - what is the classical limit here?

Let's begin by noting that the first "kinetic" term in the Lagrangian (58) is proportional to $\hbar$. Then it follows that

$$\left.\begin{aligned} G(\underline{n}_2,\underline{n}_1;t_2t_1) &= \int_1^2 \mathcal{D}n(t)\, e^{iS\int_1^2 d\underline{n}\cdot\underline{A}}\, e^{-\frac{1}{2}\int_1^2 dt\, \mathcal{H}(S\underline{n})} \\ &\equiv \int_1^2 \mathcal{D}n(t)\, e^{iS\omega_{21}[\underline{n}]}\, e^{-\frac{1}{2}\int_1^2 dt\, \mathcal{H}(S\underline{n})} \end{aligned}\right\} \quad (96)$$

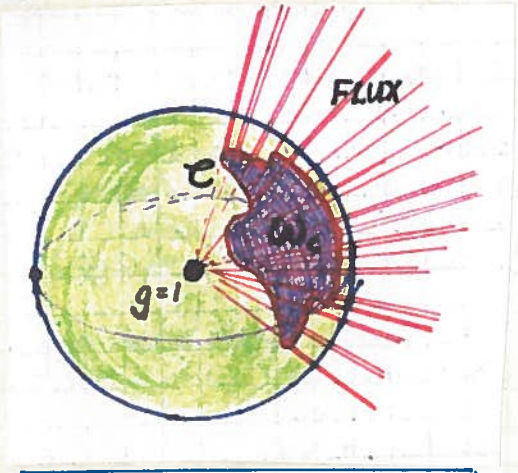
where the number $\omega_{21}[\underline{n}]$ is defined as

$$\omega_{21}[\underline{n}] = \int_{\underline{n}_1}^{\underline{n}_2} d\underline{n}(t)\cdot\underline{A} \quad (97)$$

and is just the contribution from the field of the monopole in the Bloch sphere - the phase is thus completely analogous to the Aharonov-Bohm phase we saw in Chapter 4. To make the analogy complete, let's imagine that the vector $\underline{n}(t)$ accomplishes a circuit, ie., that the spin rotates along some path and then comes back to where it started. One simple way to do this is by making the spin precess, ie., to apply a constant field $\underline{B}_0$ to the spin, so that the Hamiltonian is

$$\left.\begin{aligned} \mathcal{H} &= -\gamma\, \underline{B}_0\cdot\underline{S} \\ &\equiv -\gamma S\, \underline{B}_0\cdot\underline{n} \end{aligned}\right\} \quad (98)$$

which is a simple problem to analyze; but there are other more interesting things one can do, as we will see in the next sub-section.




The Bloch sphere, with monopole at its centre; the enclosed flux by a curve $\mathcal{C}$ is the solid angle $\Omega_C$

If we allow the spin to accomplish a complete circuit $\mathcal{C}$, then the gauge-dependent and path-dependent number $\omega_{21}[\underline{n}]$ now becomes a topological phase (using a suggestive notation for an integration over circuits):

$$\left.\begin{aligned} G(\underline{n},\underline{n};t) &= \oint_{\underline{n}(0)}^{\underline{n}(t)} \mathcal{D}n'(t)\, e^{iS\omega_{\mathcal{C}}[\underline{n}']}\, e^{-\frac{1}{2}\int_0^t dt\, \mathcal{H}(S\underline{n}')} \\ &\equiv \oint \mathcal{D}\mathcal{C}(\tau)\, e^{iS\omega_C}\, e^{-\frac{1}{2}\int d\tau\, \mathcal{H}(S\underline{n}(\mathcal{C}(\tau)))} \end{aligned}\right\} \quad (99)$$

where the phase is just a Berry phase $\Phi_B$, ie.

$$\Phi_B = S\omega_C \quad (100)$$

and we can think of $\Omega_C$, which is just the solid angle enclosed by the circuit $C$, as a measure of the flux through the circuit (compare eqtn. (60)). Then we have the precise analogy with the Aharonov-Bohm effect (compare eqtns (2) and (36) in chapter B.4).

Now to get a classical limit for this, we need to take a large spin, ie., we need to minimize the action

$$S[\underline{n}] = \int_{t_1}^{t_2} dt \left[ \hbar S \underline{A} \cdot \dot{\underline{n}}(t) - \mathcal{H}(S\underline{n}) \right] \tag{101}$$

in the limit $S \gg 1$. Thus we are looking for a path $\underline{n}_0(t)$ on the Bloch sphere which is the solution to

$$\delta S[\underline{n}, \dot{\underline{n}}]\Big|_{\underline{n} \to \underline{n}_0} = 0 \tag{102}$$

Expanding out (100), we find:

$$\delta S = \hbar S \int_{t_1}^{t_2} d\tau \left\{ \left[ \left( \frac{\partial A^\alpha}{\partial n^\beta} \delta n^\beta \right) \dot{n}^\alpha + A^\alpha \frac{d}{dt} \delta n^\alpha \right] \Big|_{\underline{n} = \underline{n}_0} - \frac{\partial H}{\partial n^\alpha} \delta n^\alpha \Big|_{\underline{n} = \underline{n}_0} \right\} \tag{103}$$

and the first topological term in this can be rewritten, adding and subtracting a term $(\partial A^\alpha / \partial n^\beta) \dot{n}_\beta \delta n^\alpha$, to give

$$\begin{aligned} \delta S_{Top} &= \delta \int_{t_1}^{t_2} d\tau \, \hbar S \underline{A} \cdot \dot{\underline{n}}(\tau) \Big|_{\underline{n} = \underline{n}_0} \\ &= \hbar S \int_{t_1}^{t_2} d\tau \left\{ \frac{\partial A^\alpha}{\partial n^\beta} \left[ \dot{n}^\alpha \delta n^\beta - \dot{n}_\beta \delta n^\alpha \right] \Big|_{\underline{n} = \underline{n}_0} + \left[ A^\alpha \frac{d}{dt} \delta n^\alpha + \frac{\partial A^\alpha}{\partial n^\beta} \dot{n}_\beta \delta n_\alpha \right] \right\} \\ &= \hbar S \int_{t_2}^{t_1} d\tau \left[ \epsilon_{\alpha\beta\gamma} \frac{\partial A^\alpha}{\partial n_\beta} (\dot{\underline{n}} \times \delta \underline{n})^\gamma + \frac{d}{dt} (A^\alpha \delta n_\alpha) \right] \Big|_{\underline{n} = \underline{n}_0} \end{aligned} \tag{104}$$

Now the 2nd term in this last result is zero, since $\delta \underline{n} = 0$ at the end-points; and the 1st term is rewritten using (59) to give

$$\delta S = \int_{t_1}^{t_2} d\tau \left[ \hbar S \epsilon_{\alpha\beta\gamma} n^\alpha \dot{n}^\beta - \frac{\partial \mathcal{H}}{\partial n^\gamma} \right] \Big|_{\underline{n} = \underline{n}_0} \delta n^\gamma \tag{105}$$

so that we get the equation of motion for $\underline{S}_{cl} = \hbar S \underline{n}_0$ from (102) as

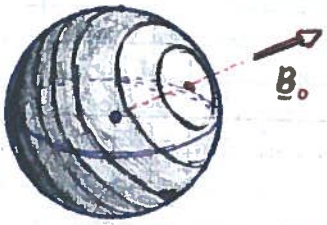
$$\dot{\underline{S}}_{cl} \equiv \underline{n}_0 \times \frac{\partial \mathcal{H}}{\partial \underline{S}_{cl}} \equiv \underline{S}_{cl} \times \frac{\partial \mathcal{H}}{\partial \underline{S}_{cl}} \tag{106}$$

which is that in (47) (putting back the factor $\gamma$).

What of fluctuations around this classical limit? We see from what we're just done that we've achieved the classical limit, not by letting $\hbar \to 0$, but instead by letting $S \to \infty$. Thus we must handle fluctuations in a "$1/S$-expansion" about

the classical path $S_{cl}$; we do this later in this chapter, when we look at $1/N$ expansions.

Before leaving the single spin for the moment, let's look at 2 other striking features of its behaviour. First, let's consider the precessional motion noted in the context of the simple Zeeman Hamiltonian (98). Suppose we think of this problem in the path integral language we have developed here. Then we see that we are dealing with a particle of charge $S$ moving on the surface of a sphere which has equipotentials like those shown in the figure; and in the field of a unit central monopole. Now notice that the motion in the presence of the monopole is completely different from what it would be without it, whether we deal with the limit $S \gg 1$ or not. Without the monopole, the particle would simply move accross the equipotentials (and settle at lowest point, in the direction of $\underline{B}_0$, provided there was a dissipative mechanism); with no dissipation it would simply oscillate around this direction. However the monopole completely changes - now it moves along the equipotentials, at least in the classical limit, as (106) shows. For small $S$, we shall see that paths going all over the Bloch sphere are allowed - but even a spin-$\frac{1}{2}$ system shows perfect precessional motion, as NMR shows. We will come back to this point.



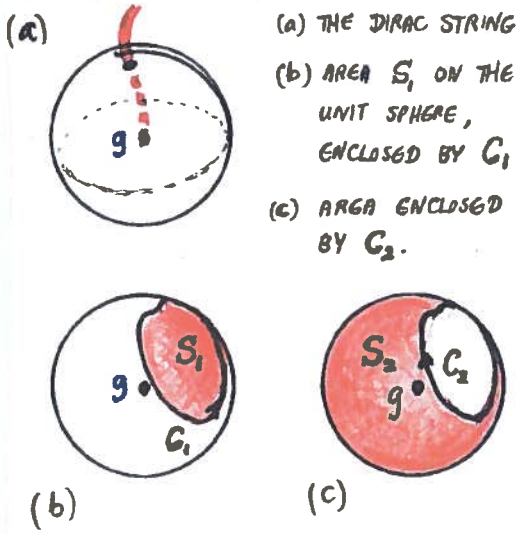

Equipotentials on the Bloch sphere for a spin in a field $\underline{B}_0$, constant in time.

The second striking feature comes when we compare integer and $\frac{1}{2}$-integer spins. The key point - and it is elementary - is that the solid angle on the Bloch sphere is only defined modulo $4\pi$. This is shown in the figure - an area is ambiguous, and can have either sign. Thus we have invariance of all physical quantities under the transformation

$$\omega_c \rightarrow \omega_c + 4\pi n\pi \qquad (107)$$

where $n = \pm 1, \pm 2$, etc. But it then follows that

$$e^{4\pi i n S} = 1 \quad \Longrightarrow \quad S = \begin{matrix} n \\ n+\frac{1}{2} \end{matrix} \qquad (108)$$




ie., that the only allowed values of $S$ are integer or half-integer. This argument does not of course tell us the connection between spin and statistics.

## LATTICE OF SPINS:

Let's now consider a lattice of identical spins, with spin $S$ each. In the real world there are 1-d, 2-d, and 3-d lattices of this kind, as well as interesting hybrids. Study of these is a huge field, partly because the variety of Hamiltonians is very large. We will only do a few things here, which are of

interest in a general field-theoretical context. We first take the lattice Hamiltonian and look at it in a long-wavelength continuum limit - this is done heuristically rather than rigourously. It is then shown how one can very easily derive simple field theories for both ferromagnetically & antiferromagnetically coupled spin systems, in a path integral framework. The models turn out to be well known in relativistic QFT, and have played a key role in strong interaction physics over the years.

To begin with let's consider a very simple isotropic exchange Hamiltonian, & how to describe the classical limit and the small fluctuations around it. This will done first of all in a condensed matter style, and then redone in a more field-theoretical style. I will keep the details reasonably brief.

We consider an isotropic exchange Hamiltonian, for a simple lattice of spins, which we will take to be a cubic lattice (in 3d) or a square lattice (in 2d). The interactions are between nearest neighbours; the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \underline{S}_i \cdot \underline{S}_j \tag{109}$$

where $\sum_{\langle ij \rangle}$ signifies a sum over nearest-neighbour sites, and we have 2 cases, viz., (i) the ferromagnetically-coupled (FM) case $J > 0$, and (ii) the antiferromagnetic (AFM) case $J < 0$. The action for the system is

$$S[\{\underline{n}_j\}] = \hbar S \sum_j \omega_{21}^j [\underline{n}_j] + JS^2 \int_1^2 dt \sum_{\langle ij \rangle} \underline{n}_i(t) \cdot \underline{n}_j(t) \tag{110}$$

where the vector potential $\underline{A}_j = \underline{A}$ is assumed to be the same for each site, and so

$$\omega_{21}^j [\underline{n}_j] = \int_1^2 d\underline{n}_j(t) \cdot \underline{A} \tag{111}$$

for the topological phase associated with the $j$-th site. The initial and final states are different for each spin, in general; we have therefore an amplitude $G(2,1)$ of form.

$$G(2,1) = \prod_{j=1}^{N} G_j(2,1) = \prod_{j=1}^{N} \int_{\underline{n}_j(t_1)}^{\underline{n}_j(t_2)} \mathcal{D}\underline{n}_j(t)\, e^{\frac{i}{\hbar} S[\{\underline{n}_j\}]} \tag{112}$$

and the vacuum-to-vacuum amplitude is

$$G(0,0) = \prod_{j=1}^{N} (2S+1)^{2N} \int \frac{d\underline{n}_j(t_2)}{4\pi} \int \frac{d\underline{n}_j(t_1)}{4\pi} \langle 0 | \underline{\mathbf{n}}(t_2) \rangle \langle \underline{\mathbf{n}}(t_1) | 0 \rangle \int_{\underline{n}_j(t_1)}^{\underline{n}_j(t_2)} \mathcal{D}\underline{n}_j(t)\, e^{\frac{i}{\hbar} S} \tag{113}$$

where $|0\rangle_N$ is the ground state of the entire system, and $|\underline{\mathbf{n}}(t)\rangle \equiv \prod_{j=1}^{N} |\underline{n}_j(t)\rangle$.

For the 2 states (FM and AFM) that we will look at, it is a fairly simple matter to guess what the relevant classical states must be; and it is also pretty simple to calculate the quadratic fluctuations we around these

classical states. We will do this in two ways:

(a) Using Holstein-Primakoff operators. These are described in Appendix B.5; for our purposes we simply need to know that once we have defined the relevant classical state, we defined the spin fluctuations around it using a set of operators $\{b_j, b_j^+\}$, which are bosons, and defined at each site. In the simplest FM case, where all spins are parallel along the z-axis in the classical ground state, we have $\underline{S}_j^{(cl)} = S\hat{\underline{z}}, \forall j$, and the fluctuations are

$$\left.\begin{aligned} S_j^z &= S - b_j^+ b_j \\ S_j^+ &= (2S)^{1/2}\left[1 - \frac{1}{2S} b_j^+ b_j\right]^{1/2} b_j \\ S_j^- &= (2S)^{1/2} b_j^+ \left[1 - \frac{1}{2S} b_j^+ b_j\right]^{1/2} \end{aligned}\right\} \quad (114)$$

where we notice the square-root reduction (this because $S_z^2 + S_x^2 + S_y^2 = S^2$) in $S_j^\pm$). The representation for the AFM case is a little more complicated, but basically the same - it is described below.

For a uniform lattice system it is more useful to Fourier transform these relations; then, eg.,

$$S^z(\underline{r}_j) = S - \frac{1}{V}\sum_{\underline{k}\,\underline{q}} e^{-i\underline{q}\cdot\underline{r}_j} b_{\underline{k}+\underline{q}}^+ b_{\underline{k}} \quad (115)$$

and so on; here $V$ is the system volume. One can then rewrite that Hamiltonian in this language, and since we already have the classical state, it is merely a question of diagonalizing the Hamiltonian to find the fluctuation spectrum — we do this below.

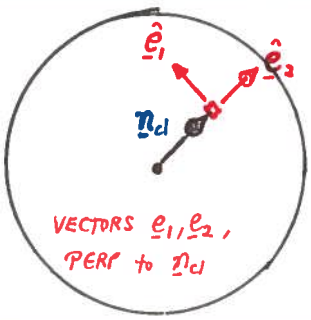
(b) Using Path Integrals. In this case we must start again from (110), and divide the vectors $\{\underline{n}_j(t)\}$ into their classical part and the fluctuation part — we will therefore write

$$\underline{n}_j(t) = \underline{n}_j^0(t)\left[1 - \underline{\phi}_j^2\right]^{1/2} + \sum_{\alpha=1,2} \hat{\underline{e}}_{j\alpha}(t)\phi_j^\alpha(t) \quad (116)$$

or, in a continuum approximation

$$\underline{n}(x) = \underline{n}_0(x)\left[1 - \underline{\phi}^2(x)\right]^{1/2} + \hat{\underline{e}}(x)\cdot\underline{\phi}(x) \quad (117)$$

where the pair of vectors $\underline{e}_1(x), \underline{e}_2(x)$ is perpendicular to the local axis $\underline{n}_0(x)$ of the spin, defined by the classical theory (see figure). The latter formulation in (117) is deliberately written in relativistic notation - the field $\underline{\phi}(x) = (\phi_1(x), \phi_2(x), \phi_3(x))$ describes small fluctuations around the classical field $\underline{n}_0(x)$; here $x = (\underline{r}, t)$ is the $(D+1)$-dimensional space in which the spin system exists. We write the gradients of these fields as




$$\left.\begin{aligned} \partial_\mu \underline{n}_{cl}(x) &= A_\mu^\alpha \hat{\underline{e}}_\alpha \\ \partial_\mu \hat{\underline{e}}^\alpha(x) &= \Gamma_\mu^{\alpha\beta}\hat{\underline{e}}_\beta - A_\mu^\alpha \underline{n}_0 \end{aligned}\right\} \quad (118)$$

where we see that the gradient of $\underline{n}_{cl}(x)$ has to be perpendicular to $\underline{n}_{cl}(x)$ itself (because $|\underline{n}(x)| = 1$ everywhere).

Note that the vector field $\underline{\phi}(x)$ is just a vector analogue to the fluctuation field $\psi(x)$ we dealt with already, in discussing the fluctuations $\psi(x)$ around the classical solution $\phi_{cl}(x)$ of the scalar field system. Here we have made it a 2-d vector, moving on the Bloch sphere. But one can also consider a quite general $N$-dimensional vector field $\underline{n}(x)$, and have it move on an $(N-1)$-dimensional hypersphere, and consider fluctuations $\underline{\phi}(x)$ on this hypersphere. Then the field $\underline{\phi}(x)$ is an $O(N-1)$ vector field; eqtns. (116) - (118) go through as before, except that now the sum $\alpha = 1, 2, \ldots N-1$ in (116).

Consider now how we might write the path integrals for this theory. It is convenient to take a specific form for the Lagrangian, so we choose the continuum form of (110), expanded to lowest order in gradients; the simplest relativistic analogue of this reads, for the generating functional $Z[\underline{J}]$ with a current $\underline{J}(x)$ coupling to $\underline{n}(x)$, as (dropping a constant $JS^2N$):

$$Z[\underline{J}(x)] = \int \mathcal{D}\underline{n}(x)\, \delta(\underline{n}^2(x)-1)\, e^{\frac{i}{\hbar}\int d^4x \left( \left[ \frac{1}{\hbar} S\omega[\underline{n}(x)] + \frac{JS^2}{2}(\partial_\mu n(x) \partial^\mu n(x)) \right] - \underline{J}(x)\cdot n(x) \right)} \tag{119}$$

where the $\delta$-fn is inserted to keep $n^2(x) = 1$ everywhere. Now we can rewrite this path integral in 4 different interesting ways:

- We can deal with the $\delta$-fn by introducing a new field $\lambda(x)$, and performing a functional Fourier transform to get

$$Z[J(x)] = \int \mathcal{D}\lambda(x) \int \mathcal{D}\underline{n}(x)\, e^{\frac{i}{\hbar}\int d^4x \left( \frac{1}{\hbar} S\omega[n(x)] + \frac{JS^2}{2}(\partial_\mu n \partial^\mu n) + \lambda(x)(n^2(x)-1) - \underline{J}(x) n(x) \right)} \tag{120}$$

so that we have a quadratic form for the Lagrangian, we a new field which is basically behaving like a Lagrange multiplier.

