B.5 : NON-PERTURBATIVE METHODS IN Q.F.T.

Although perturbation theory is an essential tool in QFT, just as it is in ordinary QM (and in classical mechanics or classical GR), it is very limited in its application. Many interesting processes that occur even in the simplest systems are not revealed by perturbative expansions, and yet can be seen using other methods. When we get to more complicated systems, these methods are essential. The list of physical phenomena that require such methods is almost endless. Just to name a few:

Topological Solitons & other Singular objects : In almost every system in Nature having any kind of order and in the presence of interparticle interactions, one can have singularities in the order parameter or field that describes these systems. This is true in classical biology (the evidence is all around you!) in classical & Q. chemistry (notably in non-equilibrium phenomena in, eg., chemical reactions), and in almost every interesting physical system (commonly cited examples are vortices, domain walls, & other singular "textures" in superfluids, magnets, and other quantum fluids like the quantum Hall system, as well as in Yang-Mills theories and in the early universe, with objects like cosmic strings being of particular interest – and of course string theory itself began with considerations like these). In classical systems ranging from hydrodynamical systems, to simple solids, all the way to General Relativity, such singular objects abound (vortices, defects, dislocations, etc., and of course Black holes). There would be almost nothing left to study in GR if we looked only perturbatively at it.

Phase Transitions : These are also ubiquitous. Until the advent of RG (ie., Renormalization Group) methods, the only source of understanding of these was a combination of crude mean field methods, inverse-temp expansions, and exact solutions. The RG method is non-perturbative in formulation, although one has typically to use loop expansions or ϵ-expansions to get results. Other interesting methods that have been developed to understand phase transitions include $1/N$ expansions (where N is the number of field components, or of neighboring lattice sites), replica methods (particularly for spin systems), and of course various wave-function ansatzes for QM phase transitions (simple mean field ones like BCS, or more complicated ones like, eg., Gutzwiller, hypernetted chain, etc...). Perturbation theory is incapable of revealing phase transitions, which are non-analytic in the relevant parameters (interaction strengths for quantum phase transitions, temperature for classical phase transitions). Some kinds of phase transition require special methods – examples include localization for disordered systems or spin glasses.

Tunneling : We are all familiar with WKB methods for quantum tunneling in simple QM. These methods are just a special case of a general class of mathematical methods coming from asymptotic analysis. In field theory this all gets a lot more interesting. In cosmology one deals with vacuum tunneling and inflation, and in particle physics with "instanton" solutions to the field equations. In condensed matter physics tunneling is ubiquitous, from the microscopic level of atoms to macroscopic tunneling (in superconductors & magnets). And of course nuclear physics and the energetics of stars depend essentially on tunneling (in fission & fusion). Need I go on? Again, the tunneling process is non-perturbative in the relevant coupling.

Asymptotic methods of various kinds, beyond WKB, are applied everywhere in physics where perturbation theory fails, ranging from high-energy or low-energy scattering, Brehmstrahlung, resonant phenomena & pair creation in QED, QCD, and cosmology as well as atomic and condensed matter systems, down to the lowest energies we look at in superfluids.

Chaotic Phenomena : The work of K.A.M. (Kolmogorov, Arnold, & Moser) finally resolved the problems raised by Laplace & highlighted by Poincaré, concerning the dynamics of non-linear systems. At first the KAM methods were applied exclusively to classical systems, in studies ranging from stellar & planetary dynamics to hydrodynamics & turbulence. But now this work has had an interesting impact on QM, in studies of "Q. Chaos". This is another class of problems where the relationship between classical & Q. phenomena, revealed by semiclassical asymptotic analysis, is extremely revealing. This work has already affected work in most areas of physics. Perturbation theory fails spectacularly here, as first emphasized by Laplace.

Strongly - Correlated Systems : Any reasonably interesting classical or Q. Many-body system will have strong inter-particle correlations. Even in weak coupling it is a major job to reveal these in perturbation expansions. For stronger couplings perturbation provides only a guide, and one really has to do other things. One can use non-perturbative graphical methods (as in Fermi liquid theory, in quantum liquids or in nuclear physics) or a host of other methods including 1/N expansions, RG methods, wave-function ansatzes like BCS or Gutzwiller, eikonal & other semiclassical expansions, etc..). Much of modern condensed matter physics focusses on questions of this kind, ranging from high-T_c superconductors, magnets, etc., to more exotic systems. Topological field theory, first devised in high-energy physics, has become essential in the last 30 yrs.