- We can also separate out the classical field $\underline{n}_0(x)$ from the fluctuation around it, and then integrate separately over the 2 fields. From (118) we see that we can eliminate the classical field entirely from the exchange term in the action, because $\underline{n}_0$ is eliminated in the calculation of $\partial_\mu \underline{n} \partial^\mu \underline{n}$:

$$\frac{JS^2}{2}\int d^4x\, (\partial_\mu \underline{n}\, \partial^\mu \underline{n}) = \frac{JS^2}{2}\int d^4x \left\{ \left[ (1-\underline{\phi}^2)\,\delta^\beta_\alpha + \phi_\alpha \phi^\beta \right] A^\alpha_\mu A^\mu_\beta + (\partial_\mu \phi^\alpha - \Gamma^{\alpha\beta}_\mu \phi_\beta)(\partial^\mu \phi^\alpha - \Gamma^{\mu}_{\alpha\beta}\phi^\beta) \right\} \tag{121}$$

so that we can reduce the functional integral over the exchange term to one over the $A^\alpha_\mu$ and $\Gamma^{\alpha\beta}_\mu$ fields. This is less useful if we also need to do the full integral over $\underline{n}$ in the other terms, but if we are only interested in the fluctuations, it gives a useful alternative.

- We can use the $CP^N$ representation. Consider the set of complex variables $\underline{z}(x) = (z_1(x), z_2(x) \ldots z_N(x))$, with the constraint that

$$|\underline{z}^2| = \sum_{j=1}^{N} |z_j(x)|^2 = 1. \tag{122}$$

If we think of $\underline{z}(x)$ as a field, we can make the assumption that is invariant under local gauge transformations, ie., invariant under $\underline{z}(x) \to \underline{z}(x) e^{i\varphi(x)}$. We then make the standard manouevre, viz., write a free field Lagrangian

$$S_0 = \frac{1}{g_0^2} \int d^4x \, (\partial_\mu + i A_\mu) \underline{z}^\dagger (\partial^\mu - i A^\mu) \underline{z} \tag{123}$$

where $\underline{z}^\dagger(x)$ is the Hermitian conjugate of $\underline{z}(x)$; and the theory is invariant under the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu \varphi(x)$. Note that there is no kinetic term for $A_\mu(x)$ in this action, so we can eliminate $A^\mu(x)$ in the classical limit by minimizing the action. This just gives

$$\frac{\delta}{\delta A_\mu} \left[ (\partial_\mu \underline{z}^\dagger)(\partial^\mu \underline{z}) - i \left( A_\mu \underline{z}^\dagger \partial^\mu \underline{z} - A^\mu \underline{z} \partial_\mu \underline{z}^\dagger \right) + A_\mu A^\mu \right] = 0 \tag{124}$$

so that

$$A_\mu(x) = -\frac{i}{2} \left( \underline{z}^\dagger \partial_\mu \underline{z} - \underline{z} \partial_\mu \underline{z}^\dagger \right) \tag{125}$$

We now make the connection with the $O(3)$ $\underline{n}$-field; letting $N = 2$ in our $CN^N$ model, we write

$$\left.\begin{aligned} \underline{n}(x) &= \underline{z}^\dagger \underline{\tau} \underline{z} \equiv z_\alpha^\dagger \tau^{\alpha\beta} z_\beta \\ &\equiv (z_1^\dagger, z_2^\dagger) \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}\right\} \tag{126}$$

so that the $z_\alpha$ now are components of a spinor, with $\underline{\tau}$ the Pauli vector matrix; we then have

$$\frac{Js^2}{2} \int d^4x \, (\partial_\mu n \, \partial^\mu n) = 2Js^2 \int d^4x \left[ (\partial_\mu + i A_\mu) \underline{z}^\dagger (\partial^\mu - i A^\mu) \underline{z} \right] \tag{127}$$

with $A_\mu$ again satisfying (125).

It is interesting to note what form the field "curvature" tensor $F_{\mu\nu}$ takes in this representation; we have

$$\left.\begin{aligned} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu &= \underline{n} \cdot (\partial_\mu n \times \partial_\nu \underline{n}) \\ &\equiv \epsilon_{\alpha\beta\gamma} n^\alpha \partial_\mu n^\beta \partial_\nu n^\gamma \end{aligned}\right\} \tag{128}$$

which, as we will now show, is just a topological charge density for the

$\underline{n}$-field. We define the scalar quantity

$$Q = \frac{1}{4\pi}\int d^2x\, \Phi(x) = \frac{1}{4\pi}\int d^2x\, \epsilon^{\mu\nu} F_{\mu\nu}(x)$$
$$= \frac{1}{4\pi}\int d^2x\, \epsilon^{\mu\nu}\, \underline{n}(x)\left(\partial_\mu \underline{n}(x) \times \partial_\nu \underline{n}(x)\right) \qquad (129)$$

which is the "winding number of a sphere onto a sphere"; for any smoothly-varying $\underline{n}(x)$, it will be an integer. $\Phi(x)$ is called "Pontryagin density", and $\Phi$ the Pontryagin index, by mathematicians. We see that if there was a kinetic term for the gauge field of the $\underline{n}$-vector, it would be

$$T \sim \int d^2x\, F_{\mu\nu}(x) F^{\mu\nu}(x) = \Phi^2 \qquad (130)$$

Note that we can also write the topological or Berry phase term in a simple way in this CPN language. It is easy to see that if we write this in the form

$$S_B = i\hbar S \sum_j \omega^j[\underline{n}_j]$$
$$= i\hbar S \sum_j \int dt\, \underline{A}\cdot\frac{d\underline{n}_j}{dt} \longrightarrow i\hbar S \int dt \sum_j \underline{z}_j^\dagger \frac{d}{dt}\underline{z}_j \qquad (131)$$

Let us, for example, pick a form for $\underline{z}_j$ in $CP_2$ that is in the same gauge as we picked before, with the Dirac string through the north pole; then we have

$$\underline{z}_j(t) = \begin{pmatrix} z_j^+(t) \\ z_j^-(t)\end{pmatrix} = \begin{pmatrix} \cos\theta_j/2\, e^{-i\phi_j} \\ \sin\theta_j/2\end{pmatrix} \qquad (132)$$

and it is easily verified that (126) is correct.

Note, incidentally, the relationship between the complex 2-component vector $\underline{z}$ and the quantity $z = e^{i\phi}\tan\theta/2$ that appears in the definition of coherent states for spin-$\frac{1}{2}$ (cf. App. B5). We see that $z = z_-/z_+$, as we would expect since $z$ refers to the Riemann projection of the $\underline{n}$-vector.

Let's now summarize all that we have found for the different representations of the path integral for a spin system. We suppose that we start with a Hamiltonian $\mathcal{H}(\{\underline{s}_j\})$, which could be of simple exchange type, or more complicated. We can then represent the system in the following ways.

(i) Using the Holstein-Primakoff expansion in (114).

(ii) In terms of the $\underline{z}$-fields, or the related coherent spin states, with an action given by

$$S[\{\underline{z}_j^\dagger, \underline{z}_j\}] = \sum_j \int dt \left[i\hbar S\, \underline{z}_j^\dagger \dot{\underline{z}}_j - \mathcal{H}(\{\underline{z}_j^\dagger, \underline{z}_j\})\right] \qquad (133)$$

where the Hamiltonian is written using (125) in terms of the $\{\underline{z}_j\}$.

(iii) Directly in terms of the $\underline{n}$-vector field, so that we can write the generating functional

$$\left.\begin{aligned} Z[\underline{J}] &= \int \mathcal{D}\underline{n}(x)\, \delta(|\underline{n}(x)|^2 - 1)\, e^{\frac{i}{\hbar}\int d^4x \left[\hbar S\, \omega[\underline{n}(x)] - \mathcal{H}[S\underline{n}(x)] - \underline{J}(x).\underline{n}(x)\right]} \\ &\equiv \int \mathcal{D}\lambda(x) \int \mathcal{D}\underline{n}(x)\, e^{\frac{i}{\hbar}\int d^4x \left[\hbar S\, \omega[\underline{n}] - \mathcal{H}[S\underline{n}] + \lambda(x)(|\underline{n}(x)|^2 - 1) - \underline{J}(x).\underline{n}(x)\right]} \end{aligned}\right\} \quad (134)$$

where we have coupled the spin to an external field $\underline{J}(x)$. This is the most obvious way to do path integrals for the spin field $\underline{n}(x)$. But we can also write things in terms of magnon "Holstein-Primakoff" operators, or in terms of the $\underline{z}(x)$ fields, and this gives real flexibility. In order to understand all of this better, we need now to look at some examples.

## (ii) QUANTUM FERROMAGNET :

This is the simplest example we can consider. Let us however look at a slightly more general example than the simple isotropic model of (109); we consider instead the Hamiltonian

$$\left.\begin{aligned} \mathcal{H} &= -\gamma \sum_j H_0 S_j^z - \sum_{\langle ij\rangle} \left[ J_{ij}^{\parallel} S_i^z S_j^z + J_{ij}^{\perp}(S_i^x S_j^x + S_i^y S_j^y)\right] \\ &\equiv -\gamma \sum_j H_0 S_j^z - \sum_{\langle ij\rangle} \left[ J_{ij}^{\parallel} S_i^z S_j^z + \tfrac{1}{2} J_{ij}^{\perp}(S_i^+ S_j^- + S_i^- S_j^+)\right] \end{aligned}\right\} \quad (135)$$

ie., we have an external field applied along the z-axis, and we have 2 different exchange constants, retaining the longitudinal and transverse terms (ie., we have "exchange anisotropy"). Note that this far from being the most general ferromagnetically-coupled Hamiltonian that one could consider *

In what follows I will not treat this problem using path integral methods, but instead use the old-style method of Holstein-Primakoff operators, which will give us a very straightforward path into the problem.** We begin by asking what the ground state of the system ought to be; this will also be our classical starting point.

Note first that in the absence of the exchange terms, the ground state of the system is obvious, no matter what is $S$ (the spin at each site); we just have the state

$$|0\rangle = \prod_{j=1}^{N} |S, m_j = S\rangle \equiv |\uparrow\uparrow\uparrow \ldots \uparrow\rangle \quad (136)$$

in which all spins are pointing up along the z-axis (for $H_0 > 0$). Now let's ask

---

* Important terms missing from (129) include:

(i) Single-ion terms, of form $\sum_j K_j^{\alpha\beta\gamma\ldots} S_j^\alpha S_j^\beta S_j^\gamma \ldots$ (a simple one is $\sum_j K_j^{\parallel} (S_j^z)^2$).

(ii) More complex "multi-site" exchange terms, of form $\sum_{ijkl\ldots} J_{ijkl\ldots}^{\alpha\beta\gamma\ldots} S_i^\alpha S_j^\beta S_k^\gamma \ldots$

(iii) Long-range inter-spin dipolar interactions, which don't even conserve total spin.

** Done by Holstein & Primakoff in 1940; one of the first uses of QFT in condensed matter.

what happens when we add the exchange term, noting that for FM ordering we require that $J_{ij}^{\parallel}$ and $J_{ij}^{\perp} > 0$. From ordinary QM it is clear that this does not disturb the state $|0\rangle$, because

$$\sum_j \left[ S_i^z ,\ S_i^x S_j^x + S_i^y S_j^y \right] \quad = \quad 0 \tag{137}$$

We therefore establish that $|0\rangle$ is still an eigenstate of the full $\mathcal{H}$. Whether it is still the ground state needs a little more investigation. Notice first that we can think of the Hamiltonian, apart from the external field Zeeman term, as a set of pairwise interactions, so it suffices to understand what happens to a single pair. Second, we see that we can also rewrite (135) as

$$\mathcal{H} = -\gamma \sum_j H_0 S_j^z - \sum_{\langle ij \rangle} \left[ J_{\perp}\, \underline{S}_i \cdot \underline{S}_j + (J_{\parallel} - J_{\perp})\, S_i^z S_j^z \right] \tag{138}$$

from which we see that if $(J_{ij}^{\parallel} - J_{ij}^{\perp}) > 0$, the last term merely increases the stability of the state $|0\rangle$, and so all we have to worry about is the isotropic term, which we can rewrite as

$$\begin{aligned} \mathcal{H}_{\text{isotr.}} &= -\sum_{\langle ij \rangle} \frac{J_{\perp}}{2} \left[ (\underline{S}_i + \underline{S}_j)^2 - \underline{S}_i^2 - \underline{S}_j^2 \right] \\ &= -\sum_{\langle ij \rangle} \frac{1}{2} J^{\perp} \left[ 2S(S+1) - (\underline{S}_i + \underline{S}_j)^2 \right] \end{aligned} \tag{139}$$

and the ground state of this is obviously $|0\rangle$ as well, since the energy is minimized by making the spins parallel. Thus the only real difficulty arises when $(J_{\parallel} - J_{\perp}) < 0$; and in fact one can destabilize the simple ground state to other more complicated states under these circumstances. Thus we have the following possibilities.

(a) When $H_0 = 0$, and $J_{\parallel} = J_{\perp} = J$, so that

$$\mathcal{H} = -\sum_{\langle ij \rangle} J\, \underline{S}_i \cdot \underline{S}_j \tag{140}$$

the ground state has all spins parallel; however, this state is actually $(2NS+1)$-fold degenerate (where $N$ is the number of spins), because the total spin $\mathbf{S} = \sum_j \underline{S}_j$ can point in any direction; the system is invariant under rotations in spin space, so that the correct ground state is a symmetric combination of all of these, so that in a coherent state representation we just have

$$|0\rangle_{\text{symm}} = \int \frac{d\Omega_N}{4\pi} |\underline{N}\rangle \quad \equiv \quad \prod_{j=1}^{N} \int \frac{d\eta_j}{4\pi} |\underline{\eta}_j\rangle\, \delta(\underline{N} - \underline{\eta}_j) \tag{141}$$

where $|\underline{N}\rangle$ is the state defined after (113), and $\underline{N}$ is the direction of $|\underline{N}\rangle$. Basically what we have here is a coherent superposition of magnetizations pointing in all directions. — very far from being a

classical state! However...

(b) When $H_0 \neq 0$, we immediately change this situation. Thus for the isotropic exchange Hamiltonian with applied field, ie., for

$$\mathcal{H} = -\gamma \sum_j H_0 S_j^z - \sum_{<ij>} J \underline{S}_i \cdot \underline{S}_j \qquad (142)$$

the ground state $|0\rangle$ in (130) quickly drops below all the other states involved in the superposition. To see what the energy gap is to the first excited state, we might think of trying the state

$$|1_k\rangle = \left( \prod_{i \neq k}^{N} |\uparrow_i\rangle \right) |\downarrow_k\rangle \qquad (143)$$

which has an energy $2\gamma S$ higher than $|0\rangle$; but it is physically obvious (and will become quantitatively clear below) that we can get an excited state with much lower energy by delocalizing the flipped spin in (143). To properly discuss the definition of the classical state and the excited states for this system, we need to look at fluctuations, which we do below.

(c) Finally, for the full Hamiltonian in (135), $|0\rangle$ will remain the ground state provided $(J_{\parallel} - J_{\perp}) > 0$; otherwise things get more complex.

**HOLSTEIN-PRIMAKOFF TREATMENT**: Let's look first at the isotropic system (109), with an applied field - the full case in (135) involves exactly the same manoeuvres, but is more messy. By substituting the Holstein-Primakoff operators into (109), and making the expansion of the square roots, so that, eg.,

$$\begin{aligned} S_j^+ &= (2S)^{1/2} \left[ 1 - \frac{1}{2S} b_j^\dagger b_j \right]^{1/2} b_j \\ &= (2S)^{1/2} \left[ b_j - \frac{1}{4S} b_j^\dagger b_j b_j + O(1/S^2) \right] \\ &\equiv \left( \frac{2S}{N} \right)^{1/2} \left[ \sum_q e^{-i\underline{q}\cdot\underline{j}} b_q - \frac{1}{4NS} \sum_{q_1 q_2 q_3} \delta e^{i(q_1 - q_2 - q_3)} b_{q_1}^\dagger b_{q_2} b_{q_3} + O(1/S^2) \right] \end{aligned} \qquad (144)$$

we can write the Hamiltonian in lattice space as

$$\mathcal{H} = \gamma H_0 \sum_j b_j^\dagger b_j - 2S^2 \sum_{<ij>} J_{ij} \qquad (145)$$

where we subtract off the constant ground state energy, given by

$$E_0 = -\gamma N S H_0 - \frac{1}{S} \sum_{<ij>} S^2 J_{ij} \left( b_i^\dagger b_j - b_j^\dagger b_i \right) + \cdots \qquad (146)$$

where the higher terms are higher order in $1/S$. We can also write this in terms

of the Fourier transformed operators $b_q, b_q^\dagger$; defining the Fourier transform of the exchange interaction as

$$J_q = \sum_{j \neq i} J_{ij} e^{i q \cdot (r_i - r_j)} \xrightarrow[n.n.]{} J z \gamma_q \tag{147}$$

where

$$\gamma_q = \frac{1}{z} \sum_{\langle nn \rangle} e^{i q \cdot r_{nn}} \tag{148}$$

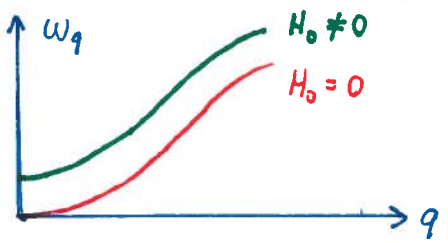
is the "form factor" and $z$, the coordination number, is the number of nearest neighbor around any given lattice site; we can now write (145) as

$$\mathcal{H} = \gamma H_0 \sum_q n_q + z S \sum_q J (1 - \gamma_q) n_q \tag{149}$$

where to simplify things we've assumed inversion symmetry in the lattice, so that $J_q = J_{-q}$, and where $n_q = b_q^\dagger b_q$. Thus we can write $\mathcal{H}$ in the form

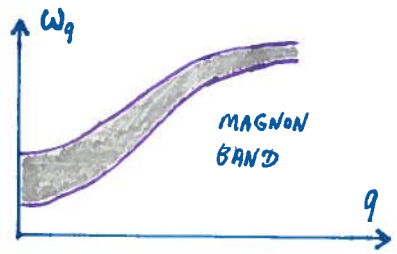
$$\mathcal{H} = \sum_q \omega_q^0 b_q^\dagger b_q \tag{150}$$

where the magnon dispersion relation goes like




ABOVE: MAGNON DISPERSION IN ISOTROPIC SYSTEM

BELOW: FOR ANISOTROPIC SYSTEM




$$\omega_q^0 = z S J (1 - \gamma_q) + \gamma H_0 \tag{151}$$

which for small $q$ must be $\propto |q|^2$; in fact we have (with $c \sim O(1)$):

$$\left.\begin{aligned} \omega_q^0 &\sim \gamma H_0 + \tfrac{1}{2} \sum_{\langle nn \rangle} (q \cdot r_{nn})^2 + O(q r_{nn})^4 \\ &\sim \gamma H_0 + c J S (q a_0)^2 + O(q a_0)^4 \end{aligned}\right\} \tag{152}$$

so that we always have a mass gap induced by the applied, which goes to zero with the field, and a (non-relativistic) form for the magnon dispersion above this gap.

The detailed form for $\gamma_q$ depends on the crystal structure. Here are a few important examples:

1 dimension ($z = 2$): Then we have: $\gamma_q = \cos q a_0$ (153)

2 dimensions: ($z = 4$): For a square lattice $\gamma_q = \frac{1}{2}(\cos a_0 q_x + \cos a_0 q_y)$ (154)

3 dimensions: The most common are:

- Simple cubic ($z = 6$): $\gamma_q = \frac{1}{3}(\cos q_x a_0 + \cos q_y a_0 + \cos q_z a_0)$
- B.C.C. ($z = 8$): $\gamma_q = \frac{1}{8}\left(\cos \frac{q_x a_0}{2} \cos \frac{q_y a_0}{2} \cos \frac{q_z a_0}{2}\right)$
- F.C.C. ($z = 12$): $\gamma_q = \frac{1}{3}\left[\cos \frac{q_x a_0}{2} \cos \frac{q_y a_0}{2} + \cos \frac{q_x a_0}{2} \cos \frac{q_z a_0}{2} + \cos \frac{q_y a_0}{2} \cos \frac{q_z a_0}{2}\right]$

(155)

These results illustrate one key general result, and also raise a key question. The key result is that as $\gamma H_0 \to 0$, we end up with a set of "Goldstone modes", viz., modes whose frequency $\to 0$ as $q \to 0$. We will not discuss this point in great detail here - it is analyzed later in the context of broken symmetries and order parameters. However, very roughly speaking, we can say that in a situation where a symmetry of the underlying Hamiltonian is "spontaneously broken" in the ground state, then there will be a branch of excitations which has frequency $\to 0$ as $q \to 0$. This is a delicate point, because we saw that when $H_0 = 0$ rigorously, the ground state is actually a superposition of form (141), ie, there is no broken symmetry. The way this is conventionally formulated is using a specific order of limits; we say that

$$\lim_{H_0 \to 0} \lim_{N \to \infty} |0\rangle = |\uparrow\uparrow\uparrow \cdots \uparrow\rangle \qquad (156)$$

to replace (141), where we take the thermodynamic limit $N \to \infty$ before taking the applied field to zero. In the opposite case, where we first take $H_0$ to zero, we have

$$\lim_{N \to \infty} \lim_{H_0 \to 0} |0\rangle = |0\rangle_{symm} = \int \frac{d\Omega_{\hat{N}}}{4\pi} |\hat{N}\rangle \qquad (157)$$

ie., the totally symmetric state in (141).

The question raised by these results is simply this - how trustworthy are they? We notice that they are obtained to leading order in a $1/S$ expansion, so we might expect them to be more accurate for large $S$, and this indeed so. But things are actually more subtle than this. In fact a more careful statement of the Goldstone result is

> "If an underlying symmetry is spontaneously broken, and the interparticle/interspin couplings are sufficiently short-ranged, then a Goldstone mode ($\omega_q \to 0$ as $q \to 0$) will appear". (158)

However this does not tell us when the symmetry will be broken. We will return again to this point later, once we have looked at the AFM system; but suffice it to say that the result depends both on the form of the interactions and on the dimensionality of the system. To give a foretaste of what we will see, and to clarify a little the meaning of (158), let's extend very slightly the treatment being given here, in 2 ways:

(a) HIGHER MAGNON TERMS: The effective Hamiltonians in (149) and (150) are produced by the lowest-order expansion in $1/S$; they give a set of non-interacting magnons, ie, a free field theory with small oscillations around the ground state, in which these oscillations correspond to a quadratic form in the $b_q, b_q^\dagger$ and are therefore uncoupled. From this it is completely obvious how to write the action, propagators, and diagram rules for this free field theory, with and without the field $H_0$.

However as soon we go to higher order in $1/S$, interactions between the magnons are generated. The form of these terms and of their interaction kernels is messy, & depends in a complicated way on the interactions in the original Hamiltonian. Because of this complexity we only make a few brief remarks:

* If we simply take the isotropic exchange FM, then further expansion in powers of $1/S$ produces interactions between even numbers of magnons only. Thus we get a series sum of form

$$\mathcal{H} = \sum_{q} \hbar\omega_q^0 b_q^\dagger b_q + \sum_{n=2}^{\infty} \sum_{q_1 \cdots q_{2n}} \Gamma^{(2n)}_{q_1 \cdots q_{2n}} b_{q_1}^\dagger b_{q_2}^\dagger \cdots b_{q_n}^\dagger b_{q_{n+1}} \cdots b_{q_{2n}} \delta\left(\sum_{\alpha} q_\alpha\right) \tag{159}$$

where the vertices $\Gamma^{(2n)}_{q_1 \cdots q_{2n}}$ have to be calculated for a given lattice symmetry and form for $J_{ij}$. The results can be complicated; thus, for a simple cubic lattice and nearest-neighbor $J$, one finds

$$\left.\begin{aligned} \Gamma^{(4)}_{q_1 \cdots q_4} &= \frac{Jz}{2S}\left[\left(\gamma_{q_1} + \gamma_{q_4} - 2\gamma_{q_2 - q_4}\right) + \frac{1}{8}\left(\gamma_{q_1} + q_{q_4}\right)\right] \\ \Gamma^{(6)}_{q_1 \cdots q_6} &= \frac{Jz}{16 S^2}\left[\gamma_{q_1} + \gamma_{q_6} - 2\gamma_{q_3 - q_5 - q_6}\right] \end{aligned}\right\} \tag{160}$$

and so on. Thus, diagrammatically, we have generated a theory with an infinite set of interactions, reminiscent of our discussion of phonons in a solid.

* As soon as rotational symmetry about the magnetization axis is broken, vertices with odd numbers of magnons also enter the effective Hamiltonian. Thus, a much more general Hamiltonian for the system involves external fields, exchange & dipolar interactions, and both exchange & single-ion anisotropy terms; a simple example would be one with 2-site exchange and quadratic single-ion anisotropy:

$$-\mathcal{H} = -\sum_j \gamma_j^{\alpha\beta} H_0^\alpha S_j^\beta + \sum_j K_j^{\alpha\beta} S_j^\alpha S_j^\beta - \sum_{ij} J_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta + \sum_{ij} V_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta \tag{161}$$

where $\gamma_j^{\alpha\beta} = \mu_B g^{\alpha\beta}$ is an anisotropic $g$-factor for the spins, $K_j^{\alpha\beta}$ is the single-ion anisotropy tensor, and the dipolar interaction $V_{ij}^{\alpha\beta}$ takes the form

$$\left.\begin{aligned} V_{ij}^{\alpha\beta} &= \frac{\mu_B^2}{r_{ij}^3}\left[\delta_{\mu\nu} - 3\frac{r_{ij}^\mu r_{ij}^\nu}{r_{ij}^2}\right] g_i^{\mu\alpha} g_j^{\nu\beta} \\ &\xrightarrow[\text{limit}]{\text{isotropic}} \left[\delta_{\alpha\beta} - 3\frac{r_{ij}^\alpha r_{ij}^\beta}{r_{ij}^2}\right] V_0(r_{ij}) \end{aligned}\right\} \tag{162}$$

where in the isotropic limit, $\mu_B g^{\alpha\beta} \to \mu_B g \delta^{\alpha\beta} = \gamma$; then in this

limit, the total dipolar interaction takes the familiar form:

$$V_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta \xrightarrow[\text{isotropic}]{} V_0(r_{ij}) \left[ \underline{S}_i \cdot \underline{S}_j - 3 \frac{(\underline{r}_{ij} \cdot \underline{S}_i)(\underline{r}_{ij} \cdot \underline{S}_j)}{r_{ij}^2} \right] \tag{163}$$

with $\underline{r}_{ij} = \underline{r}_i - \underline{r}_j$, and the interaction strength: $V_0(\underline{r}) = \gamma^2 / |\underline{r}|^3$ (164)

If we calculate a magnon expansion with a Hamiltonian like (155) we get very messy results, which can be found in specialized papers and reviews. They are of no particular interest to us here, except to note that in general they do not conserve magnon number. Thus, e.g., suppose we consider an isotropic exchange interaction and an isotropic dipolar interaction, but with no single-ion terms and no external field. Then the result for the new effective Hamiltonian becomes

$$\left.\begin{aligned} \mathcal{H} = \sum_q \left[ \omega_q^0 b_q^+ b_q + (U_q b_q b_{-q} + \text{H.c.}) \right] \\ + \sum_{q_1 q_2 q_3}{}' \left( \Gamma^D_{q_1 q_2 q_3} b_{q_1} b_{q_2} b_{q_3}^+ + \text{H.c.} \right) \delta(q_1 + q_2 - q_3) + \cdots \end{aligned}\right\} \tag{165}$$

where the coefficients $U_q$, $\Gamma^D_{q_1 q_2 q_3}$, etc., are complex and depend in a complicated way on the sample shape, the lattice structure, and the anisotropic g-factor. The non-conservation of magnon number reflects the fact that the dipolar interaction itself does not even conserve total spin. One can of course rediagonalize the "bare" magnon term, by a rotation in the space of operators $b_q$ and $b_{-q}^+$; but the result is a set of magnons which are superpositions of the two (and thus, a superposition of $\hat{S}_q^+$ and $\hat{S}_{-q}^-$), rather than just $\hat{S}_q^+$ and $\hat{S}_q^-$ separately.*

(b) DEMAGNETIZATION FIELDS & BROKEN SYMMETRY: It is intuitively and physically obvious that when the system of spins polarized, there is an extra magnetic field in the system which is generated by the spins themselves. This term is absent from the Hamiltonian in explicit form, and absent from the effective Hamiltonians given up to (159), because its microscopic cause is to be found in the dipolar interaction. Thus, the extra field – which for historical reasons is called the demagnetization field – arises self consistently from the magnetization itself. In the continuum limit we can rewrite the dipolar term as

$$\left.\begin{aligned} \mathcal{H}_{dip} &= \sum_{ij} V_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta \\ &= -\mu_0 \int d^3r \, \underline{H}_{DM}(\underline{r}) \cdot \underline{M}(\underline{r}) \\ &= \mu_0 \int d^3r \int d^3r' \, \nabla_{\underline{r}'} \cdot \underline{M}(\underline{r}') \frac{1}{|\underline{r} - \underline{r}'|} \nabla_{\underline{r}} \cdot \underline{M}(\underline{r}) \end{aligned}\right\} \tag{166}$$

* This transformation, introduced by Holstein & Primakoff in 1940, was re-used in 1947 in modified form by Bogolubov to superpose particle and anti-particle boson operators, and by BCS to superpose particle & hole fermion operators. It is now known as a Bogolubov transformation.

and we see that demagnetization field $\underline{H}_{DM}(\underline{r})$ has a very interesting non-local form, depending on the integral of $M(r)$ throughout the sample:

$$\underline{H}_{DM}(\underline{r}) = -\int d^3r' \, \underline{M}(\underline{r}') \cdot \nabla_{\underline{r}'} \frac{1}{|\underline{r}-\underline{r}'|} \nabla_{\underline{r}} \qquad (167)$$

The non-local character arises from the long-range nature of the dipolar interaction, and the form of (166), with the dependence on $(\underline{r}-\underline{r}')$, arises directly from the specific form of (163) and (164).

We can now look at this result in the context of the statement about Goldstone bosons given in (158). The basic point is this — if, in the absence of dipolar interactions, we reduce $\underline{H}_0 \to 0$, then depending on the order of limits in (156) and (157), we may or may not get ordering of the spins into a macroscopic magnetization, but this magnetization will not feed back on the system, and we will have a set of gapless Goldstone bosons, in the form of magnons with dispersion $\omega_q^0 \propto |\underline{q}|^2$. However, regardless of how we let $\underline{H}_0 \to 0$, once the dipolar interaction is introduced, the long-range nature of this will lead to a feedback effect in which a net field is produced by the spins, in a self-consistent way in which spins act to line each other up. We are all familiar with this — it is the phenomenon which leads to ordinary magnets.

However we now notice that this result has had a profound effect. First, it has destroyed the gapless Goldstone modes — the demagnetization field now induces a gap in the magnon spectrum, even when $\underline{H}_0 = 0$. This is quite consistent with (158), with its proviso that forces need to be short-ranged.

Second, we notice that what this field $\underline{H}_{DM}(\underline{r})$ will be cannot be specified unless one specifies what is the shape of the system. This crucial point is often ignored or misunderstood in the literature — it means that the system has no well-defined thermodynamic limit! If we let $N \to \infty$, the physical properties still depend on the shape of the system, via (166). Thus you should beware of much of what passes for proof in the statistical mechanics or QFT literature, which often ignores boundary conditions at infinity — given that gauge fields are intrinsically long-ranged, and Nature is described in a fairly deep way by gauge field theories, we are rarely entitled to be so cavalier about what happens at boundaries.

Notice, however, that the existence of this additional field does have a very useful effect. It provides a mechanism to stabilize the classical solution to our QFT; the gap in the spectrum makes it much more difficult for quantum fluctuations to alter the ground state. To see this in detail, and to prove it, is not all that simple, but we shall see that at least in some cases — including this one, for dimensionality $d \geq 2$, in the absence of other destabilizing influences — the idea works rather well, and the classical field will simply be the local magnetization $\underline{M}(\underline{r},t)$.

Note also that we have ignored one other key point. This is that one can develop non-trivial inhomogeneous configurations in $M(\underline{r},t)$, which are

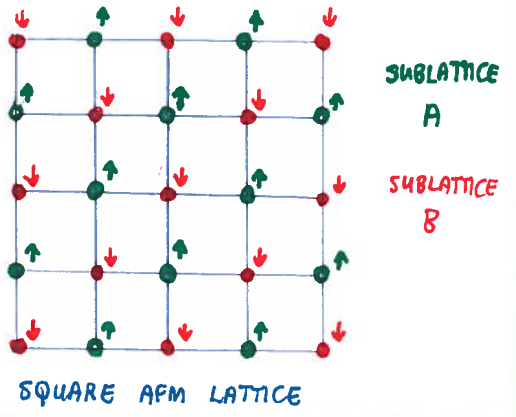
topologically stable. For a FM system, we have already seen how this can work in 1 dimension; the 1-d Sine-Gordon system is a very simple example of a system in which gapped small oscillations exist in an ordered system, which can also possess topological solitons. If we go to 2 spatial dimensions, the system can support both line domain walls and vortex solutions; and in 3 dimensions we have sheetlike domain walls, vortex lines, and "Bloch point" or "hedgehog" monopole soliton solutions.

The role and behaviour of soliton and instanton solutions in a QFT is studied in more detail later in this section.

## (iii) QUANTUM ANTIFERROMAGNET :

The quantum FM is an example of a system where the classical solution to the QFT works very well - indeed, we were able to perform a fluctuation expansion around this theory which works very well for dimensionality $\geq 2$. The main reason this worked so well was that the classical field stabilized itself, using the demagnetization field it generated.

The antiferromagnetically-ordered (AFM) state is quite different, in a number of different ways, which will lead us into further insights into general features of QFT. The classically ordered state was first discussed by Néel in 1932.




Néel's basic idea can be seen in the figure, which shows a 2-d square AFM array of spins, with nearest neighbours linked by vertical or horizontal lattice vectors. We see that if we divide the system into 2 sublattices, then any spin on sublattice A only interacts with spins on sublattice B, and vice-versa. With AFM interactions, ie., with $J < 0$, then it is clearly classically advantageous to pick opposite orientations for the spins on the 2 sublattices, to minimize the energy; indeed, for the simple system shown in the figure, we have a classical ground state energy $E_0 = \frac{1}{2} z J N S^2$, where $z = 4$ is the number of nearest neighbours, and we assume $J_{ij} = J$ for nearest neighbours, and zero otherwise.

However there are some obvious problems with this simple picture. The first is that, unlike the FM case, the dynamics of a spin in such an environment are not consistent with the ground state we have chosen. We can see this by just by considering a single pair of spins with AFM coupling; then the Hamiltonian is

$$\mathcal{H}_{12} = -J \underline{S}_1 \cdot \underline{S}_2 \qquad (J<0) \tag{168}$$

which has the ground state

$$|0\rangle = \frac{1}{\sqrt{2}} \left[ |S,-S\rangle - |-S,S\rangle \right]_s \longrightarrow \frac{1}{\sqrt{2}} \left[ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right] \quad \text{spin-}\tfrac{1}{2} \tag{169}$$

where the notation $|S,S'\rangle_{\hat{z}} \equiv |S,s\rangle|S,s'\rangle$, ie., we only write the projection along $\hat{z}$ inside the ket. The energy of the ground state is

$$E^{\circ}_{12} = JS(S+1) \qquad (J<0) \tag{170}$$

Now we see that the classical state $|S,-S\rangle$ is neither the ground state nor an eigenstate; the classical energy is $JS^2 > E^{\circ}_{12}$, and

$$\mathcal{H}_{12}\,|S,-S\rangle_{\hat{z}} = J\,|-S,S\rangle_{\hat{z}} \tag{171}$$

or, for spin ½,

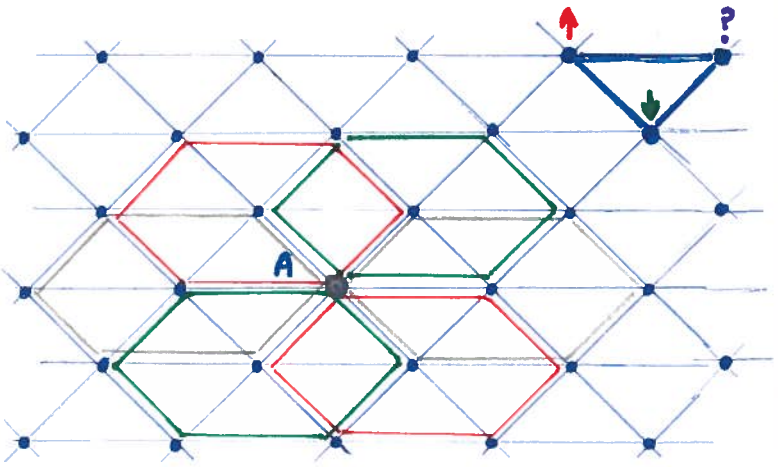
$$\mathcal{H}_{12}\,|\uparrow\downarrow\rangle = J\,|\downarrow\uparrow\rangle \tag{172}$$

These remarks simply underline the role of quantum fluctuations; the ground state is a superposition of the two possible classical states $|S,-S\rangle$ and $|-S,S\rangle$, and indeed simply resonates/oscillates between them with frequency $J/\hbar$. To put it another way, we are forming a coherent superposition of oppositely oriented spin states in which, however, the direction of the spins is entirely undetermined. Thus, in the absence of some other field which can stabilize the orientation, the classical state is a bad approximation.

At first glance these remarks seem to parallel those made about the FM state (cf. eqtn. (139) and discussion thereafter). However, the AFM situation is very different, because in the FM case, the triplet state $|\uparrow\uparrow\rangle$ is an eigenstate as well as being the classical solution. Thus, the interactions are not destabilizing the FM structure at the microscopic level, as in the AFM case.

For this reason the proposal of Néel in 1932 was initially treated rather sceptically by theorists, particularly those of the London school. We will come back to this point, which can be put in the form of a question, viz., why should the classical AFM state be stable at all? Yet, remarkably, it is – in both 3-d and 2-d systems – classical analysis works quite well, as shown by experiments. However the story turns out to have an interesting subtle twist.

The second obvious problem with the classical argument given above is that, unlike the case of the FM, it relies heavily on the lattice structure – for many lattices one cannot find an obvious classical AFM state. Consider, eg., the hexagonal lattice shown in the figure. To figure out a sublattice structure is not so obvious (and would depend on the precise form of the $J_{ij}$); we note that any particular lattice site partakes of 6 different hexagonal plaquettes. Suppose that $J_{ij} = J$ for nearest neighbors, with $J<0$. Consider now the energy minimization for a single triangle




HEXAGONAL LATTICE: ANY LATTICE POINT A PARTAKES OF 6 DIFFERENT HEXAGONS. EACH TRIANGLE SHOWS MAGNETIC FRUSTRATION FOR NEAREST NEIGHBOUR AFM INTERACTIONS.

of 3 spins. It is clear that we cannot minimize the energy by choosing oppositely oriented spins (if one is up, the other down, then how do we choose the 3rd one?), and in fact the classical energy is minimized by having the spins oriented at mutual angles of 120°. Again, however, the ground state of this system is not this classical state, but a superposition of them. Moreover, as one easily sees, any attempt to fit these "triangle" states together into an ordered lattice with spins oriented at 120° between nearest neighbours, is again frustrated by the lattice structure. This "frustration" effect is of great importance in magnetism — it prevents us from finding an ordered classical state.

Investigation of these issues over the last 60 yrs has led to some very interesting results. In what follows we will look at just 2 of them in detail, viz., (i) the classical spin wave theory for ordered AFM systems, and (ii) the non-linear sigma model for AFM systems, and the destabilization of the AFM state by quantum fluctuations, with the emergence of an extra "topological Θ-term" in the effective action.

**AFM SPIN WAVES:** The theory of AFM spin waves is an expansion of small oscillations, ie., of quantum fluctuations, about the classical Néel state. As such we are holding our nose and pretending that the problem just discussed does not exist. Now since the ratio of the quantum fluctuation energy to the classical energy decreases as $S$ increases, we can hope that this tactic may work for large $S$, (and for large $z$); we shall see later that this is so.

Let's first quickly do things using a Holstein-Primakoff technique. We generalize (114) to cover 2 sublattices, so that

$$\left.\begin{aligned} S_{jA}^{+} &= (2S)^{1/2}\left[1-\frac{1}{2S}a_j^{+}a_j\right]^{1/2}a_j \\ S_{jB}^{+} &= (2S)^{1/2}\,b_j^{+}\left[1-\frac{1}{2S}b_j^{+}b_j\right]^{1/2} \end{aligned}\right\} \qquad (173)$$

and

$$\left.\begin{aligned} S_{jA}^{z} &= S - a_j^{+}a_j \\ S_{jB}^{z} &= b_j^{+}b_j - S \end{aligned}\right\} \qquad (174)$$

where the swapping between $S^+$ and $S^-$ in the defns of these operators for the 2 sublattices, and between $\pm S$ for the defn of $S^z$, comes because the spins are oppositely oriented in the 2 cases.