One could continue this. However the present chapter will be about methods. I will cover methods involving "eqtns of motion" (eg., Schwinger-Dyson eqtns) and the associated Ward identities, graphical expansions, etc.; and then discuss 4 key methods, viz.,

- 1/N expansions
- semiclassical expansions (including eikonal & loop expansions)
- gradient expansions
- WKB / instanton expansions.

From what is said above, it will be clear that I will only be scratching the surface here. You can look at any decent book on QFT, either for particle physicists or condensed matter physicists, to find out more - these non-perturbative methods are really central to the whole of physics. Non-linearity and all it entails is so fundamental to subjects like hydrodynamics or General Relativity that one often finds no discussion of perturbative methods in texts on these subjects!

B.5 (a) EQUATIONS OF MOTION

We have seen how in ordinary QM, we can associate an eqtn of motion, the Schrodinger eqtn, with a path integral which describes the time evolution of the propagator $G(\underline{r},t;\underline{r}'t')$; indeed, we have that for a CLOSED SYSTEM,

$$\left.\begin{aligned} \int_1^2 \mathcal{D}q\, e^{\frac{i}{\hbar}S_0[q]} &= G_{21} = \langle 2| e^{-i\hat{H}(t_2-t_1)/\hbar} |1\rangle \\ \text{and} \qquad \langle r_2 t_2 | 2\rangle &= \int dr_1\, G_{21} \langle r_1 t_1 | 1\rangle \end{aligned}\right\} \qquad (1)$$

is equivalent to the Schrodinger eqtn of motion for $\psi(r,t)$, in the form

$$\left.\begin{aligned} (\mathcal{H} - i\hbar\partial_t)\, G(2,1) &= i\hbar\, \delta(2,1)\, \Theta(t_2-t_1) \\ (\mathcal{H} - i\hbar\partial_t)\, \psi(r,t) &= 0 \end{aligned}\right\} \qquad (2)$$

Moreover, we have also seen that if we have a system which is acted upon by some EXTERNAL SOURCE $J(t)$, then we can write an equation of motion for a propagator $\mathcal{G}(x,x'|J(t))$ which is a functional of the external source $J(t)$, of the form

$$\left[\left.\frac{\delta S_0[q]}{\delta q(t)}\right|_{q=-i\hbar\,\delta/\delta J} + J(t) \right] \mathcal{G}(2,1|J(t)) = 0 \qquad (3)$$

where $S_0[q]$ in (1) and (3) are the action without any external field. Eqtn (3) makes clear that the source $J(t)$ affects the equation of motion as an external force.

So, how is this all modified for a field theory. In what follows let us see how this works, first for scalar field theory, and then look at some of the consequences of this.

(i) SCHWINGER-DYSON EQUATIONS:

Actually, eqtn (3) is the blueprint for the "Schwinger-Dyson" hierarchy of eqtns. of motion in QFT., which can be written for both relativistic fields and for condensed matter systems. The analogous quantity to $G(2,1|J)$ in QFT is provided by taking the appropriate time limits to $\pm\infty$, to get $Z[J]$, and then we find that for a QFT, such as ϕ^4 theory:

$$\boxed{\left[\left.\frac{\delta S[\phi]}{\delta \phi(x)}\right|_{\phi=-i\hbar\,\delta/\delta J(x)} + J(x) \right] Z[J] = 0} \qquad (4)$$

which is the eqtn of motion for the field $\phi(x)$; the derivation of this exactly parallels that for ordinary QM. If we split up the Lagrangian for the system into a free-field part and an interaction, so that

$$L = L_0 + L_V = \tfrac{1}{2}\phi \hat{K}_0 \phi - V(\phi) \tag{5}$$

where for ϕ^4 theory we have

$$\left.\begin{aligned} \hat{K}_0 &= -(\partial^2 + m^2) \\ V(\phi) &= g/4!\, \phi^4(x) \end{aligned}\right\} \tag{6}$$

and then we easily find that

$$\boxed{\left[i\hbar \hat{K}_0 \,\delta/\delta J(x) + \frac{\partial V(\phi)}{\partial \phi}\Big|_{\phi = -i\hbar\, \delta/\delta J(x)} + J(x) \right] Z[J] = 0} \tag{7}$$