We will choose as our Hamiltonian a system with both single ion anisotropy and an external field, so that

$$\mathcal{H} = -\gamma \underline{H}_0 \cdot \sum_{j=1}^{N} \underline{S}_j - \sum_{i\neq j}\sum J_{ij}\,\underline{S}_i\cdot\underline{S}_j - \tfrac{1}{2}\sum_j K_2\,(S_j^z)^2 \qquad (175)$$

$$(J_{ij} < 0\,; \quad K_2 > 0)$$

so that we have "easy-axis" anisotropy; we can rewrite this as

$$\mathcal{H} = -\gamma \underline{H}_0 \cdot \Big(\sum_{i\in A}\underline{S}_i + \sum_{j\in B}\underline{S}_j\Big) - \sum_{i\in A}\sum_{j\in B} J_{ij}\,\underline{S}_i\cdot\underline{S}_j - \tfrac{1}{2}K_2\Big(\sum_{i\in A}(S_i^z)^2 + \sum_{j\in B}(S_j^z)^2\Big) \qquad (176)$$

with separate sums over the 2 sublattices $A$ and $B$; we notice that we are now using the classical Néel state as a point of departure. Now, with a bit of algebra, we can rewrite this Hamiltonian as

$$\left.\begin{aligned}\mathcal{H} = \tilde{E}_0 + \gamma H_0 \Big(\sum_{i\in A} a_i^\dagger a_i - \sum_{j\in B} b_j^\dagger b_j\Big) - \sum_{i\in A}\sum_{j\in B} S J_{ij}\,(a_i^\dagger a_i + b_j^\dagger b_j + a_i b_j + a_i^\dagger b_j^\dagger) \\ + K_2\Big(\sum_{i\in A} a_i^\dagger a_i + \sum_{j\in B} b_j^\dagger b_j\Big)\end{aligned}\right\} \quad (177)$$

and again, it is useful to Fourier transform here, now writing

$$a_q^\dagger = \Big(\frac{2}{N}\Big)^{1/2} \sum_{i\in A} e^{i\underline{q}\cdot\underline{r}_i} a_j^\dagger \qquad b_q^\dagger = \Big(\frac{2}{N}\Big)^{1/2} \sum_j e^{-i\underline{q}\cdot\underline{r}_j} b_j^\dagger \quad (178)$$

and defining the form factor $\gamma_{\underline{q}}$ via

$$z J_0 \gamma_q = \sum_{j\in B} J_{ij}\, e^{-i\underline{q}\cdot(\underline{r}_i - \underline{r}_j)} \quad (179)$$

so that we have

$$\mathcal{H} = \tilde{E}_0 - 2J_0 z S \sum_{\underline{q}} \left[A_0 a_q^\dagger a_q + B_0 b_q^\dagger b_q + \gamma_q (a_{\underline{q}} b_{\underline{q}} + a_{\underline{q}}^\dagger b_{\underline{q}}^\dagger)\right] \quad (180)$$

and we have defined the following constants:

$$\left.\begin{aligned}\tilde{E}_0 &= \tfrac{1}{2}(J_0 z - K_2) N S^2 \\ A_0 &= 1 + \frac{1}{J_0 z S}(K_2 S + \gamma H_0) \\ B_0 &= 1 + \frac{1}{J_0 z S}(K_2 S - \gamma H_0)\end{aligned}\right\} \quad (181)$$

Now (180) is a quadratic form in the space of operators $a_q, a_q^\dagger, b_q, b_q^\dagger$; we need to find a complex rotation in this 4-d space to diagonalize this form. This is the Bogoliubov transformation*

$$\left.\begin{aligned} a_q &= \alpha_q \cosh\theta_q - \beta_q^\dagger \sinh\theta_q & b_q &= \beta_q \cosh\theta_q - \alpha_q^\dagger \sinh\theta_q \\ a_q^\dagger &= \alpha_q^\dagger \cosh\theta_q - \beta_q \sinh\theta_q & b_q^\dagger &= \beta_q^\dagger \cosh\theta_q - \alpha_q \sinh\theta_q \end{aligned}\right\} \quad (182)$$

so that

$$\tanh 2\theta_q = \frac{\gamma_q}{1 + (K_2/J_0 z)} \quad (183)$$

and we see that $\theta_q$ is a rotation in a "Lorentzian space" of the original operators

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* Again, due to Holstein-Primakoff (1940); Bogoliubov was in 1947 and 1958.

The final form for the Hamiltonian is then

$$\mathcal{H} = E_0 + \hbar \sum_q \left[ \omega_q^+ \left(\alpha_q^+ \alpha_q + \tfrac{1}{2}\right) + \omega_q^- \left(\beta_q^+ \beta_q + \tfrac{1}{2}\right) \right] \tag{184}$$

where the eigenfrequencies are given by

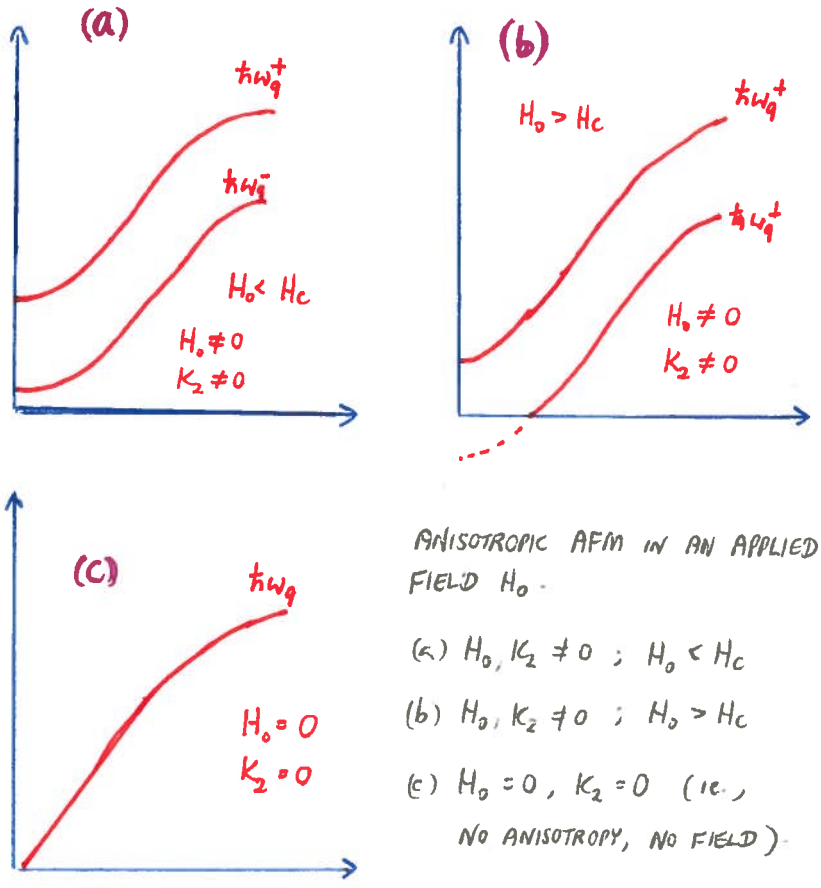
$$\left.\begin{aligned} \hbar\omega_q^\pm &= \left(\Delta_0 + (1-\gamma_q^2) J_0^2 z^2 S^2\right)^{1/2} \pm \gamma H_0 \\ &\xrightarrow[\text{small } q]{} \left(\Delta_0 + c_0^2 q^2\right)^{1/2} \pm \gamma H_0 \end{aligned}\right\} \tag{185}$$

where the "gap" $\Delta_0$ is

$$\Delta_0 = \left(K_2^2 + 2 J_0 z K_2\right)^{1/2} S \tag{186}$$

and the constant $c_0$ depends on the detailed form of $\gamma_q$. Notice what happens here when the anisotropy $K_2 \to 0$; then we have a LINEAR dispersion for the AFM spin waves, instead of the quadratic dispersion found in the FM case, i.e., we have

$$\text{AFM spin waves:} \qquad \hbar\omega_q^\pm \xrightarrow[K_2 = 0]{} c_0 |q| \pm \gamma H_0 \tag{187}$$




ANISOTROPIC AFM IN AN APPLIED FIELD $H_0$.

(a) $H_0, K_2 \neq 0$ ; $H_0 < H_c$

(b) $H_0, K_2 \neq 0$ ; $H_0 > H_c$

(c) $H_0 = 0$, $K_2 = 0$ (i.e., NO ANISOTROPY, NO FIELD)

Now there are some subtleties associated with these results. The first is to do with the sign of $\omega_q^-$. We notice that if there is no anisotropy, then when the applied field $H_0 = 0$, the spectrum is linear and gapless (see Fig (c) in the figure). However if $H_0 \neq 0$, then the energy $\hbar\omega_q^-$ becomes NEGATIVE when $|q|$ is less than a critical value.

The same is true even when $K_2 \neq 0$. Thus for sufficiently small $H_0$, the anisotropy stabilizes both $\omega_q^+$ and $\omega_q^-$, and keeps them both gapped and positive. But when $H_0$ exceeds a critical field $H_c$, the same thing happens; i.e., $\omega_q^-$ goes negative for small momentum. This critical field is given by

$$H_c^2 = (H_J + H_k) H_{ic} = (\Delta_0/\gamma)^2 \tag{188}$$

where the 2 fields $H_J$ and $H_k$ are given by

$$\left.\begin{aligned} \gamma H_J &= |J| z S \\ \gamma H_k &= K_2 S \end{aligned}\right\} \quad (189)$$

and are commonly referred to as the exchange & anisotropy fields respectively; we can rewrite (185) as

$$\left.\begin{aligned} \hbar\omega_q^{\pm} &= \gamma\left(H_k(H_J+H_k) + (1-\gamma_q^2)H_J^2\right)^{1/2} \pm \gamma H_0 \\ &= \gamma\left(H_c^2 + (1-\gamma_q^2)H_J^2\right)^{1/2} \pm \gamma H_0 \end{aligned}\right\} \quad (190)$$

Now, although we will not take the time to demonstrate it here, this instability is one in which the spins "cant over" in response to the field. To see what is meant by this, let's now rewrite things in a slightly different form, which brings out the fact that because pairs of spins are involved in the AFM ordering, we actually have 2 independent vectorial degrees of freedom involved in the low-energy spin dynamics, and the best way to describe these is not using the 2 individual sublattice magnetizations, which are strongly coupled, but instead by writing $\underline{S}_j(t)$ in the form (here $\alpha_j = \pm$ identifies sublattice A or B):

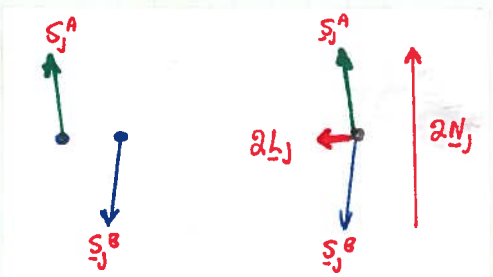
$$\underline{S}_j(t) = S\alpha_j\left[1 - |\underline{m}_j(t)|^2\right]^{1/2} \underline{l}_j(t) + S\underline{m}_j(t) \quad (191)$$

where now the Néel vector $S\underline{l}(t)$ and the canted magnetization vector $S\underline{m}(t)$ represent "difference and sum variables for the 2 sublattices. Note: in the case where the 2 sublattices can be uniquely identified and we can pair off spin $\underline{S}_j^A$ with $\underline{S}_j^B$ in the corresponding mirror lattice, we can also define

$$\left.\begin{aligned} \underline{M}_j &\to \tfrac{1}{2}(\underline{S}_j^A + \underline{S}_j^B) \\ \underline{L}_j &\to \tfrac{1}{2}(\underline{S}_j^A - \underline{S}_j^B) \end{aligned}\right\} \quad (192)$$

However, as defined in (191), $\underline{m}_j(t)$ is zero in the ordered AFM phase where the 2 sublattices are antiparallel; we are not measuring a real magnetization when we define $\underline{M}_j(t)$. Now suppose we start off in zero field, $H_0 = 0$. Then if $K_2 = 0$ also, the Néel vector can point in any direction (provided it is even defined, ie., there is some AFM ordering, which we recall is one of the questions at issue), and we expect $\underline{L}_j = 0$. However as soon as we apply a field, it is energetically advantageous for $\underline{l}_j(t)$ to rotate into the plane perpendicular to $\underline{H}_0$, and a small $\underline{m}_j$ will form, parallel to the field; we note that in general,

$$\left.\begin{aligned} \underline{l}_j \cdot \underline{m}_j &= 0 \\ |\underline{l}_j(t)| &= 1 \end{aligned}\right\} \quad (193)$$




DEFINITION OF THE NÉEL & THE CANTED MAGNETIZATION VECTORS

so that the 2 vectors are always perpendicular. If $K_2 \neq 0$, then in low fields, $\underline{\ell}_j(t)$ will align along the $\pm \hat{z}$ direction, but when $H_0 > H_c$, canting will again occur. In general the phase diagram, & the spin orientations, can become a little complex.

Now we see the physics behind the instability in the spin wave spectrum. In writing our HP operators, we have assumed that the state is ordered, with A and B sublattice spins antiparallel. But the spin wave calculation tells us that for $H_0 > H_c$, there exist disturbances/fluctuations away from this state with negative energy, ie., it is unstable – and now we see why, because the system wants to go to a new canted classical ground state.

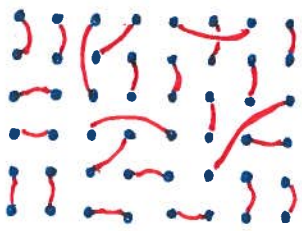
**QUANTUM DISORDER & THE HALDANE GAP :** All of the above reasoning was basically classical – we started from a classical ground state and then looked at small oscillations about it. But how good is this reasoning?

To answer this, and to illustrate how one may attack the problem using non-perturbative path integral methods, we will reformulate the problem in a different way. We will find a result which is of considerable theoretical interest, because it shows how a topological "θ-term" can appear in the action for the system, and determine the properties.

Before doing this, let's begin with a few more physical questions – this is always a good idea when you are approaching a new topic. We have already seen (cf. eqtns. (169)-(172) that for a pair of spins, there is no reason to expect anything like Néel ordering – so why would we want to use this as a starting point for our theory? This question was posed repeatedly over the years by P.W. Anderson, who argued that one might try a quite different starting point for a ground state, the "RVB" state (where RVB means "Resonating Valence Bond"). The basic idea here is simple – we have seen that for a pair of spins, the singlet state is the lowest energy state for AFM coupling – and of course this state is nothing but the standard valence bond state from elementary chemistry. Let us therefore define the valence bond state between 2 sites $i$ and $j$ as

$$|VB\rangle_{ij} \equiv |\phi_{ij}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle\right) \qquad (194).$$

Notice that if a site $i$ is in a completely entangled singlet state with site $j$, then neither $i$ nor $j$ can entangle with another site $k$ (this, in quantum info theory, is given the grand name of the "quantum monogamy theorem"). This means that we can define an RVB configuration like that shown in the figure, in which every site in the lattice is paired off with another one. But of course this configuration is only one of a huge number – without developing any formalism here, we can imagine enumerating all such different configurations, and then summing over them with a different amplitude for each one. One can imagine a huge range of possibilities here, depending on how much one emphasizes short-range or long-range bonds. Thus one extreme has the bonds all between nearest neighbors – this is a "dimerized state", or "valence bond solid". On the



A RESONATING BOND CONFIGURATION.

other hand one can imagine in which much more weight is given to long-range bonds (eg., one in which the weighting is independent of bond length). In this case on expects, as larger & longer bonds come in, to go through a sequence of quite disordered states, and ultimately end up with a Néel-type state, with AFM ordering *

To decide what is the correct ground state (as well as the excited states above it) is very hard in this kind of theory. One can argue that when $\mathcal{Z}$ is large, the large number of AFM couplings involved will stabilize the AFM state (first look at small clusters of spins, then couple these to larger ones, and so on). Certainly it is true that a strong AFM coupling will try and keep neighbouring spins as antiparallel as possible, whether this be in a singlet state, or some other state in which such an AFM correlation is present. But how is one to proceed? And for small $\mathcal{Z}$, can we still make such an argument?

One other way of thinking about this turns out to be very fruitful. We have, when doing fluctuations à la HP, been perturbing around a classical state in which all spins in the system are fixed rigidly in an AFM lattice, in a ground state with long-range order. But we can choose a much more general class of states, in which neighbouring pairs of spins are almost exactly anti-aligned, but where we allow the ordering axis itself to fluctuate over longer length scales. To do this in ordinary field theory, where we must specify the ground state before we start, this is very hard - which state do we choose? But in path integral formulations of the problem, it is easy - we simply assume that the vectors $\underline{\ell}_j(t)$ and $\underline{m}_j(t)$ in (191) are classical, and then the assumption of slow variation of $\underline{\ell}_j(t)$ implies, because of (193), that $\underline{m}_j(t) \ll 1$ everywhere (nearest neighbour spins are almost exactly antiparallel).

Let us therefore rewrite the action in terms of $\underline{\ell}_j(t)$ and $\underline{m}_j(t)$. To keep things simple we assume $H_0 = 0$, $K_2 = 0$, and then we have

$$S_{AFM}[\underline{\ell}_j, \underline{m}_j] = S_B[\{\underline{\ell}_j, \underline{m}_j\}] - \int dt\, \mathcal{H}[\{\underline{\ell}_j, \underline{m}_j\}] \tag{195}$$

where the 1st term (the Berry phase or topological term) is

$$\left.\begin{aligned} S_B &= \hbar S \sum_j \int dt\, \underline{A} \cdot \dot{\underline{n}}_j(t) \\ &= \hbar S \sum_j \int dt \left\{ \underline{A} \cdot (\alpha_j \dot{\underline{\ell}}_j(t) + \dot{\underline{m}}_j(t)) + \dot{\ell}_j^\mu \frac{\partial A_\mu}{\partial \ell^\nu} m^\nu(t) \right\} \end{aligned}\right\} \tag{196}$$

where to get the 2nd expression we use $|\underline{m}(x)| \ll 1$, and expand to 1st order in $\underline{m}(t)$. To deal with the latter term in this expression, we first integrate by parts, and then use $(\nabla \times \underline{A}) \cdot \underline{\ell}_j(t) = 1$ (compare (59)), and then get

$$S_B = \hbar S \sum_j \int dt \left\{ \alpha_j \underline{A} \cdot \dot{\underline{\ell}}_j(t) - (\underline{\ell}_j \times \dot{\underline{\ell}}_j) \cdot \underline{m}_j \right\} \tag{197}$$

in which we now have both $\underline{\ell}_j$ and $\underline{m}_j$; a total time derivative has been dropped

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* See, e.g., Liang, Doucot, & Anderson, PRL 61, 365 (1988)

here as well.

The Hamiltonian can be handled similarly. We again choose the isotropic nearest-neighbour exchange Hamiltonian, and so we get (cf. (109)):

$$\left.\begin{aligned}\mathcal{H} &= -S^2 J \sum_{\langle ij\rangle} \underline{n}_i \cdot \underline{n}_j \\ &= -JS^2 \sum_{\langle ij\rangle} \left\{ \alpha_i \alpha_j \left[ 1 - \tfrac{1}{2} (\underline{r}_{ij} \cdot \underline{\nabla} \underline{\ell}_i)^2 \right] + 2 \underline{m}_i^2 \left[ 1 - \alpha_i \alpha_j \right] \right\} + \cdots \end{aligned}\right\} \quad (198)$$

where we again expand to lowest order in the gradients of $\underline{n}_i$ (and hence lowest order in $\underline{m}_i$).

Now for nearest neighbours, $\alpha_i \alpha_j = -1$ always, so we can now write the Hamiltonian in the continuum approximation (valid because we are looking here at slow variations in space) in the form (dropping constants)

$$\mathcal{H} = -JS^2 \int d^D r \; \tfrac{1}{2} \left[ (\nabla \underline{\ell}(x))^2 + 8 \underline{m}^2(x) \right] \quad (199)$$

so that now we can write the generating functional of the system as

$$Z[\underline{j}, \underline{h}] = \int \mathcal{D}\underline{\ell}(x) \int \mathcal{D}\underline{m}(x) \; e^{\frac{i}{\hbar}\left[ S_{eff}[\underline{\ell}, \underline{m}] + \int d^D x (\underline{j} \cdot \underline{\ell} + \underline{m} \cdot \underline{h}) \right]} \quad (200)$$

where $\underline{j}(x)$ and $\underline{h}(x)$ are external fields coupling to $\underline{\ell}(x)$ and $\underline{m}(x)$, and the effective action $S_{eff}[\underline{\ell}, \underline{m}]$ is

$$\left.\begin{aligned} S_{eff}[\underline{\ell}, \underline{m}] = \hbar S \int dt \Big\{ & \sum_j \alpha_j \underline{A} \cdot \dot{\underline{\ell}}_j(t) - \int d^D r \; (\underline{\ell} \times \dot{\underline{\ell}}) \cdot \underline{m} \\ & - \tfrac{1}{2} JS^2 \int d^D r \left[ (\nabla \underline{\ell})^2 + 8 \underline{m}^2 \right] \Big\} \end{aligned}\right\} \quad (201)$$

where

We now need to integrate over 2 fluctuating variables, $\underline{\ell}(x)$ and $\underline{m}(x)$, which are orthogonal to each other (cf. (193)), so our job seems a little complicated at first, since the 2 fluctuations are coupled via the 2nd term in the Berry phase.

However, we have already seen that the fluctuations $\underline{m}(x)$ are small, so it does not make sense to go beyond lowest order in these. So we can integrate this out by minimizing the effective action $S_{eff}$ under variations of $\underline{m}(x)$; this immediately gives

$$\underline{m}(x) = \frac{1}{8JS} \left( \underline{\ell}(x) \times \dot{\underline{\ell}}(x) \right) \quad (202)$$

We then notice that $(\underline{\ell} \times \dot{\underline{\ell}})^2 = (\partial_t \underline{\ell})^2$, because of the constraint in (193), so finally we have

$$S_{eff}[\underline{\ell}] = \int dt \left\{ \hbar S \sum_j \alpha_j \underline{A} \cdot \dot{\underline{\ell}}_j - \tfrac{1}{2} \rho_s^0 \int d^D x \left[ (\nabla \underline{\ell}(x))^2 + \frac{1}{c_0^2} (\partial_t \underline{\ell}(x))^2 \right] \right\} \quad (203)$$

where we have introduced 2 constants, viz., the bare "spin stiffness" $\rho_s^0$ and the AFM spin wave velocity $c_0$, given by

$$\left.\begin{aligned} \rho_s^0 &= JS^2 \\ c_0^2 &= 8J^2S^2 \end{aligned}\right\} \quad (204)$$

Now let us see what we have here. The first term in (203) is what is left of the Berry phase - we shall look at it in a moment. The rest of the action is just the action of the non-linear sigma model, which we already introduced previously (cf. eqns. (119) - (127)). As noted already, this model is one of great interest in many different fields; we shall use it as a toy model to study renormalization in the next chapter. The spin stiffness $\rho_s^0$ is the analogue, for a spin system, of the superfluid density - this just measures the resistance of the spin system to the bending of the spins away from their most favourable state of anti-alignment (for nearest neighbour spins).

The Berry term in (203) has been left for the moment in its lattice form; but we would like to have a continuum form. Now it turns out that we have to do this with care; and the answer depends on the number of spatial dimensions, as follows

1-d AFM Chain: This is the famous case elucidated by Haldane in 1983, leading to the prediction of the "Haldane gap", and of a key difference between integer and ½-integer spin chains*. Suppose we have a function that is varying slowly as we go from one lattice site to the next. Calling this function $f_j$, we see that

$$\sum_j (-1)^j f_j \qquad \tfrac{1}{2}\int dx\, (\partial f/\partial x) \quad (205)$$

plus terms $\sim O(f''(x), (f'(x))^2, \text{etc})$. Applying this to the Berry phase term in (203), which for a 1-d lattice just has $\alpha_j = (-1)^j$, we get the Haldane result, viz.,

$$\left.\begin{aligned} S_B[\underline{l}] &= \hbar S \int dt \sum_j (-1)^j \underline{A}\cdot\dot{\underline{l}}_j \\ &\to \tfrac{1}{2}\hbar S \int dt \int dx\; \underline{l}\cdot(\partial_x \underline{l} \times \partial_t \underline{l}) \\ &\equiv \tfrac{1}{2}\hbar S \int dt \int dx\; F_{xt}(x,t) \;=\; 2\pi\hbar S\, Q_{xt} \end{aligned}\right\} \quad (206)$$

where the topological term $Q_{\mu\nu}$ is, in $D$ dimensions

$$Q_{\mu\nu} = \frac{1}{4\pi}\int d^D x\, F_{\mu\nu}(x) = \frac{1}{4\pi}\int d^D x\; \underline{l}(x)\cdot(\partial_\mu \underline{l}(x) \times \partial_\nu \underline{l}(x)) \quad (207)$$

(compare eqns. (128) and (129)). Now eqn (206) is an important result, so let's unpack it in several different ways.

- The first way is to use your geometrical intuition. Let's look at the 1st

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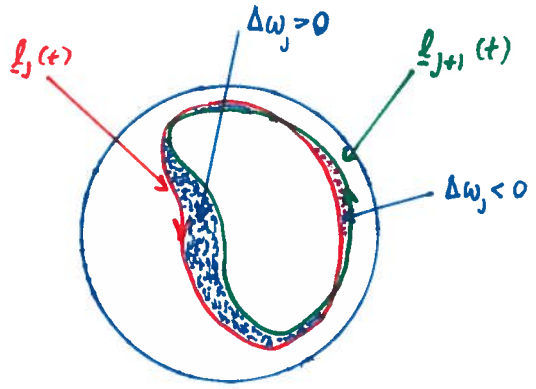
* F.D.M. Haldane, Phys. Rev. Lett. 50, 1153 (1983); and Phys. Lett. 93A, 464 (1983)

line of (206), and ask how to think about it geometrically. Now for each site, we see that the action $S_B$ is measuring the Berry phase / solid angle accumulated by the motion, ie., we have

$$S_B[\underline{\ell}] = \hbar S \sum_j (-1)^j \omega_j[\underline{\ell}_j] \tag{208}$$

and so if we now look at adjacent pairs of $\underline{\ell}$ vectors, we are simply looking at the difference between $\omega_j[\underline{\ell}_j]$ and $\omega_{j+1}[\underline{\ell}_{j+1}]$. To see this geometrically, let's look at 2 closed curves, and define

$$\left.\begin{aligned} \Delta\omega_j &= \omega_j[\underline{\ell}_j] - \omega_{j+1}[\underline{\ell}_{j+1}] \\ &= \int \underline{A} \cdot (d\underline{\ell}_j - d\underline{\ell}_{j+1}) \end{aligned}\right\} \tag{209}$$




CURVES FOR $\underline{\ell}_j(t)$ AND $\underline{\ell}_{j+1}(t)$ ON BLOCH SPHERE. THE DIFFERENCE $\Delta\omega_j = \oint \underline{A}\cdot(d\underline{\ell}_j - d\underline{\ell}_{j+1})$ IS ALSO SHOWN

which for 2 closed curves just gives the picture shown in the figure - we are looking at the difference between the 2 solid angles (between the red and green curves). If we now assume this difference is small, we see that we are integrating the rate of change " of $\omega(x)$ as we go down the 1-d system (in a continuum approximation), ie., we are calculating the integral

$$S_B = \tfrac{1}{2}\hbar S \int dt \int dx \, \frac{d}{dx}\left(\underline{A}(\underline{\ell})\cdot\frac{d}{dt}\underline{\ell}(x,t)\right) \tag{210}$$

where I write $\underline{A}(\underline{\ell})$ to signify that the vector potential $\underline{A}$ is referring to the vector $\underline{\ell}$, and that since $\underline{\ell}\cdot(\nabla\times\underline{A}(\underline{\ell})) = 1$, we can also write that

$$\underline{\ell} \equiv (\nabla_{\ell}\times\underline{A}(\underline{\ell})) \tag{211}$$

(compare eqtn (59)).

- A second way to think about (206) is to subtract off a term which is actually zero, and which allows us to rewrite the action in a more symmetric form. So let us write, from (210), that

$$\left.\begin{aligned} S_B[\underline{\ell}] &= \tfrac{1}{2}\hbar S \int dt \int dx \left\{ \frac{d}{dx}\left(\underline{A}(\underline{\ell})\cdot\frac{d}{dt}\underline{\ell}(x,t)\right) - \frac{d}{dt}\left(\underline{A}(\underline{\ell})\cdot\frac{d}{dx}\underline{\ell}(x,t)\right)\right\} \\ &= \tfrac{1}{2}\hbar S \int dt \int dx \left[ \left(\partial_x \underline{A}(\ell)\cdot\partial_t\underline{\ell}\right) - \left(\partial_t\underline{A}_{\ell}\cdot\partial_x\underline{\ell}\right)\right] \\ &= \tfrac{1}{2}\hbar S \int dt \int dx \left[(\nabla_{\ell}\times\underline{A}(\underline{\ell}))\cdot(\partial_x\underline{\ell}\times\partial_t\underline{\ell})\right] \\ &= \tfrac{1}{2}\hbar S \int dt \int dx \left[\underline{\ell}\cdot(\partial_x\underline{\ell}\times\partial_t\underline{\ell})\right] \end{aligned}\right\} \tag{212}$$

which reproduces (206). Then finally

- We can simply subtract off a total time derivative from our initial formula, as follows:

$$\left.\begin{aligned} S_B[\underline{\ell}] &= \tfrac{1}{2}\hbar S \int dt \sum_j (-1)^j \underline{A}\cdot\dot{\underline{\ell}}_j \\ &= \tfrac{1}{2}\hbar S \int dt \int dx \left[ \frac{d}{dx}\left(\underline{A}(\ell)\cdot\frac{d}{dt}\underline{\ell}(x,t)\right) - \frac{d}{dt}\left(\underline{A}(\underline{\ell})\cdot\frac{d}{dx}\underline{\ell}(x,t)\right)\right] \end{aligned}\right\} \quad (213)$$

where we assume boundary conditions such that $\underline{A}\cdot(d\underline{\ell}/dx)$ is unchanged between the initial and final states. It is then straightforward, using the same manoeuvres as above, to get the final answer in (206).

We can now summarize all of this by saying that for a 1-dimensional AFM coupled spin chain, the action takes the form

$$\left.\begin{aligned} S_{eff}[\underline{\ell}] &= S^{NLS}[\ell] + S^{Top}[\ell] \\ S^{NLS}[\underline{\ell}] &= -\tfrac{1}{2}\rho_s^0 \int dt \int dx \left[ (\partial_x \underline{\ell}(x,t))^2 + \tfrac{1}{c_0^2}(\partial_t \underline{\ell}(x,t))^2 \right] \\ S^{Top}[\underline{\ell}] &= \frac{\theta}{4\pi}\int dx \int dt\ \underline{\ell}\cdot(\partial_x\underline{\ell}\times\partial_t\underline{\ell}) \\ &\equiv \tfrac{1}{2}\hbar S \int dx \int dt\ \underline{\ell}\cdot(\partial_x\ell\times\partial_t\ell) \equiv \theta Q_{xt} \end{aligned}\right\} \quad (1d) \qquad (214)$$

where the "topological $\theta$-parameter" $\theta = 2\pi\hbar S$.

Now to fully elucidate the properties of this system, we need to know how to treat the non-linear sigma term $S_{NLS}[\underline{\ell}]$; and this we can't do until we have looked more fully at renormalization and the renormalization group. However we can look at the "$\theta$-term", or Berry phase term, in more detail - it is simple to analyse.

Consider first the Pontryagin number $Q_{xt}$; it is quantized, and in fact we see that the total topological term is just going to be

$$S_{Top}[\underline{\ell}] = n\theta \equiv 2\pi\hbar n S \qquad (215)$$

so that we get a phase in the action given by $2\pi n S$. As we have already seen, this distinguishes critically between integer and ½-integer spins (cf. eqns. (107), (108)); in the case where $S$ is an integer, it makes no difference at all to the physics, since $\exp\{4\pi i S\} = 1$. However for ½-integer spin, we can get contributions in the action of $\pm 1$, coming from even/odd $n$ in (215). When we sum over these, it is not obvious what we will get.

To answer this question, as noted above, we need to deal with the other term $S_{NLS}[\underline{\ell}]$ in the total action. However we can simply state here what is found: For ½-integer spins, the effect of the Berry phase $\theta$-term is to make the system gapless (as one finds using simple spin wave theory). However when $S$ = integer, the system develops a large gap in the spin wave spectrum, even when there is no anisotropy or external field. This gap is called the "Haldane gap";

and we will return to it in the next chapter.

<u>AFM lattice, dimension $d \geq 2$</u> : A very obvious question we can ask is – does the $\Theta$-term exist for higher dimensions? This question was of great interest in the early days of high-$T_c$ superconductivity, until it was discovered that the answer is, at least for most cases, no.

To see this, let's repeat the same sort of analysis as we did for the 1-d case, but now for the (2+1)-dimensional case, ie., for spatial dimension $d = 2$. Then instead of (204) we have

$$S_B[\underline{\ell}] = \hbar S \int dt \sum_{ij} (-1)^{i+j} \underline{A} \cdot \dot{\underline{\ell}}_{ij}(t) \tag{216}$$

where now $i$ and $j$ refer to lattice sites measured along the $x$ and $y$ directions. In the same way as we went to the continuum limit in (205) for a chain, we can do the same in 2-d for a lattice function $f_{ij}$:

$$\sum_i \sum_j (-1)^{i+j} f_{ij} \longrightarrow \tfrac{1}{4} \int dx \int dy \left( \frac{\partial^2 f}{\partial x \, \partial y} \right) \tag{217}$$

and so then (216) transforms in this limit to

$$\left.\begin{aligned} S_B[\underline{\ell}] &\longrightarrow \tfrac{1}{2}\hbar S \int dt \int dx \int dy \; \partial_y \left[ \partial_x \left( \underline{A} \cdot \frac{d}{dt} \underline{\ell}(x,y;t) \right) \right] \\ &= 2\pi\hbar S \int dy \; \partial_y \left( Q_{xt}(y) \right) \end{aligned}\right\} \tag{218}$$

where the quantity $Q_{xt}(y)$ is the topological term as a function of $x$ and $t$, fixed at a specific $y$:

$$Q_{xt}(y) = \frac{1}{4\pi} \int dx \int dx \; \underline{\ell}(x,y;t) \cdot \left( \partial_x \underline{\ell}(x,y;t) \times \partial_t \underline{\ell}(x,y;t) \right) \tag{219}$$

Now, however, the problem appears. For a given fixed $y$, $Q_{xt}(y)$ is a fixed topological charge; and changing $y$ will not change this. Thus – this is the standard argument – $\partial_y Q_{xt}(y) = 0$. It then follows that for a 2-d lattice (and also for $d = 2$), the Berry term is zero. We will return to this point later, because this argument can fail if there are singularities in the $\underline{\ell}(r,t)$ field.

It turns out that all of this theory agrees well with experiment, once one has handled the non-linear sigma term (see next chapter). We therefore conclude that even in 1 dimension, we can describe the AFM-coupled spin system as though it has short-range order; but there may well be no long-range order (a conclusion which will be demonstrated in the next chapter). The effect of the topological term is profound, because it lifts the gap in what would have otherwise been a Goldstone boson mode. This conclusion doesn't yet follow from what we've done – again, it requires renormalization group methods.

# B.6.2 : FAST & SLOW VARIABLES

A central element in the formulation of all physical theories is the separation of physical processes into different scales - both time scales and length scales. In what follows, we emphasize that ALL theories in physics involve this - that there is no such thing as a genuinely "microscopic" theory, and that all Hamiltonians or Lagrangians are "effective".

The standard mantra in physics, which is formulated in the ideas of "renormalization", and "integrating out" degrees of freedom, is that when we write down an effective Lagrangian or effective Hamiltonian, we have already performed a kind of "coarse graining" of the system, in which very fast and very small-scale processes and structures have been eliminated from the description, and incorporated into an effective theory into which these fast, small objects have been absorbed into larger and slower entities which now form the basic elements of the theory. To illustrate this, let's consider 2 examples:

(1) A Hydrogen Atom in a Box: Suppose we have a container, an absolutely impermeable box of volume $V_0$ (assume $V_0 = L_0^3$), containing a single H atom, along with photons in a thermal distribution at temperature $T$. How then should we describe this system? It is clear that if the temperature satisfies $kT \ll E_0$, where $E_0$ is the ionization energy, we can write a Hamiltonian

$$H = P^2/2M + \sum_q \hbar\omega_q \left(b_q^\dagger b_q + 1/2\right) + V_{int}\left(\Phi, b_q, b_q^\dagger\right) \quad (220)$$

where $M = M_0 + m_e$ is the combined mass of the proton and electron, $P$ is the momentum of the H atom, $b_q$ and $b_q^\dagger$ are photon operators, and $V_{int}$ is the weak interaction between the photons and the position $\Phi$ of the atom.

But what if the photons are at a much higher temperature, say $10^5$ K? Then we must write

$$\mathcal{H} = \frac{P_N^2}{2M_0} + \frac{P_e^2}{2m_e} + \frac{e^2}{|\underline{R}_N - \underline{r}_e|} + \sum_q \hbar\omega_q \left(b_q^\dagger b_q + 1/2\right) + \tilde{V}_{int}\left(\underline{R}_N, \underline{r}_e, b_q, b_q^\dagger\right) \quad (221)$$

where we are still working with a non-relativistic Hamiltonian, but now with Coulomb interactions between electron and proton, and the much stronger couplings between these 2 charged particles and the photons. Notice that even the vacua of these 2 Hamiltonians are different; in the first case we have a photon vacuum along with the vacuum for a single uncharged particle, whereas in the second, the vacuum is that for 2 charged particles plus photons.

If we continue to raise the energy we can go up to $kT \sim m_e c^2$, i.e., $kT \sim 5 \times 10^9$ K. At this point we must switch to a Lagrangian

$$L(x) = \bar{\psi}_N(x)\left[i\gamma^\mu D_\mu - M_0\right]\psi_N(x) + \bar{\psi}_e\left[i\gamma^\mu D_\mu - m_e\right]\psi_e + F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (222)$$

and, to fix the constraint that the net electronic charge in the system is unity (and likewise for the proton), we can add Lagrange multipliers to fix these, in the action. Now the vacuum is the full QED vacuum, with both nucleon and electron fields, and it will show strong vacuum fluctuations, with electron-positron pairs appearing all over the place.

One can of course continue this to even higher $T$, but the point is clear; each of these descriptions involves the assumption of some set of building blocks with effective couplings between them, and all more microscopic processes are incorporated implicitly into these.

(ii) A simple Fluid: Consider a gas of, eg., $^3$He atoms. At a temperature of, eg., 100 K, we can describe this by a simple effective Hamiltonian of form

$$\mathcal{H} = \sum_j \frac{\underline{P}_j^2}{2M_3} + \sum_{i \neq j} V(\underline{r}_i - \underline{r}_j) \qquad (\text{fast}) \tag{223}$$

where $M_3$ is the mass of the $^3$He atoms, and $V(\underline{r}_i - \underline{r}_j)$ the non-relativistic effective interaction between them (with a short-range strong repulsion, and long-range weak attraction).

However if we are only interested in the "long-wavelength" properties of the system at this temperature — ie., those occurring at length scales $\gg n^{-1/3}$, where $n$ is the particle density, and timescales $\gg \tau_0$ (the mean free time between collisions) then we can go over to the continuum description embodied in the Navier-Stokes equation of motion, viz.,

$$\frac{d}{dt}\underline{v}(\underline{r},t) + \nu \underline{\nabla}^2 \underline{v}(\underline{r},t) = -\underline{\nabla} p(\underline{r},t) \qquad (\text{slow}) \tag{224}$$

where $p(\underline{r},t)$ is the pressure, and $\nu$ the dynamical velocity; this is known as a "hydrodynamic" description, and we will be discussing it in much more detail later on.

If we now cool the $^3$He to, say, 10 mK, it will liquefy (and solidify if the pressure is high enough) and then we go over the Landau Fermi Liquid Theory description, with an effective Hamiltonian

$$\mathcal{H} = \sum_{\underline{k}\sigma} \varepsilon_{\underline{k}\sigma} c^+_{\underline{k}\sigma} c_{\underline{k}\sigma} + \sum_{\underline{k}\underline{k}'\underline{q}\sigma\sigma'} f^{\sigma\sigma'}_{\underline{k}\underline{k}'} c^+_{\underline{k}+\underline{q},\sigma} c_{\underline{k}'\sigma'} c^+_{\underline{k}'-\underline{q},\sigma} c_{\underline{k}\sigma} \tag{225}$$

involving quasiparticle operators $c^+_{\underline{k}\sigma}$, $c_{\underline{k}\sigma}$, a quasiparticle energy $\varepsilon_{\underline{k}\sigma}$, and effective FLT interactions $f^{\sigma\sigma'}_{\underline{k}\underline{k}'}$. If we go to still lower temperatures, we go through a superfluid transition, and the effective Hamiltonian becomes very complicated indeed. Note again that the vacua of these different descriptions are all quite different from each other.

In what follows we will discuss various techniques for separating out the fast and slow processes in a DFT, and extracting the key information about each. We note that we will be focussing on a single system at a given temperature, and switching between two energy/length scales — this is different from

what happens when we change the temperature, which will change completely the nature of the system itself.