We will look at more complicated theories below. Now let's consider how we can use this eqtn of motion. We are, in QFT, not terribly interested in $Z[J]$ except as a generating functional for the correlation functions. So let's substitute $Z[J]$ into (7), to get eqtn of motion for the correlation functions; we have

$$Z[J] = \sum_{n=0}^{\infty} \left(\tfrac{i}{\hbar}\right)^n \tfrac{1}{n!} \prod_{j=1}^{n} \int dx_j\, G_n(x_1 .. x_n)\, J(x_j) \tag{8}$$

and using

$$[J(x), -i\hbar\, \delta/\delta J(x')] = i\hbar\, \delta(x-x') \tag{9}$$

we simply differentiate (7) repeatedly with respect to $J(x)$. We then get the coupled set of eqtns:

$$\begin{aligned} (\partial^2 + m^2)\, G_1(x) + g/6\, G_3(x,x,x) &= 0 \\ (\partial^2 + m^2)\, G_2(x,x') + g/6\, G_4(x,x,x,x') &= -i\hbar\, \delta(x-x') \end{aligned} \tag{10}$$

and for $n > 3$

$$(\partial^2 + m^2)\, G_n(x, x_1' .. x_{n-1}') + g/6\, G_{n+2}(x,x,x; x_1' .. x_{n-1}') = -i\hbar \sum_{j=1}^{n-1} \delta(x - x_j')\, \widetilde{G}_{n-2}(\{x_j'\}) \tag{11}$$

where

$$\widetilde{G}_{n-2}(\{x_j\}) \equiv G_{n-2}(x_1', x_2' .. x_{j-1}', x_{j+1}', .. x_{n-1}') \tag{12}$$

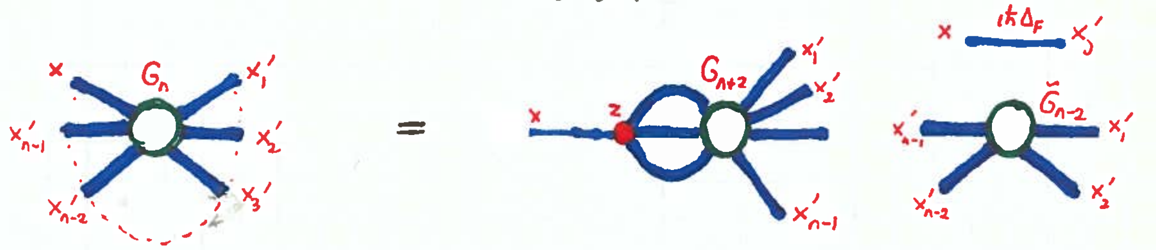
Thus we have a hierarchy of coupled eqtns of motion for the correlation functions, which is exactly analogous to the BBGKY hierarchy for an interacting classical system. We can rewrite these eqtns without the differential operator $\partial^2 + m^2$ by simply multiplying to the left by its inverse operator $\Delta_F(x-x')$; this gives

$$\boxed{G_n(x, x_1' .. x_{n-1}') - \frac{g}{6} \int dz\, \Delta_F(x-z)\, G_{n+2}(zzz, x_1' .. x_{n-1}') - i\hbar \sum_{j=1}^{n-1} \Delta_F(x - x_j')\, \widetilde{G}_{n-2}(\{x_j'\}) = 0} \tag{13}$$

and in particular for $G_2(x,x')$ we have

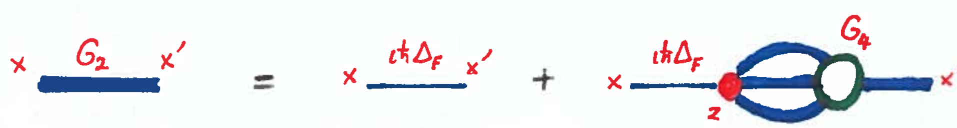
$$G_2(x,x') - \frac{g}{6}\int d^4z\, \Delta_F(x-z)\, G_4(zzz,x') - i\hbar\Delta_F(x-x') = 0 \qquad (14)$$

These equations can of course be depicted graphically; this is most easily done for eqtn (13), which has the following graphical representation:



(15)

and which for eqtn (14) becomes



(16)

The physical meaning of these eqtns is obvious if we start with (14), shown in (16). It says that the full propagator/correlator for a single field excitation, going from x' to x', is given by summing the free propagator (where no interactions come into play) and then adding all possible 4-point interaction processes. Thus (14) and (16) are nothing but the generalization of the usual Dyson eqtn., which we have seen many times already, to a field theory.