# B.6.2 (a) GRADIENT EXPANSIONS

Let us begin by recalling that a semi-classical expansion about the classical theory, introduced in section B.6.1 (b) and in typical WKB form in eqtns (67)-(73), is an expansion in powers of $\hbar$; but it also involves an expansion in spatial and temporal gradients, as we see from the first few terms in the expansion, and from the general recursion relation in (69). One sees this also in the lowest "eikonal" term in the expansion of the Schrodinger eqtn; if we write

$$\psi(r,t) = A(r,t)\, e^{iS/\hbar} \tag{226}$$

and substitute into the Schrodinger eqtn., viz.,

$$\left[-\frac{\hbar^2}{2m} + V(r)\right]\psi = i\hbar\,\partial_t\,\psi \tag{227}$$

we find, assuming that $A(r,t)$ is slowly-varying, the lowest-order eqtn (of order $\hbar^0$, ie., independent of $\hbar$):

$$\frac{1}{2m}\left(\nabla S(r,t)\right)^2 + V(r) = -\partial_t S(r,t) \tag{228}$$

which is just the Hamilton-Jacobi eqtn for $S(r,t)$, the action. One can continue this along the lines given in eqtns (67)-(69).

Let's now start off by doing some explicit gradient expansions, in several different ways.

## (i) GRADIENT EXPANSION FOR DRIVEN SCALAR FIELD

We have already discussed what happens when a scalar field (or for that matter, an ordinary particle in, eg., a harmonic well) is subject to an external field - whether this field just be some applied force, or whether it be some time- and space-varying potential. Thus, eg., we could pick an action

$$S = \frac{1}{2}\int d^3x\left[\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) + V(x)\,\phi^2(x)\right] \tag{229}$$

in which the external field acts on $\phi(x)$ in addition to the constant mass. If $V(x)$ is just treated as some external field, then we leave everything at the end as a functional of $V(x)$, so that, eg., the 2-point propagator takes the form $\mathcal{G}(x,x'|V)$, and the partition function / generating functional can be

written as a functional $Z[J,V]$ of both the potential $V(x)$ and the applied field $J(x)$.

The idea now is to pick out the long-wavelength, slow parts of this, in powers not of the coupling constant but of the spacetime gradients. This means that the results ought to be valid away from weak coupling, if we do things properly.

To illustrate this let's start with a problem we already understand really well, which is a driven free scalar field, with the driving coming from a current source, giving

$$S[J] = \int d^4x \, \tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - J(x)\phi(x) \tag{230}$$

so that we deal with

$$Z[J] = \int \mathcal{D}\phi(x)\, e^{i S[J]} \tag{231}$$

We know, of course, how to deal with this problem exactly; but let's look at it now in a gradient expansion. The 1-particle propagator satisfies the eqtn. of motion

$$\left(\partial^2 - m^2 + J(x)\right) G(x,x'|J) = -\delta(x-x') \tag{232}$$

Now if $J(x)$ is considered to be slowly varying, we expect $G(x,x'|J)$ to be a slowly-varying function of the "centre of mass" coordinate $\tfrac{1}{2}(x+x')$, but not necessarily of $x-x'$ (to convince yourself of this, consider the simple example of an oscillator, subject to a force $J(t)$; if $J(t)$ is a constant, all we do is shift the oscillator, and if it is slowly-varying compared to the natural frequency $\omega_0$ of the oscillator, then it will hardly disturb the oscillator; but the oscillator will still have natural oscillations on a timescale $2\pi/\omega_0$, which will be reflected in the Green function $G(t,t'|J(t))$).

Thus we see that if we have the condition

$$\left|\frac{\partial_\mu J}{J}\right| \ll m. \tag{233}$$

we can assume slow variation. Let's now Fourier transform with respect to $x-x'$, so that

$$G_k(x|J) = \int d^4(x-x')\, G(x,x'|J)\, e^{-ik\cdot(x-x')} \tag{234}$$

& so that the eqtn of motion now reads

$$\left[(k_\mu + i\partial_\mu)(k^\mu + i\partial^\mu) + m^2 - J(x)\right] G_k(x|J) = 1 \tag{235}$$

where for simplicity we write things in terms of $x$ rather than $\tfrac{1}{2}(x+x')$. We are now going to deal with the formal solution of this eqtn in 2 different ways. We have formally that

$$G_k(x) = \frac{1}{\left(k^2 - \partial_x^2 + 2ik_\mu\partial_x^\mu + m^2 - J(x)\right)} \tag{236}$$

and the idea is to now use the slow variation of $J(x)$, and of the gradient operators in this result, as the basis for an expansion. How we do this depends on what we choose for our unperturbed result, and what we choose as the expansion parameter.

(a) **FLUCTUATION EXPANSION** : Suppose we treat the gradient operators as the perturbation, and so write the propagator eqtn of motion as

$$\left[ G_0^{-1}(k,x|J) - W_k(\partial_x^\mu) \right] \mathcal{G}_k(x|J) = 1 \tag{237}$$

where we have

$$G_0(k,x|J) = \frac{1}{k^2+m^2-J(x)} \tag{238}$$

and

$$W_k(\partial_\mu) = \partial_x^2 - 2ik_\mu \partial_x^\mu \tag{239}$$

where we have written $\partial_x^\mu$ and $\partial_x^2$ to explicitly indicate that the operator $\partial_\mu$ is acting on the variable $x$ (ie, the variable upon which things depend slowly).

We can now write a Dyson expansion for the eqtn of motion; we have

$$\left.\begin{aligned} \mathcal{G}_k(x|J) &= \left[ G_0^{-1}(k,x|J) - W_k(\partial_\mu) \right]^{-1} \\ &= G_0(k,x|J) + G_0(k,x|J)\, W_k(\partial_\mu)\, G_0(k,x|J) + \cdots \end{aligned}\right\} \tag{240}$$

Note that this form of the expansion is only useful if we have a simple form for $G_0(k,x|J)$. To see how this works let's determine the lowest corrections to $G_0$ in this expansion. We have

$$\left.\begin{aligned} \delta^{(1)}\mathcal{G}_k(x|J) &= G_0 W G_0 \\ &= \frac{1}{k^2+m^2-J(x)} \left(\partial^2 - 2ik_\mu\partial^\mu\right) \frac{1}{k^2+m^2-J(x)} \\ &= G_0^3(k,x|J)\left[\left(\partial^2 - 2ik_\mu\partial^\mu\right)J(x)\right] \quad \sim \; -2i\,G_0^3(k,x|J)\,k_\mu\partial^\mu J(x) \end{aligned}\right\} \tag{241}$$

and then for the 2nd correction

$$\left.\begin{aligned} \delta^{(2)}\mathcal{G}_k(x|J) &= G_0 W G_0 W G_0 \\ &= G_0^3(k,x|J)\left[\left(\partial^2 - 2ik_\mu\partial^\mu\right)^2 J(x)\right] + 3\,G_0^5(k,x|J)\left[\left(\partial^2 - 2ik_\mu\partial^\mu\right)J(x)\right] \\ &\sim \; -4\,G_0^3(k,x|J)\,k_\mu\partial^\mu k_\nu\partial^\nu J(x) - 6i\,G_0^5(k,x|J)\,k_\mu\partial^\mu J(x) \end{aligned}\right\} \tag{242}$$

where in the approximate result we assume that $|k_\mu\partial^\mu J(x)| \gg \partial^2 J(x)$, ie, that the important values of $k$ extend over a much wider range than the Fourier transform of $J(x)$ (because things vary more rapidly with $x-x'$ than with $x$).

If we wish we can further Fourier transform $\mathcal{G}_k(x|J)$, so that everything is now

written in terms of the Fourier transform of $J(x)$, ie., in terms of

$$J_q = \int d^4x\, e^{-iqx} J(x) \tag{243}$$

$$\mathcal{G}_{kq}[J_q] = \int d^4x\, e^{-iqx}\, \mathcal{G}_k(x|J) \tag{244}$$

We then have

$$\left[(k^2+q^2+2k_\mu q^\mu) + m^2 - J_q\right] \mathcal{G}_{kq}[J_q] = 1 \tag{245}$$

and we can then calculate everything from this.

We can of course represent this Dyson eqtn diagrammatically, as shown in the figure; the extend perturbing fields are now the gradient operators


so that we can think of the extend gradient operator $W_{k_\mu}(\partial^\mu)$ as a self-energy operator.

## (b) FRADKIN-STYLE WKB EXPANSION

: A somewhat more obvious tactic to employ in making a gradient expansion of a function like $\mathcal{G}_k(x|J)$ is to separate off everything but the free field term, ie., write

$$\left[G_0^{-1}(k) - U_k(\partial_\mu, J(x))\right] \mathcal{G}_k(x|J) = 1 \tag{246}$$

where

$$G_0(k) = \frac{1}{k^2+m^2} \tag{247}$$

and

$$U_k(\partial_\mu, J(x)) = \partial_x^2 - 2ik_\mu\partial^\mu + J(x) \tag{248}$$

is the perturbation. Now there is a nice way to represent the gradient expansion which makes it look much more like the WKB expansion we discussed in eqtns. (67)-(73); this uses what is usually called the "Schwinger proper time" or "5th-time" representation (although it is actually due to V. Fock), in a form discussed by Fradkin. We begin by noting we can write the Feynman propagator as

$$G_0(k) = \frac{1}{k^2+m^2-i\delta} = i\int_0^\infty ds\, e^{-is(k^2+m^2-i\delta)} \equiv i\int_0^\infty ds\, G_0(k,s) \tag{249}$$

and now write the total Green function in the form

$$\mathcal{G}_{ic}(x|J) = i\int_0^\infty ds\, e^{i(k^2+m^2-i\delta)} \mathcal{F}_k(x,s|J) \tag{250}$$

where the slowly-varying effects are all encapsulated in the functional $\mathcal{F}_k(x,s|J)$. The parameter $s$ is called the proper time, and we see why this is so if we note that $G_0(k,s)$ satisfies the differential eqtn

$$(k^2+m^2-i\delta)\, G_0(k,s) = i\partial_s G_0(k,s) \tag{251}$$

We can now think of the functional $\mathcal{F}_k(x,s|J)$ as a weighting functional over the free particle propagator; note that when $J(x) = 0$, $\mathcal{F}_k(x,s|J) = 1$.

It is now time to put the factors of $\hbar$ back into all of these eqtns; if we do so, we find that we must replace $U_k(\partial_\mu, J(x))$ by

$$U_k(\partial_\mu, J(x)) = \partial_x^2 - 2ik_\mu\partial^\mu + \hbar J(x) \tag{252}$$

Let us now, in the spirit of WKB, write $\mathcal{F}_k(x,s|J)$ in the form

$$\mathcal{F}_k(x,s|J) = e^{\Phi(x)/\hbar} = e^{1/\hbar \sum_n \hbar^n \psi_n(x)} \tag{253}$$

and then substitute this into the eqtn of motion (compare the substitution of (68) into (67). We find the following differential eqtn for $\psi_n(k,x;s|J)$:

$$(\partial^2 + 2ik_\mu\partial^\mu)\psi_n + \sum_{r=1}^{n-1} \partial_\mu\psi_r\,\partial^\mu\psi_{n-r} + J(x)\,\delta_{n1} = -i\partial_s\psi_n \tag{254}$$

which should be compared with eqtn. (69); note that $\psi_0 = 0$.

We then find the leading correction $\psi_1(k,x;s|J)$ from the differential eqtn.

$$(\partial^2 + 2ik_\mu\partial^\mu + i\partial_s)\,\psi_1(k,x;s|J) = -J(x) \tag{255}$$

This is the same kind of eqtn we run into when trying to solve for any function defined in terms of a Green function with a source $J(x)$; you should easily find that

$$\psi_1(k,x;s|J) = i\int_0^s ds' \sum_q e^{iqx}\,\phi_1(q,x;s'|J) \tag{256}$$

with

$$\phi_1(qx;s'|J) = J_q\, e^{-is'(q^2-2k_\mu q^\mu)} \tag{257}$$

and $J_q$ the Fourier transform of $J(x)$ defined above. The higher terms are given by more complicated forms. The general form for the higher terms can be found, if one wishes, by either using the set of coupled eqtns

in (254), or by rewriting it, using the result we found for $\psi_1$ in (256), in the form

$$\psi_n(k,x;s|J) = i\int_0^s ds'\, e^{i(s-s')(\partial^2 - 2ik_\mu\partial^\mu)} \sum_{r=1}^{n-1} \partial_\mu \psi_r(k,x;s|J)\,\partial^\mu \psi_{n-r}(k,x;s'|J) \quad (258)$$

Now these results are a little academic, in that we already know how to write a solution to this problem – it is just the well-worn Gaussian result that we've been using since section B.1. Recall that we have

$$Z_0[J] = e^{-\frac{i}{2\hbar}\int d^4x \int d^4x'\, J(x)\Delta_F(x-x')J(x')} \quad (259)$$

and the propagator $G_2(x,x'|J)$ is (cf. eqtn B.1 (67)):

$$G_2(x,x'|J) = -\hbar^2 \frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(x')} \quad (260)$$

where now we do <u>not</u> set $J(x) = 0$ after taking the 2nd functional derivative; we have

$$\left.\begin{aligned} G_2(x,x'|J) = {} & \left[\int d^4y \int d^4y'\, \Delta_F(x-y)J(y)\Delta_F(x'-y')J(y') - \Delta_F(x-x')\right] \\ & \times e^{-\frac{i}{2\hbar}\int d^4x \int d^4x'\, J(x)\Delta_F(x-x')J(x')} \end{aligned}\right\} \quad (261)$$

It is an interesting exercise to expand this result directly in a gradient expansion, and compare with what we have found above.

The real use of gradient expansions comes in when we have more than one field. This can happen either when we are dealing with a coupled field system (which we will come to presently, in the form of QED), or when we split a single field into 2 components, viz., a slowly-varying and a rapidly-varying one. This tactic forms the basis of the renormalization and renormalization group approaches (next chapter); it is also at the basis of the "background field method", to which we come very soon.

## (ii) GRADIENT EXPANSION for Q.E.D.

We now turn to a real physical theory, where the use of long-wavelength analyses has been of key importance over a long period of time. The best-known problem in QED of this kind is that of infra-red divergences and the contribution of soft photons to electron-photon scattering – this is usually known as the "Bremsstrahlung" problem, where it refers to the deceleration of electrons or other charged particles by photon emission, when they are in the field of some other charged object which changes their trajectory (a charged particle moving in free space cannot emit photons – this is kinematically impossible). It is process

of key importance in nuclear physics, astrophysics, plasma physics, and solid-state physics. The photons are emitted in all directions, with intensity $\propto m^{-4}$ (for radiation perpendicular to the motion of the particle, of mass $m$), and $\propto m^{-6}$ (for radiation parallel to the motion. This means that light charged particles like electrons lose energy very quickly to this process.

The theoretical problems arise when one tries to analyze this process perturbatively in QED, and it quickly becomes apparent that the process is non-perturbative — indeed, as the energy of the emitted photons goes to zero, the number of photons carrying emitted energy diverges, and one must sum diagrams to infinite order to properly understand it. This phenomenon was called the "IR catastrophe" by Bloch & Nordsieck in 1937, who first analyzed it. In solid-state physics in the late 1960's and 1970's it was called the "X-ray edge" catastrophe, or the "Anderson orthogonality" catastrophe, and it figured large in the discussion of the Kondo problem; it was not finally tamed until the development of renormalization group methods.

That the problem is serious is obvious if we consider that it affects not only all scattering processes involving charged particles, but also the calculation of the one-particle propagator $\mathcal{G}_2(x,x')$ for the electrons — one needs to understand the Bremsstrahlung contribution to find the low-energy form for this propagator.

In what follows we will see how to do a gradient expansion on this problem — it turns out that this does the job, and is exact in the low-energy limit.

## ONE-PARTICLE PROPAGATOR $\mathcal{G}_2(x,x'|A)$:

If we try to calculate the Bremsstrahlung process and its effect on $\mathcal{G}_2(x,x')$ using a perturbative expansion in the coupling, we find that we cannot get any sensible results without summing diagrams to infinite order — but which diagrams should we sum?

So instead we adopt the following tactic, which we already developed back in section B.3. We calculate $\mathcal{G}(x,x'|A_\mu(z))$, where we have frozen $A_\mu(z)$ to begin with; and then we functionally average $\mathcal{G}_2(x,x'|A)$ over all different configurations of $A_\mu(x)$, with the appropriate weighting factor.

If we followed the development of section B.3, we would first calculate $Z[\eta,\bar\eta;J_\mu]$, and then derive $\mathcal{G}_2(x,x'|A_\mu)$ from this by functional differentiation. But from what we've done above, we see there is no need to do this — we can simply calculate $\mathcal{G}_2(x,x'|A_\mu)$ directly. We have the eqtn. of motion for the fermion propagator, given by

$$\left[\gamma^\mu(i\partial_\mu + eA_\mu) - m\right]\mathcal{G}_2(x,x'|A_\mu) = -\delta(x-x') \tag{262}$$

in the presence of some fixed or "frozen" background field $A_\mu(x)$, which we will later unfreeze. In the early days of QED, when the first attempts were being made to understand the IR divergence problem, this eqtn. was approximated by the simpler eqtn.

$$\left[c^\mu(i\partial_\mu + eA_\mu) - m\right]\mathcal{G}_2(x,x'|A_\mu) = -\delta(x-x') \tag{263}$$

where the $c^\mu$ are numbers, instead of the gamma matrices $\gamma^\mu$ in (262), and which

satisfy the relation $$g_{\mu\nu} c^\mu c^\nu = 1 \tag{264}$$

In this case the free particle propagator $G_0(xx')$ obeys the eqtn.

$$(ic^\mu \partial_\mu - m) G_0(xx') = \delta(x-x') \tag{265}$$

with solution $$G_0(x,x') = e^{ik(x-x')} \frac{i\hbar}{c^\mu k_\mu - m - i\delta} \tag{266}$$

instead of

$$(i\gamma^\mu \partial_\mu - m) S_F(x,x') = -\delta(x-x') \tag{267}$$

with solution $\mathcal{G}_0 = i\hbar S_F$ given by

$$\mathcal{G}_0(x-x') = \sum_k e^{-ik(x-x')} \frac{i\hbar}{\gamma^\mu k_\mu - m + i\delta} \tag{268}$$

(cf chapter B.2, eqtns (52)-(55)).

Let's now do the same thing here as we did for the scalar field theory, finding in proper time representation the result for $\mathcal{G}_2(x,x'|A)$, and for its partial Fourier transform $\mathcal{G}_k(x|A)$. Without going through all the details, we see that we can write

$$\left.\begin{aligned} \mathcal{G}_2(x,x'|A) &= i\int_0^\infty ds\ \mathcal{G}_s(xx'|A) \\ &= i\int_0^\infty ds\ e^{is/\hbar[\gamma^\mu(i\partial_\mu + eA_\mu) - m]}\ \delta(x-x') \end{aligned}\right\} \tag{269}$$

so that

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\,\mathcal{G}_s(xx'|A) = -i\hbar\,\partial_s\,\mathcal{G}_s(x,x'|A) \tag{270}$$

with the boundary condition $\mathcal{G}_s(x,x'|A) \longrightarrow \delta(x-x')$, as $s \to 0$.

Now we Fourier transform the relative coordinate as before, ie., write the $\delta$-function in (269) as

$$\delta(x-x') = \sum_k e^{ik(x-x')} \equiv \int \frac{d^4p}{(2\pi\hbar)^4} e^{\frac{ip}{\hbar}(x-x')} \tag{271}$$

so that we can write $\mathcal{G}_s(xx'|A)$ as

$$\mathcal{G}_s(x,x'|A) = \sum_k e^{-ik(x-x')}\,\mathcal{G}_k(x;s|A) \tag{272}$$

and then write $\mathcal{G}_k(x;s|A)$ in the form (cf. eqtns. (250) and (253)):

$$\mathcal{G}_k(x;s|A) = e^{is/\hbar(\gamma^\mu p_\mu - m + i\delta)}\, e^{i\Phi_s(x|A)} \tag{273}$$

where now we put back the usual $i\delta$ factor in the free propagator, and define $p_\mu = \hbar k_\mu$. The phase factor $\Phi_s(x|A)$ in (273) then satisfies a differential eqtn., which for low energy, such that $p^2 \ll m^2$, taken the simplified

form:

$$\gamma^\mu \left( \partial_\mu \Psi_s + e A_\mu \right) = \partial_s \Psi_s \tag{274}$$

with the boundary condition: $\Psi(s=0) = 0$.

The solution to this differential eqtn. is simple; in analogy with the previous discussion (cf. eqtn (255) - (257)) we Fourier transform to $q$-space, and get

$$\Psi_s(x|A) = \int_0^s ds' \sum_q e^{-iqx} e^{is'p^\mu q_\mu} e p_\mu A^\mu(q) \tag{275}$$

where $A_\mu(q)$ is the Fourier transform of $A_\mu(x)$. Thus we can write out the complete solution for $G_s(x,x'|A)$ in the form (using (275), (273), and (272)):

$$G_2(x,x'|A) = i \sum_k e^{-ik(x-x')} \int_0^\infty ds\, e^{is/\hbar (\gamma^\mu p_\mu - m + i\delta)} e^{ie \int_0^s ds' \sum_q e^{is'q^\mu p_\mu} e^{-iqx} p^\mu A_\mu(q)} \tag{276}$$

Notice 2 things about this solution, in comparison with the discussion of the scalar field. First, we do not need to go to an expansion in powers of $\hbar$ in the phase $\Psi_s$, and so there is no set of recursive eqtns as in (254). This is because the low-E Dirac eqtn is only linear in derivatives $\partial_\mu$, instead of quadratic like scalar field theory (or the Schrödinger eqtn). Thus (276) is a closed solution. The 2nd thing we notice is that the phase factor $\Psi_s(x|A)$ does not depend on the relative momentum $k$; this is because of the simplicity of the low-E Dirac eqtn., so that apart from the linear derivative $\partial_\mu$ we only have the function $A_\mu(x)$.

Notice also that we have yet to do any kind of gradient expansion; this will come when we go over to the functional integration over $A_\mu(x)$.

## INTEGRATED PROPAGATOR $G_2(x,x')$ IN IR REGIME:

We now wish to integrate out the $A$-field, to get an expression for $G_2(x,x')$. This is the real point of deriving the result in (276); it forms a platform for deriving an expression for the propagator which does not explicitly involve a sum of graphs. Thus we will be evaluating an expression of form

$$G_2(x,x') = \int \mathcal{D}A^\mu \, \mathcal{P}[A^\mu] \, G_2(x,x'|A) \tag{277}$$

where $\mathcal{P}[A^\mu]$ is a weighting functional for different configurations of $A^\mu(x)$; essentially a probability amplitude distribution.

What do we choose for the weighting functional? Let's recall what we found in previous chapters for the generating functional & propagators in QED. The generating functional can be written as (cf chapter 3.4, eqtn. (168)):

$$Z_{EM}[J_\mu] = \int \mathcal{D}A_\mu \, e^{\frac{i}{\hbar}\left\{ \sum_q \frac{1}{2} A^\mu_q (D^0_{\mu\nu}(q))^{-1} A^\nu_{-q} - J_\mu(q) A^\mu_{k+q} \right\}} \tag{278}$$

where one assumes the existence of a fixed source $J_\mu(x)$ which we can, if we wish,

associate with the Dirac fermions via the usual relationship between $\bar{\psi}(x)$, $\psi(x)$, and the Dirac current $J_\mu(x)$.

From a comparison of (277) and (278), we might naively suppose that the weighting factor $\mathcal{P}[A^\mu]$ is then just given by

$$\mathcal{P}[A^\mu] = \mathcal{P}_0[A^\mu] = e^{\frac{i}{2\hbar}\sum_q A^\mu_q (D^0_{\mu\nu}(q))^{-1} A^\nu_{-q}} \tag{279}$$

Now actually this cannot be right in general, because we know that the photon propagator itself is renormalized by the coupling to fermions - we will return to this point below. But let's first just try this out, to see what happens. Putting (279) into (277), we have

$$\mathcal{G}_2(x,x') = \int \mathcal{D}A^\mu \, e^{\frac{i}{2\hbar}\sum_q A^\mu_q (D^0_{\mu\nu}(q))^{-1} A^\nu_{-q}} \; \mathcal{G}_2(x,x'|A^\mu_q) \tag{280}$$

or, if we want to be more sophisticated, the renormalized photon propagator $D_{\mu\nu}(q)$ in place of $D^0_{\mu\nu}(q)$ in (280).

Notice that we have no Fadeev-Popov determinant in (280); again, I will return to this point below.

Let's now see what we get from this. The Gaussian functional integral is trivial; writing:

$$\begin{aligned} \mathcal{F}_S(p,q|A) &= f_S^\mu(p,q)\, A_\mu(q) \\ &= \left(e\int_0^S ds'\, e^{is'p^\nu q_\nu} p^\mu\right) A_\mu(q) \end{aligned} \tag{281}$$

we have

$$\begin{aligned} \mathcal{G}_2(p) &= i\int_0^\infty ds \int \mathcal{D}A^\mu_q \; e^{\frac{i}{2}\sum_q A^\mu_q (D^0_{\mu\nu}(q))^{-1} A^\nu(-q) + i f_S^\mu(p,q) A_\mu(q)} \; e^{isG_0^{-1}(p)} \\ &= i\mathcal{N}\int_0^\infty ds\; e^{isG_0^{-1}(p)} \; e^{-\frac{i}{2}\sum_q f_S^\mu(p,q) D^0_{\mu\nu}(q) f_S^\nu(p,-q)}. \end{aligned} \tag{282}$$

where the normalization factor $\mathcal{N}$ will drop out later, so we henceforth ignore it. We can write the interaction term in (282) in various ways, viz.,

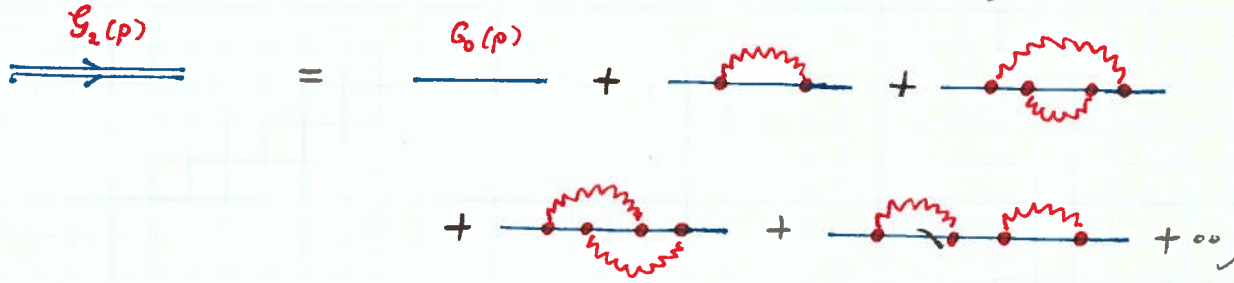
$$\begin{aligned} \chi_S(p) &= \frac{1}{2}\sum_q f_S^\mu(p,q) D^0_{\mu\nu}(q) f_S^\nu(p,-q) \\ &= \frac{e^2}{2}\sum_q \int_0^S ds_1 \int_0^S ds_2 \; e^{i(s_1-s_2)p^\mu q_\mu} \; p^\mu D^0_{\mu\nu}(q) p^\nu \\ &\equiv \frac{e^2}{4}\sum_q \int_0^S ds_1 \int_0^{s_1} ds_2 \; e^{i(s_1-s_2)p^\mu q_\mu} \; p^\mu D^0_{\mu\nu}(q) p^\nu \\ &= e^2 \sum_q \frac{1-e^{isp^\mu q_\mu}}{(p^\mu q_\mu)^2} \; p^\mu D^0_{\mu\nu}(q) p^\nu \end{aligned} \tag{283}$$

and we will use the Ikot form, to write

$$G_2(p) = i\int_0^\infty ds\, e^{isG_0^{-1}(p)}\, e^{-i\chi_s(p)}$$

$$= i\int_0^\infty ds\, \exp\left\{ i\left[ (\gamma^\mu p_\mu - m + i\delta) - e^2 \sum_q \frac{1 - e^{isp^\mu q_\mu}}{(p^\mu q_\mu)^2}\, p^\mu D^0_{\mu\nu}(q)\, p^\nu \right]\right\} \qquad (284)$$

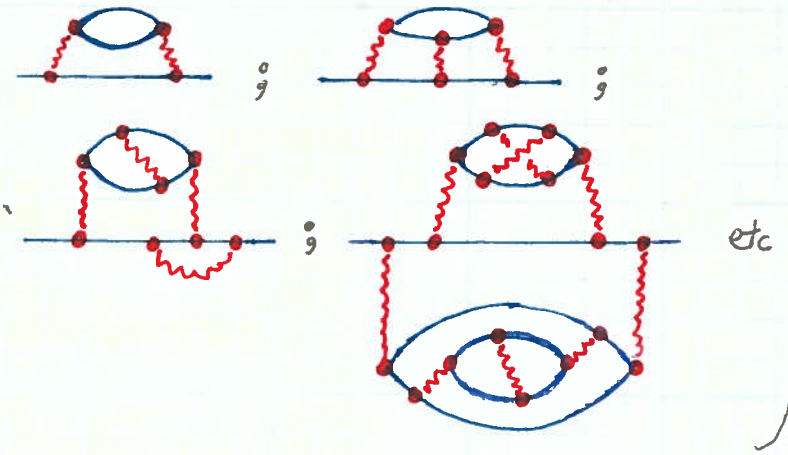
Now let's stop and look at this result, since it is actually quite interesting. The phase factor $\chi_s(p)$ is the sort of thing one sees often in calculations of wave propagation through a medium - we do not pause here to explore this - and from the point of view of perturbation theory it has the effect of summing over an infinite set of graphs, as we see in the figure.




(285)

This graphical sum is easily demonstrated starting from the 3rd form for $\chi_s(p)$ in (283). We notice that, since all the internal fermion lines are bare fermion lines, we sum over both reducible and irreducible graphs in (285). The graphical sum also makes clear what was noted above, viz., that we are only using free photons in the calculation. Thus all the following graphs, shown in the figure, are missing from the result:

**MISSING GRAPHS:**




(286)

We are also missing all the disconnected graphs, but these are cancelled by the normalization factor anyway.

Thus we see that we have a certain approximation here for the fermion propagator, in which we have an arbitrary number of photon loops, but no fermion loops at all.

Why is this result not exact? The reason can be seen if we compare with what we would do if we wanted to get an exact result from this

tactic of integrating over all configurations of $A^\mu(q)$. This would involve the kind of calculation we did in chapter 8.3, where we coupled 2 scalar fields, and then functional integrated over one of them while freezing the other. Now if we did this, we would get the same result as above for $p^2 \ll m^2$, but for for larger fermionic momentum the results would be more complicated, because then higher momentum terms would enter into the results. To see this in this WKB formalism, note that the Fourier-transformed function $\mathcal{G}_2(p,x|A)$ satisfies

$$[\gamma^\mu(i\partial_\mu - p_\mu + eA_\mu) - m]\,\mathcal{G}_p(x|A) = -1 \qquad (287)$$

and to get the result for $\mathcal{I}_s$ given above we have assumed that $\gamma^\mu p_\mu - m$ is small. If we drop this assumption, then the results are more messy, and indeed we find an infinite hierarchy of eqtns again for $\mathcal{I}_s$, when expanded in powers of $e$.

Physically one can understand what is going on here in a fairly simple way. When the photon momenta are small, an electron moving through a vacuum cannot couple to virtual electron-positron pairs, because these require an energy $\geq 2mc^2$ to create. Thus none of the graphs with the internal fermion loops can play a role, and all that is left is the set of graphs shown in (285). However once the threshold energy at $2mc^2$ is reached, the imaginary part of the fermion self-energy will start to be strongly affected by this process.

We also notice that in this calculation, I have paid no attention to the whole problem of pathologies of the functional integration over $A_\mu(q)$, with the necessity for Fadeev-Popov determinants, etc. A proper calculation would start from a determination of $\mathcal{Z}$, with the appropriate introduction of the FP determinant, followed by functional differentiation, etc. But, the key is that for this calculation it doesn't matter. This is because ultimately we have to fix a gauge when we do the calculation of $\mathcal{G}_2(p)$; it is not a gauge-invariant quantity anyway. What we are actually after is the singularity structure of the propagator, which we expect to be gauge-independent because it is physical.

The next step in this discussion would be the calculation of (284), by direct integration over $\int d^4q$, followed by the integration $\int ds$ over proper time. Actually we do not do this here, since it is rather lengthy and unilluminating, but simply quote the answer:

$$\mathcal{G}_2(p) \xrightarrow[p^2 \ll m^2]{} \frac{1}{\gamma^\mu p_\mu - m + i\delta}\left|1 - \frac{\gamma^\mu p_\mu}{m}\right|^\beta \qquad (288)$$

where the parameter $\beta$ is given by

$$\beta = (3-\alpha)\frac{\alpha_{QED}}{2\pi} \qquad (289)$$

where $\alpha$ is the gauge fixing parameter in the photon propagator, and $\alpha_{QED}$ is just the dimensionless coupling in QED (i.e., $\alpha \approx 1/137$).

Because $\alpha_{QED}$ is so small, the effect of the IR divergence on the electron propagator at low energy is rather small. However, as noted above, the effect is rather noticeable in astrophysics and in hot charged plasmas – it is also crucial to accelerator design.

A more detailed analysis reveals that what is going on here is the emission of very large numbers of soft photons in the IR limit. Notice that as $q \to 0$ in (284), the integral $\Sigma$ is proportional to $\int d^4q/q^4$, giving the log divergence that leads to the power law in (283), upon exponentiation. This means that at an energy $q$ we have a number of $1/q$ photons. This is why the classical calculation of Bremsstrahlung is exact in the low energy limit, because a bosonic field becomes classical when its modes are highly populated.

**QED for a FINITE-DENSITY FERMIONIC SYSTEM** : Up to now we have been looking at a single Dirac fermion coupled to photons. But a much more realistic and interesting problem involves a finite density of fermions – this can describe situations ranging from the early universe, the interior of stars (conventional, white dwarf and neutron stars) and other plasmas on earth or in space, down to conducting condensed matter systems (ie., metallic systems).

The situation now changes radically, for 2 reasons. First, we break Lorentz invariance – the Fermi sea fixes a reference frame. Second, there is no longer an energy gap for the production of particle/anti-particle pairs – as we saw in the discussion of Fermi liquids, we can excite particle-hole pairs of arbitrarily low energy.

To deal with these changes we need to both modify the photon propagator, and modify the eqtn. of motion for the fermion propagator $G_2(xx'|A)$ in the presence of some gauge configuration $A_\mu(x)$. Let's first look at the photon propagator – the fully renormalized form for this we have already seen, and is given by

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\alpha}(q)\,\Pi_{\alpha\beta}(q)\,D_{\beta\nu}(q) \qquad (290)$$

where the polarization part $\Pi_{\mu\nu}(q)$ was discussed briefly in the last chapter.

In what follows I don't give any detailed discussion of $\Pi_{\mu\nu}(q)$, which for a charged liquid of fermions is a fairly lengthy task, but simply remark that the charged part of this function (ie., $\Pi_{00}(q)$, and $\Pi_{0j}(q)$, where $j$ runs over the spatial indices) are short-range in real space, because of charge screening. However the magnetic part $\Pi_{ij}(q)$ is long-ranged – one cannot easily screen magnetic fields (there is no "Faraday cage" for magnetic fields). This means that the low-$q$ behaviour of $\Pi_{ij}(q)$ is important, and indeed it can be shown fairly easily that it will yield the form

$$D_{ij}(q) = (\delta_{ij} - \hat{q}_i\hat{q}_j)\,\frac{|\underline{q}|}{\chi|\underline{q}| - i\gamma|\omega|} \qquad (291)$$

in the Landau gauge ($\alpha = 0$) where $\chi$ and $\gamma$ are dimensionful functions which

depend on the details of the liquid of fermions involved: the densities and masses of these fermions, the spatial dimension, and so on. We note that the components $\Pi_{00}(q)$ and $\Pi_{01}(q)$ have been eliminated here (playing no role in the $q \to 0$ IR regime), and we are dealing here purely with transverse photons (ie., magnetic degrees of freedom). One can calculate $\chi$ and $\gamma$ most easily in a RPA, in which $\Pi_{\mu\nu}(q) \to \Pi_0(q)$, the simple particle-hole bubble; a more sophisticated calculation changes $\chi$ and $\gamma$, but not the form of (291). Nor is it changed if one changes the spatial dimension, nor even if one substitutes heavy quarks instead of the electrons, and, eg., gluons instead of the photons. Thus the following calculation is just as relevant to QCD as it is to QED, and can be applied to high-density astrophysical phenomena, as well as to the early universe.

In what follows we discuss a simple example of the kind of calculation that can be done. To make the integrations easier, we reduce the number of spatial dimensions to 2; this has the benefit of making the calculation relevant to systems like 2DEG's (2-dimensional electron gases, used in electronic devices) or the FQHE liquids (FQHE = Fractional Quantum Hall Effect); the model has also been discussed for strongly-correlated systems in 2 dimensions such as high-$T_c$ superconductors.

The effective Hamiltonian we are interested in for the system is then

$$\mathcal{H}_{\text{eff}} = \int d^D r \left\{ \frac{1}{2m} \psi^\dagger(x) \left[ (i\underline{\nabla} - g_0 \underline{A}(x))^2 - \mu \right] \psi(x) + \left[ (\partial_t \underline{A})^2 + (\nabla \times \underline{A})^2 \right] \right\} \tag{292}$$

where all electric field effects have been eliminated, so we deal with $\underline{A}(x)$. Then the 1-particle fermion propagator satisfies the eqtn.

$$\left[ i\partial_t - \frac{1}{2m} (-i\underline{\nabla} - g_0 \underline{A}(x))^2 + \mu \right] \mathcal{G}_2(xx'|\underline{A}) = -\delta(x-x') \tag{293}$$

so that its partial Fourier transform over the relative variable $x-x'$, defined by

$$\left.\begin{aligned} \mathcal{G}_2(xx'|\underline{A}) &= \sum_{\underline{k}} \int \frac{d\epsilon}{2\pi} \mathcal{G}(\underline{k}, \epsilon; x|\underline{A})\, e^{i[\underline{k}\cdot(\underline{x}-\underline{x}') - \epsilon(t-t')]} \\ &\equiv \sum_k \mathcal{G}_k(x|\underline{A})\, e^{ik(x-x')} \end{aligned}\right\} \tag{294}$$

which satisfies (again defining $p = \hbar k$):

$$\left[ i\partial_t - \epsilon - \epsilon_p^0 - \tfrac{p}{m}(-i\underline{\nabla} - g_0 \underline{A}) - \frac{1}{2m}(-i\underline{\nabla} - g_0 \underline{A})^2 \right] \mathcal{G}_k(x|\underline{A}) = -1 \tag{295}$$

Then we go through the same manoeuvres as before - writing

$$\mathcal{G}_k(x|A) = i \int_0^\infty ds\, \mathcal{G}_k(x;s|A) \tag{296}$$

we write

$$\mathcal{G}_k(x;s|A) = e^{i[s G_0^{-1}(k) + \mathcal{V}_s(k, x|\underline{A})]} \tag{297}$$

and expand $\Psi$ in powers of the bare coupling $g_0$, as

$$\Psi_s(k,x|\underline{A}) = \sum_n g_0^n \psi_n(s;kx|\underline{A}) \tag{298}$$

Then, upon substitution, we find that

$$\psi_n = \int_0^s ds' e^{-i(s-s')\left[i\partial_t + \frac{\nabla^2}{2m} + i\frac{k_0\cdot\nabla}{m}\right]} \sum_{r=1}^{n-1}\left[\underline{\nabla}\psi_r(s)\cdot\underline{\nabla}\psi_{n-r}(s') - 2\underline{A}\cdot\underline{\nabla}\psi_{n-1}(s)\right] - \underline{A}^2\delta_{n2} \tag{299}$$

with the lowest term given by

$$\psi_1(s;kx|\underline{A}) = \frac{g_0}{m}\sum_q e^{i(\underline{q}\cdot\underline{x}-\omega t)}\frac{1-e^{-is(\omega-\Omega_{\underline{p}}(\underline{q}))}}{\omega-\Omega_{\underline{p}}(\underline{q})}\underline{p}\cdot\underline{A}(x) \tag{300}$$

where we define:

$$\left.\begin{aligned}\Omega_{\underline{p}}(\underline{q}) &= \epsilon^0_{\underline{p}+\underline{q}} - \epsilon^0_{\underline{p}} = \frac{1}{m}\left(\underline{p}\cdot\underline{q} + \tfrac{1}{2}|\underline{q}|^2\right)\\ &\sim \underline{p}\cdot\underline{q}/m\end{aligned}\right\} \tag{301}$$

and drop the $|\underline{q}|^2$ term because we assume $q^2$ is small — again, we work in the IR regime.