The full Schwinger-Dyson eqtns, given in (11), (13), and (15), generalize this to an arbitrary correlator – we can think of it as the n-point correlator involving both the free propagation of one of the excitations (with the others doing what they want) added to the repeated interaction together of all of them.

There are many things one may do with these eqtns – one could write a whole book about the use of eqtns of motion in QFT. Note that if we suppress the interactions in (7), we immediately get back our familiar form for $Z_0[J]$; the solution to

$$\left[i\hbar(\partial^2+m^2)\frac{\delta}{\delta J(x)} + J(x)\right] Z_0[J] = 0 \qquad (17)$$

is obviously

$$Z_0[J] = e^{-\frac{i}{2\hbar}\int d^4x \int d^4x\, J(x)\Delta_F(x-x')J(x')} \qquad (18)$$

and if we wish, we can then get back our usual formula for $Z[J]$ by starting from this, and applying (7) with the interactions added back. To do this we write the ansatz (see next page):

$$Z[J] = \frac{1}{Z[0]} e^{i/\hbar \, \Phi(-i\hbar \, \delta/\delta J)} Z_0[J] \tag{19}$$

where $Z[0] \equiv Z[J=0]$ is the full generator, with interactions, with $J(x)$ set to zero (so as to normalize $Z[J]$), and Φ is a phase coming from the interactions. We then easily find that (cf. section 8.1, eqtn (79)):

$$Z[J] = \frac{1}{Z[0]} e^{-\frac{i}{\hbar}\int d^4x \, V(-i\hbar \, \delta/\delta J(x))} Z_0[J] \tag{20}$$

NON-RELATIVISTIC FERMIONS: The above derivation shows how things work in a simple relativistic field theory. One can also develop the Schwinger-Dyson eqtn hierarchy for non-relativistic many-body systems. Rather than carry out a general derivation, let's do it for our favourite condensed matter problem, viz., the interacting fermion system, with the usual Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{N} -\frac{\hbar^2 \nabla_j^2}{2m} + \sum_{i<j} V(\underline{r}_i - \underline{r}_j) \tag{21}$$

and then define the $2n$-point correlator as (here $x \equiv (\underline{r},t)$):

$$G_{2n}(x_1 \cdots x_n | x_1' \cdots x_n') = (-i\hbar)^n \langle 0 | T\{\psi(x_1) \cdots \psi(x_n) \psi^+(x_1') \cdots \psi^+(x_n')\} | 0 \rangle \tag{22}$$

Since this is a non-relativistic system, we can separate out time and space variables. Schrodinger's eqtn is now:

$$\left[\left(i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m}\right) - \int d\underline{r}'^3 \rho(\underline{r}';t)\right] \psi(\underline{r},t) = 0 \tag{23}$$

where $$\rho(\underline{r},t) = \psi^+(\underline{r},t)\psi(\underline{r},t) \tag{24}$$

Now to get the SD hierarchy, instead of differentiating with respect to $J(x)$ as we did for ϕ^4 theory, we differentiate with respect to its analogue here, which is to say, we repeatedly differentiate (22) w.r.t. one or other of its time variables. This then gives, for $n=1$ (and for FERMIONS):

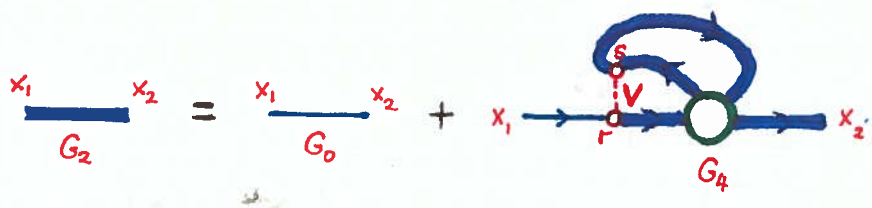
$$\left(i\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2m}\right) G_2(x,x') = \hbar \, \delta(x-x') - i\hbar \int d^3\underline{s} \, V(\underline{r}-\underline{s}) \, G_4(\underline{s},t; \underline{r},t | \underline{s},t; \underline{r}'t') \tag{25}$$

which is a little different from (10), and will be interpreted graphically below; and

$$\left(i\hbar \partial_t + \frac{\hbar^2 \nabla_\ell^2}{2m}\right) G_{2n}(x_1 \cdots x_n | x_1' \cdots x_n') = \hbar \sum_{\ell'=1}^{n} (-1)^{\ell+\ell'} \hat{G}_{2n-2}^{\ell\ell'}(\{x_j, x_j'\}) \, \delta(\ell - \ell')$$
$$- i\hbar \int d^3\underline{s} \, V(\underline{r}-\underline{s}) \, G_{2n+2}(\underline{s}t_j; x_1 \cdots x_n | \underline{s}t_j, x_1' \cdots x_n') \tag{26}$$

Anyway, we can now again multiply on the left by the inverse operator, ie., by the bare 1-particle Green function $G_0(x_1-x_2)$. This gives us, for the 2-point correlator

$$G_2(x_1-x_2) \;=\; -iG_0(x_1-x_2) \;-\; \int dt \int d^3\underline{r} \int d^3\underline{s}\; G_0(\underline{r}_1-\underline{r}, t_1-t)\, V(\underline{r}-\underline{s})\, G_4(\underline{s},t;\underline{r},t\,|\,\underline{s},t;\underline{r}_2 t_2) \qquad (27)$$



in which we see that an excitation can either propagate freely, or interact repeatedly with a particle-hole pair. Actually (27) has the same basic structure as (16); there are just the following differences in detail:

(i) The fermion loop factor $(-1)^L$, if we deal with fermions.

(ii) The existence of an interaction $\hat{V} = V(\underline{r}-\underline{r}')\,\delta(t-t')$ which is local in time, but non-local in space. If the interaction were also local in space, it would collapse to a point in the graph, and look just like (16).

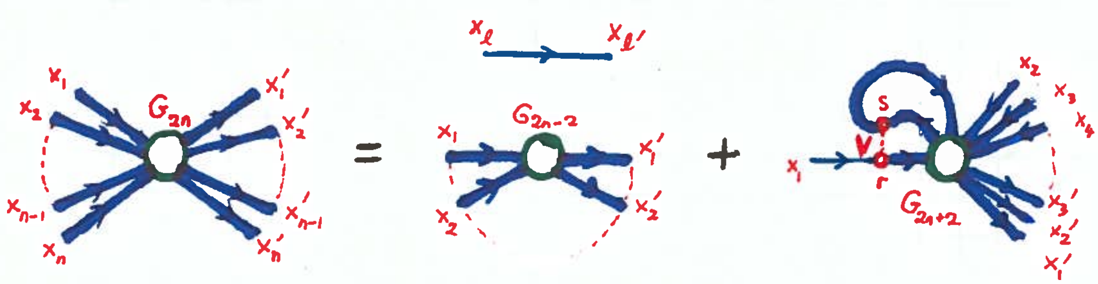
In the same way we derive the general equation:

$$\boxed{\begin{aligned} G_{2n}(\{x_j,x_j'\}) \;=\; & -\sum_{\ell} G_0(x_\ell - x_{\ell}')\, \tilde{G}^{\ell\ell'}_{2n-2}(x_j,x_j'\}) \\ & - \int dt \int d^3\underline{r} \int d^3\underline{s}\; G_0(\underline{r}_1,t_1;\underline{r},t)\, V(\underline{r}-\underline{s})\, G_{2n+2}(\underline{s},t;\underline{r}t\,|\,\underline{s},t;x_2..x_n;\{x_j'\}) \end{aligned}} \qquad (28)$$

where we write $\tilde{G}_{2n-2}(\{x_j,x_j'\})$ explicitly as

$$\tilde{G}^{\ell\ell'}_{2n-2}(\{x_j,x_j'\}) \;=\; G_{2n-2}(x_1 \ldots x_{\ell-1}, x_{\ell+1} \ldots x_n\,;\, x_1' \ldots x_{\ell-1}', x_{\ell+1}' \ldots x_n') \qquad (29)$$

and diagramatically we have:



(30)

which is the most useful form of this eqtn; note that the factor $(-1)^{\ell+\ell'} = 1$. In principle we could take the classical limit of this result, to get the coupled BBGKY hierarchy of eqtns of motion for the dynamics of a classical system of N interacting particles – but this would take us too far afield.