Again, to recover a result for $\mathcal{G}_2(p) = \mathcal{G}_2(p,\epsilon)$ we functionally integrate over the gauge field $\underline{A}(x)$; but this time we use a weighting factor

$$\mathcal{P}[A^\mu] = e^{i/2h\sum_q A^i(q)D^{-1}_{ij}(q)A^j(-q)} \tag{302}$$

where $D_{ij}(q)$ is the renormalized transverse propagator in (291). Carrying out the Gaussian integration as before, and using the lowest approximation $\psi_1$ in (300) for $\Psi_s(k,x|\underline{A})$, we find then

$$\mathcal{G}_2(\underline{p},\epsilon) = i\int_0^\infty ds\, e^{-is\,G_0^{-1}(\underline{p},\epsilon)}\exp\left\{i\frac{g_0^2}{m^2}\sum_{\underline{q}}\int\frac{d\omega}{2\pi}D_{ij}(q)p_ip_j\,\frac{1-e^{-is(\omega-\Omega_{\underline{p}}(\underline{q}))}}{(\omega-\Omega_{\underline{p}}(\underline{q}))^2}\right\} \tag{303}$$

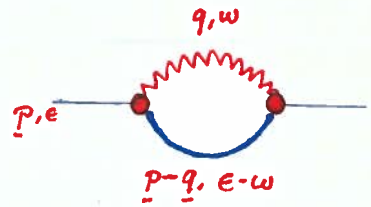
which we can compare with (284).

Although (303) looks very similar to (284), the subsequent development proceeds rather differently because the propagator $D_{ij}(q)$ is very different from the bare photon propagator.

2ND-ORDER PERTURBATION THEORY: Before doing the full phase integral in (303) (for that is what it is), it is very useful to get a quick picture of what happens in the lowest-order diagram in our theory. Thus we consider the self-energy contribution

$$\Delta\Sigma'_{\underline{p}}(\epsilon) = \frac{g_0^2}{m^2}\sum_{\underline{q}}\int\frac{d\omega}{2\pi}D_{ij}(\underline{q},\omega)\left(p_i+q_i/2\right)\left(p_j+q_j/2\right)G_0(\underline{p}-\underline{q},\epsilon-\omega) \tag{304}$$

which can be derived from standard diagrammatic rules for the diagram shown in the figure (see Appendix on diagrams).




$$= \Delta\Sigma_{\underline{p}}(\epsilon) \qquad (305)$$

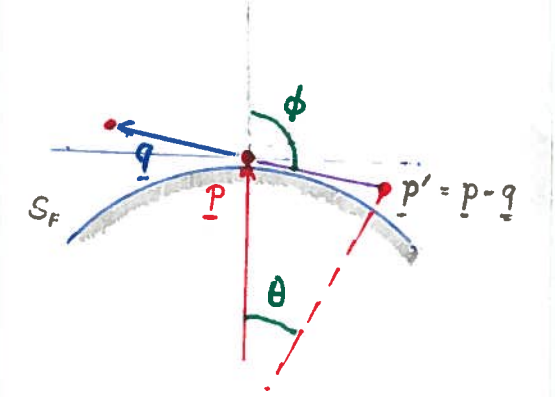
The finite-T generalization of this result, for a complex frequency $z$, is given in the usual way by

$$\Delta\Sigma_{\underline{p}}(z) = \frac{g_0^2}{N^2} \sum_{\underline{q}} \int \frac{dw}{2\pi} D_{ij}(\underline{q},w)\,(p_i + q_i/2)(p_j + q_j/2) \left[ \frac{1 - f_{\underline{p}-\underline{q}} + n(w)}{z - \epsilon^0_{\underline{p}-\underline{q}} - w} + \frac{f_{\underline{p}-\underline{q}} + n(w)}{z - \epsilon^0_{\underline{p}-\underline{q}} + w} \right] \qquad (306)$$

We can now make a number of simplifications here. Notice first that

$$(\delta_{ij} - \hat{q}_i \hat{q}_j)\left[(p_i + q_i/2)(p_j + q_j/2)\right] = |\underline{p} \times \hat{\underline{q}}|^2 = p^2 \sin^2\phi \qquad (307)$$

where the geometry of the fermion scattering from $\underline{p}$ to $\underline{p}-\underline{q}$ is shown in the figure - the Fermi surface $S_F$ is shown, along with its mirror reflection across the line running through the point $\underline{p}$ in momentum space, perpendicular to the vector $\underline{p}$, measured from the centre of the Fermi sphere. Since we are working in 2-d, the Fermi sphere becomes a circle.




GEOMETRY OF FERMION SCATTERING IN $\Delta\Sigma_{\underline{p}}(\epsilon)$

It is assumed in this calculation that the fermion with momentum $\underline{p}$ is above the Fermi energy $\mu$; the intermediate state with momentum $\underline{p}-\underline{q}$ must also be. If the particle with momentum $\underline{p}$ is close to the Fermi surface $S_F$ (and we are interested in precisely this case), then we see that the angle $\phi$ is roughly $90°$, and therefore, defining

$$\left.\begin{aligned} q_{\parallel} &= q\cos\phi \\ q_{\perp} &= q\sin\phi \end{aligned}\right\} \qquad (308)$$

we see that

$$\left.\begin{aligned} |\underline{q}| \sim q_{\perp} &\sim 2p_F \sin\theta/2 \sim p_F\theta \\ \sin^2\phi &\sim 1 \\ \Omega_{\underline{p}}(\underline{q}) &\sim \frac{\underline{p}\cdot\underline{q}}{m} = v_F q_{\parallel} \end{aligned}\right\} \qquad (309)$$

Now we don't actually want to do an exact calculation of $\Delta\Sigma_{\underline{p}}(\epsilon)$ in (306), just an approximate one, in order to see what sort of answer we get, and to help us see how to exclude the more complicated integrals in (303). To do this approximate evaluation of $\Delta\Sigma_{\underline{p}}(\epsilon)$ (or indeed an exact one) it is easier to first

find $\mathrm{Im}\,\Sigma_p(\epsilon)$, and then deduce the behaviour of $\mathrm{Re}\,\Sigma_p(\epsilon)$ from this. It then helps to write $\mathcal{D}_{ij}(q,\omega)$ in the form

$$\left.\begin{aligned}
\mathcal{D}_{ij}(\underline{q},\omega) &= (\delta_{ij}-\hat{q}_i\hat{q}_j)\,\mathcal{D}_\perp(\underline{q},\omega) \equiv (\delta_{ij}-\hat{q}_i\hat{q}_j)\,\frac{|\underline{q}|}{\chi|\underline{q}|-i\gamma|\omega|} \\
\mathcal{D}_\perp(\underline{q},\omega) &= \int_{-\infty}^{\infty}\frac{d\Omega}{\pi}\,\mathrm{Im}\,\mathcal{D}_\perp(\underline{q},\Omega)\,\frac{1}{\Omega-(\omega+i\delta)} \\
&\equiv \int_0^\infty \frac{d\Omega}{\pi}\,\mathrm{Im}\,\mathcal{D}_\perp(\underline{q},\Omega)\left[\frac{1}{\Omega-\omega-i\delta}-\frac{1}{\Omega+\omega+i\delta}\right]
\end{aligned}\right\} \quad (310)$$

where the spectral function of $\mathcal{D}_\perp(\underline{q},\omega)$ is, from (309):

$$\mathrm{Im}\,\mathcal{D}_\perp(\underline{q},\Omega) \simeq \frac{-\gamma\Omega q_\perp}{\chi^2 q_\perp^6+\gamma^2\Omega^2} \quad (311)$$

It then follows that we can write on the mass shell (i.e., for $\epsilon\to\epsilon_p^\circ$), that

$$\left.\begin{aligned}
\mathrm{Im}\,\Delta\Sigma_{\underline{p}}(\epsilon_p) &= -\pi\left(\frac{g_0 p_F}{m}\right)^2 \sum_{\underline{q}} \sin^2\phi \int_0^\infty \frac{d\Omega}{\pi}\,\mathrm{Im}\,\mathcal{D}_\perp(\underline{q},\Omega)\Big[\big(1-f_{\underline{p}-\underline{q}}+n(\Omega)\big)\,\delta\big(\Omega_{\underline{p}}(\underline{q})-\Omega\big) \\
&\qquad + \big(f_{\underline{p}-\underline{q}}+n(\Omega)\big)\,\delta\big(\Omega_{\underline{p}}(\underline{q})+\Omega\big)\Big] \\
&\xrightarrow{|\underline{p}|\geq p_F} -\pi\left(\frac{g_0 p_F}{m}\right)^2 \sum_{\underline{q}} \int_0^\infty \frac{d\Omega}{\pi}\,\mathrm{Im}\,\mathcal{D}_\perp(\underline{q},\Omega)\,\big(1-f_{\underline{p}-\underline{q}}+n(\Omega)\big)\,\delta(v_F q_\parallel-\Omega)
\end{aligned}\right\} \quad (312)$$

where the latter form drops the term $\propto \delta(\Omega_{\underline{p}}(\underline{q})+\Omega)$, which only plays a role when $|\underline{p}|<p_F$. Let's now go to the limit $kT\to 0$, and also simplify the momentum integration by writing

$$\sum_{\underline{q}} = \frac{1}{(2\pi)^2}\int q\,dq\int d\phi = \frac{N(0)}{2\pi p_F}\int dq_\perp \int d\epsilon' \quad (313)$$

with $\epsilon' \equiv \epsilon^\circ_{\underline{p}-\underline{q}} \simeq p_F/m\,(p'-p_F)$, and where the Fermi surface density of states is given in 2-dimensions by $N(0) = m/2\pi$. Because $kT=0$, we can then write

$$\left.\begin{aligned}
\int_{-\infty}^{\epsilon} d\epsilon'\,\big(1-f(\epsilon')+n(\Omega)\big)\,\delta(\epsilon-\epsilon'-\Omega) &= f(\Omega-\epsilon)+n(\Omega) \\
&\xrightarrow{kT=0} \theta(\Omega)\,\theta(\epsilon-\Omega)
\end{aligned}\right\} \quad (314)$$

so we can now simplify (312) to

$$\mathrm{Im}\,\Delta\Sigma_{\underline{p}}(\epsilon_p) \simeq \left(\frac{g_0}{m}\right)^2 \frac{N(0)\,p_F}{2\pi}\int_0^{\epsilon_p} d\Omega \int_0^\infty dq_\perp\,\mathrm{Im}\,\mathcal{D}_\perp(q_\perp,\Omega) \quad (315)$$

which we can then calculate using (311). The integrals are easily done, and we get the result

$$\mathrm{Im}\,\Delta\Sigma_p(\epsilon_p) \sim \tilde{g}^2 |\epsilon_p|^{2/3} \tag{316}$$

with the renormalized coupling $\tilde{g}$ given by

$$\tilde{g}^2 = g_0^2 \, \bar{C} \, \frac{p_F}{m \chi^{2/3} \gamma^{1/3}} \tag{317}$$

with $\bar{C} = \int_0^\infty dx \, (1+x^6)^{-1}$.

Using dispersion relations we can then deduce the form of $\mathrm{Re}\,\Delta\Sigma_p(\epsilon)$. The conclusion of this analysis is that

$$\left.\begin{aligned} \mathrm{Im}\,\Delta\Sigma_p(\epsilon) &\sim \tilde{g}^2 |\epsilon|^{2/3} \\ \mathrm{Re}\,\Delta\Sigma_p(\epsilon) &\sim \tilde{g}^2 |\epsilon|^{2/3} \end{aligned}\right\} \quad \Delta\Sigma_p(\epsilon) \sim e^{i\phi} \tilde{g}^2 |\epsilon|^{2/3} \tag{318}$$

where $\tan\phi = \mathrm{Im}\,\Delta\Sigma_p(\epsilon) / \mathrm{Re}\,\Delta\Sigma_p(\epsilon)$.

What does this result tell us? Several things. First, the result completely evades all the arguments that we went through in chapter 4, which indicated that the effect of interactions on fermions near the Fermi surface was simply to renormalize things like the fermion mass ($m \to m^*$), with a quasiparticle decay rate $\sim \mathrm{Im}\,\Sigma_p(\epsilon) \sim O(\epsilon^2)$, so that the quasiparticles were well-defined as $\epsilon \to 0$, ie., near the Fermi surface. Here we have $|\mathrm{Im}\,\Delta\Sigma_p(\epsilon)| > |\epsilon|$, ie., the decay rate becomes greater than the quasiparticle energy, and the quasiparticles are no longer well defined.

However, one can immediately ask whether this result is credible - after all, if the lowest-order graph contributing to $\mathcal{G}_2(p)$ is infra-red divergent, surely we need to go to higher order? Indeed from (318) all we can infer is that

$$\mathcal{G}_2(p=p_F, \epsilon) \sim \frac{1}{\epsilon}\left[1 - e^{i\phi} \tilde{g}^2 |\epsilon|^{-1/3} + O(\tilde{g}^4) \ldots\right] \tag{319}$$

and from our experience so far, we can expect that the higher-order graphs will be even more singular. Notice that in 4-d QED, the higher-order contributions went like $g^{2n}(\ln\epsilon)^n$; here it looks like they are going to go like $\tilde{g}^{2n}|\epsilon|^{-n/3}$, a much worse power law divergence - in fact, we will see in the next chapter that such contributions are rather dangerous to the theory.

Thus we now really need a non-perturbative treatment, and this is what (303) provides us with.

EVALUATION OF FULL PHASE INTEGRAL: If we look at (303), we see immediately that it has the form of a "singular phase integral", ie., an integral over a phase (a function the proper time s) which is divergent, because of the energy denominators in the exponent. We should not be too bothered by this - after all, if we consider the simple path integral for an electron moving in the field of an attractive proton, the propagator is

$$\mathcal{G}_{\text{Coulomb}}(2,1) = \int_1^2 \mathcal{D}\underline{r}(t) \, e^{i/\hbar \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \dot{\underline{r}}^2 + e^2/|\underline{r}|\right]} \tag{320}$$

and the phase $\frac{1}{\hbar} S[\underline{r}]$ is definitely divergent for paths which approach the origin!

But we don't worry about this because we know that such paths make little contribution to the path integral – they involve extremely fast oscillations (corresponding to the diverging kinetic energy in this regime) which average to zero. Moreover, we would never think of expanding the exponential as a power series, ie., writing

$$G_{Coulomb}(2,1) = \int_1^2 \mathcal{D}r(t)\, e^{\frac{i}{\hbar}\int_1^2 dt\, \frac{m}{2}\dot{\underline{r}}^2(t)} \left[ 1 + \int_{t_1}^{t_2} dt\, \frac{e^2}{|\underline{r}(t)|} + \frac{1}{2}\int_1^2 dt \int_1^2 dt'\, \frac{e^4}{|\underline{r}(t)||\underline{r}(t')|} + \text{etc} \right] \tag{321}$$

unless we were feeling silly – every term is formally divergent, and so the series makes no sense. Thus the integral (320) is well-defined, but not the series expansion (321).

In the same way the series in (319) should not be taken seriously, but merely as a signal that we need to go back to the full phase integral – that perturbation theory is not well-behaved and not meaningful for this problem.

From our evaluation of the momentum integrals for the 2nd-order perturbative self-energy $\Delta\Sigma_p(\epsilon)$, we see that we can write the phase $\varphi_s(p)$ in the formula (303) for $\mathcal{G}_2(p)$, viz.,

$$\mathcal{G}_2(p) = i\int_0^\infty ds\, e^{-is\mathcal{G}_0^{-1}(p)}\, e^{i\varphi_s(p)} \tag{322}$$

in the form

$$\left.\begin{aligned} \varphi_s(p) &= \frac{g_0^2}{m^2}\sum_{\underline{q}}\int\frac{dw}{2\pi}\, \mathcal{D}_{ij}(\underline{q},w)\, p_i p_j\, \frac{1-e^{-is(w-\Omega_{\underline{p}}(\underline{q}))}}{(w-\Omega_{\underline{p}}(\underline{q})+i\delta)^2} \\ &= \frac{1}{(2\pi)^2}\left(\frac{g_0 p_F}{m}\right)^2 \int\frac{dw}{2\pi}\int_0^\infty dq_\parallel \int_0^\infty dq_\perp\, \frac{1-e^{-is(w-q_\parallel v_F)}}{(w-q_\parallel v_F+i\delta)^2}\, \mathcal{D}_\perp(q_\perp,w) \end{aligned}\right\} \tag{323}$$

so that the momentum integrals factorize into separate integrals over $q_\parallel$ and $q_\perp$. If we carry out these two integrals, we find that

$$\varphi_s(p) = \frac{1}{(2\pi)^2 v_F}\,\frac{1}{(\chi^2\gamma)^{1/3}}\,\frac{e^{-i\pi/6}}{3^{3/2}}\left(\frac{g_0 p_F}{m}\right)^2 \int_0^\infty \frac{dw}{w^{1/3}}\left[\frac{1-e^{isw}}{w} - is\left(Ci(sw)+iSi(sw)\right)\right]$$

$$= \bar{g}^2 (is)^{1/3} \tag{324}$$

with the renormalized coupling

$$\bar{g}^2 = g_0^2\left[\frac{\Gamma(2/3)}{8\pi^2\sqrt{3}}\,\frac{1}{(\chi^2\gamma)^{1/3}}\,\frac{p_F}{m}\right] \tag{325}$$

Thus we finally have a proper time integral to do; substituting (324) into (322),

we get

$$\left.\begin{aligned} \mathcal{G}_2(p_F,\epsilon) &= i\int_0^\infty ds\, e^{is(\epsilon+i\delta)}\, e^{-\tilde{g}^2(is)^{1/3}} \\ &= -i\frac{\pi}{\epsilon}\left[\frac{d^2}{dz^2}\,Hi(z)\right]\Big|_{z=e^{4\pi i/3}\tilde{g}^2\epsilon^{-1/3}} \end{aligned}\right\} \quad (326)$$

where the function $Hi(z)$ is defined in terms of the Airy functions $Ai(z)$ and $Bi(z)$ as

$$\left.\begin{aligned} Hi(z) &= \frac{1}{\pi}\int_0^\infty dx\, e^{-(x^3/3 - xz)} \\ &= \frac{2}{3}Bi(z) + \int_0^z dx\,[Ai(x)Bi(z) - Ai(z)Bi(x)] \end{aligned}\right\} \quad (327)$$

This result shows that the renormalized photons in the system have a drastic effect on the fermion propagator near the Fermi surface. The Airy functions have an essential singularity $\sim \exp\{\frac{2}{3}z^{3/2}\}$ as $z\to\infty$, but a perturbative expansion as $z\to 0$. However, as we see from (326), the limit $z\to\infty$ corresponds to $\epsilon\to 0$, and so we find that $\mathcal{G}_2(p_F,\epsilon)$ has an essential singularity as $\epsilon\to 0$, but a perturbative expansion at high energy. From (326) we find

$$\mathcal{G}_2(p_F,\epsilon) = \left\{\begin{array}{ll} -\pi^{-1/2}\,\tilde{g}^{3/2}\,\frac{1}{\epsilon^{5/4}}\,e^{-\frac{2}{3}\tilde{g}^3/\epsilon^{1/2}} & (\text{low } \epsilon:\ \epsilon \ll \tilde{g}^6) \\ -\frac{i\pi}{\epsilon}\,c_1\left[1 + c_2\tilde{g}^2\frac{i^{4/3}}{\epsilon^{1/3}} + \cdots\right] & (\text{high } \epsilon:\ \epsilon \gg \tilde{g}^6) \end{array}\right. \quad (328)$$

From these results we gain a better understanding of both the perturbative structure of the theory and of its essentially non-perturbative behaviour in the low-energy limit. From (328) we see that the structure of the high-energy series is what we would have guessed from the perturbative calculation (cf. (318) and (319)), except that this perturbative result was derived in the low-energy limit. Thus the perturbative result is very misleading. The true low-energy behaviour is far more radical — no summation can ever produce a truly singular result like the low-$\epsilon$ limit in (328). Thus this behaviour is truly non-perturbative, in the full sense of the term, viz. inaccessible by any sum over diagrams, infinite or not.

Firstly, let us note that we could have found the form (326) for $\mathcal{G}_2(p_F,\epsilon)$, within dimensionless constants, purely by dimensional analysis of the integrals over $q$ and $\omega$ in (323). Thus, eg., we can write

$$\int_0^\infty dq_\parallel\,\frac{1-e^{-is(\omega - q_\parallel v_F)}}{(\omega - q_\parallel v_F)^2} \sim s\int_{-s\omega}^{\infty}\frac{du}{u^2}\,(1-e^{iu}) \sim sf(s\omega) \quad (329)$$

where $f(x)$ is some function; it then follows that $\varphi_s(p) \sim s\int_0^\infty \frac{d\omega}{\omega^{1/3}} f(s\omega) \sim s^{1/3}$. If we take account of the phases in (329) we get $(is)^{1/3}$, and the rest is just a renormalization of $g_0^2$ to $\tilde{g}^2$ (which we can only get by doing the integrals properly).