QUANTUM ELECTRODYNAMICS

: The ϕ^4 theory and the theory of non-relativistic fermions involve 4-point interactions between excitations. Clearly if we looked at, eg., the electron-phonon problem we would get a rather different kind of hierarchy. Let's instead look at the simplest gauge theory, QED, which will have a similar structure to that of the electron-phonon model.

The generalization of the Schwinger-Dyson eqtn to QED can be guessed from ϕ^4 theory (I will not derive it here). We deal with the generating functional

$$Z[J^\mu; \bar{\eta}, \eta] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{\frac{i}{\hbar}(S_{QED} + \Delta S)} \tag{31}$$

with as usual

$$\left.\begin{aligned} S_{QED} &= \int d^4x \, \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4\mu_0}\int d^4x \, F_{\mu\nu}(x) F^{\mu\nu}(x) \\ \Delta S &= \int d^4x \left[J_\mu(x) A^\mu(x) + \bar{\psi}(x)\eta(x) + \psi(x)\bar{\eta}(x) \right] \end{aligned}\right\} \tag{32}$$

and in the same way as we get a single sourced eqtn of motion for Z by varying S with respect to a single field in (4), we get 3 eqtns of motion for the 3 fields in (32) by varying with respect to each of them. In particular we have, with $S = S_{QED} + \Delta S$, that

$$\boxed{\begin{aligned} &\left[\frac{\delta S[\bar{\psi},\psi;A^\mu]}{\delta A^\mu(x)} \Bigg|_{\substack{\psi = -i\hbar\,\delta/\delta\bar{\eta} \\ \bar{\psi} = i\hbar\,\delta/\delta\eta \\ A^\mu = -i\hbar\,\delta/\delta J^\mu}} + J^\mu(x) \right] Z[J^\mu, \bar{\eta}, \eta] = 0 \qquad (33) \\ &\text{and} \\ &\left[\frac{\delta S[\bar{\psi},\psi;A^\mu]}{\delta \bar{\psi}(x)} \Bigg|_{\substack{\psi = -i\hbar\,\delta/\delta\bar{\eta} \\ \bar{\psi} = i\hbar\,\delta/\delta\eta \\ A^\mu = -i\hbar\,\delta/\delta J^\mu}} + \eta(x) \right] Z[J^\mu, \bar{\eta}, \eta] = 0 \qquad (34) \end{aligned}}$$

plus a similar eqtn for $\delta S/\delta\psi(x)$. Clearly if we went through all the rigmarole that we just did for the ϕ^4 and 4-fermion models just discussed, we would fill many pages – so here we just look briefly at the consequences of (33) and (34).

Consider the 1st eqtn (33), to give the eqtn of motion for $A^\mu(x)$ and its correlators. From (32) we have, in the 't Hooft gauge, that

$$\frac{\delta S}{\delta A^\mu} = \left[\eta_{\mu\nu}\partial^2 - (1 - 1/\alpha)\partial_\mu\partial_\nu \right] A^\nu + q\bar{\psi}\gamma_\mu\psi \tag{35}$$

and so we now substitute this into (33), making the replacements indicated for

$\bar\psi$, ψ, and A^μ, and get a rather complicated eqtn., which is most usefully written in terms of $W[J_\mu, \bar\psi, \psi] = -i\hbar \ln Z[J_\mu, \bar\psi, \psi]$; we get

$$\left[\eta_{\mu\nu}\partial^2 - (1 - 1/\alpha)\partial_\mu\partial_\nu\right]\frac{\delta W}{\delta J_\nu} + g\gamma_\mu\left[\frac{\delta W}{\delta \eta}\frac{\delta W}{\delta \bar\eta} + i\hbar\frac{\delta^2 W}{\delta\eta\,\delta\bar\eta}\right] + J_\mu(x) = 0 \qquad (36)$$

Now what we'd like to do here is find the eqtn of motion for $A^\mu(x)$ with the fields $\bar\psi(x)$ and $\psi(x)$, and indeed $A^\mu(x)$, set to tend to zero so we are in the vacuum state. Thus we want to get rid of the source fields in (36), re-expressing these in terms of the fields themselves, & then set these to zero at the appropriate time. So we make a Legendre transformation of kind discussed in section B.1(c), now adapted to QED with its 3 fields, writing

$$\Gamma[A^\mu, \bar\psi, \psi] = W[J_\mu, \bar\eta, \eta] - \int d^4x\,\left(J_\mu A^\mu + \bar\psi\eta + \bar\eta\psi\right) \qquad (37)$$

and then find that

$$\left.\frac{\delta W}{\delta J_\mu}\right|_{A^\mu} = A^\mu(x) \qquad \left.\frac{\delta \Gamma}{\delta A^\mu}\right|_{J_\mu} = -J_\mu(x)$$

$$\left.\frac{\delta W}{\delta \bar\eta}\right|_{\psi} = \psi(x) \qquad \left.\frac{\delta \Gamma}{\delta \psi}\right|_{\bar\eta} = -\bar\eta(x) \qquad (38)$$

$$\left.\frac{\delta W}{\delta \eta}\right|_{\bar\psi} = -\bar\psi(x) \qquad \left.\frac{\delta \Gamma}{\delta \bar\psi}\right|_{\eta} = \eta(x)$$

Continuing with the analogy to ϕ^4 theory, we define the 2-point propagator $G_2^{\bar\psi\psi}(x_1,x_2)$ and 2-point vertex $\Gamma_2^{\bar\eta\eta}(x_1,x_2)$ for the Dirac fermionic field, as

$$G_2^{\bar\psi\psi}(x_1,x_2) = \left.\frac{\delta^2 W}{\delta\eta(x_1)\,\delta\bar\eta(x_2)}\right|_{\eta,\bar\eta=0} = \left.\frac{\delta\psi(x_2)}{\delta\eta(x_1)}\right|_{\eta,\bar\eta=0}$$

$$\Gamma_2^{\bar\eta\eta}(x_1,x_2) = \left.\frac{\delta\Gamma}{\delta\psi(x_1)\,\delta\bar\psi(x_2)}\right|_{\psi,\bar\psi=0} = \left.\frac{\delta\eta(x_2)}{\delta\psi(x_1)}\right|_{\psi,\bar\psi=0} \qquad (39)$$

and likewise for the gauge field, we have the photon propagator $D_{\mu\nu}(x_1,x_2)$ and the vertex or "polarization part" $\Pi_{\mu\nu}(x_1,x_2)$ given by

$$D_{\mu\nu}(x_1,x_2) \equiv -\left.\frac{\delta^2 W}{\delta J_\mu(x_1)\,\delta J_\nu(x_2)}\right|_{J=0} = -\left.\frac{\delta A_\nu(x_2)}{\delta J_\mu(x_1)}\right|_{J=0}$$

$$\Pi_{\mu\nu}(x_1,x_2) = \left.\frac{\delta^2\Gamma}{\delta A_\mu(x_1)\,\delta A_\nu(x_2)}\right|_{A=0} = -\left.\frac{\delta J_\nu(x_2)}{\delta A_\mu(x_1)}\right|_{A=0} \qquad (40)$$

and, as before, these functions are inverses of each other, i.e., we have

$$\int d^4y\ \mathcal{G}_2^{\psi\bar\psi}(x_1,y)\ \Gamma_2^{\bar\eta\eta}(y,x_2) = \delta(x_1-x_2)$$
$$\int d^4y\ D_{\mu\nu}(x_1,y)\ \Pi^{\mu\nu}(y,x_2) = \delta(x_1-x_2) \qquad (41)$$

Now starting from here we can derive yet another hierarchy of Schwinger-Dyson eqtns, this time for QED. The whole thing is v. complicated, because we now have 3 SD eqtns (i.e., (33) and (34), and the conjugate of (34)), and we can then functionally differentiate these eqtns repeatedly with respect to any one of the 3 currents J^μ, $\bar\eta$, and η, to get different coupled hierarchies. We can also express these in terms of vertex parts, by functionally differentiating w.r.t. the fields A^μ, $\bar\psi$, and ψ.

As an example of what one can do, let's rewrite (32) in terms of the fermion propagator $\mathcal{G}_2(x_1,x_2)$ in (37), and the quantities in (38); we get, when $\eta,\bar\eta\to 0$ (these fermion sources being irrelevant to the problem of photon propagation), that

$$[(\eta_{\mu\nu}\partial^2-(1-1/\alpha)\partial_\mu\partial_\nu]A^\nu(x)+i\hbar q\,\mathrm{Tr}\left[\gamma_\mu\ \mathcal{G}_2^{\psi\bar\psi}(x,x)\right] - \left.\frac{\delta\Gamma}{\delta A^\mu(x)}\right|_{\psi,\bar\psi=0} = 0 \qquad (42)$$

Now to get a useful eqtn for the correlator, we need to functionally differentiate again with respect to $A^\nu(x)$; this acts on the current term (last term in (42)) to give

$$D^{-1}_{\mu\nu}(x_1x_2) = \left.\frac{\delta^2\Gamma}{\delta A^\mu(x_1)\,\delta A^\nu(x_2)}\right|_{A,\bar\psi,\psi=0} \qquad (43)$$

in the form

$$D^{-1}_{\mu\nu}(x_1x_2) = [\eta_{\mu\nu}\partial^2-(1-1/\alpha)\partial_\mu\partial_\nu]\,\delta(x_1-x_2)$$
$$+\,iq\hbar\int d^4y\int d^4y'\ \mathrm{Tr}\left[\gamma_\mu\ \mathcal{G}_2(x_1,y)\,\mathcal{G}_2(x_1,y')\,\Lambda_\nu(y,y';x_2)\right] \qquad (44)$$

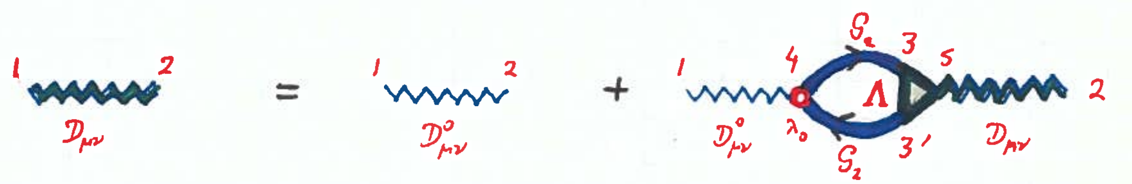
where we define the 3-point photon-electron vertex $\Lambda_\mu(y,y';x)$ by

$$\Lambda_\mu(y,y';x) = \left.\frac{\delta^3\Gamma}{\delta\bar\psi(y)\,\delta\psi(y')\,\delta A^\mu(x)}\right|_{A,\psi,\bar\psi=0} \qquad (45)$$

Now eqtn. (44) has a ready diagrammatic interpretation. Let us rewrite it schematically in the form:

$$D^{-1}_{\mu\nu}(1,2) = (D^0_{\mu\nu}(1,2))^{-1}\delta(1-2) + \int d3\int d3'\ \lambda^0_\mu\ \mathcal{G}_2(1,3)\,\mathcal{G}_2(1,3')\,\Lambda_\nu(33'2) \qquad (46)$$

where $\lambda^0_\mu = i g \gamma_\mu$. Now the inverse of this eqtn is an eqtn for $\mathcal{D}_{\mu\nu}(1,2)$, which we show as



(47)

and write as

$$\boxed{\mathcal{D}_{\mu\nu}(1,2) = \mathcal{D}^0_{\mu\nu}(1,2) + \lambda^0_\mu \int d4 \int d3 \int d3' \, \mathcal{G}(4,3)\mathcal{G}(4,3')\Lambda_\nu(33'5)\mathcal{D}_{\mu\nu}(5,2)} \quad (48)$$

What this eqtn says is that a photon coupled to electrons will either propagate without distortion, or will create a particle-hole pair, which subsequently propagates, itself able to emit or absorb photons at will.

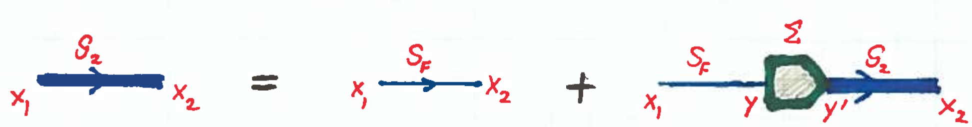
By starting from (34), and performing similar manoeuvres (now functionally differentiating w.r.t. $\psi(x)$ and $\bar{\psi}(x)$), we can also end up with another eqtn for the fermion propagator, in the form

$$(i\gamma^\mu \partial_\mu - m)\,\mathcal{G}_2(x_1 x_2) - \lambda^0_\mu \int d^4x \int d^4y \int d^4y' \left(\mathcal{D}^{\mu\nu}(x_1 x)\,\mathcal{G}_2(x_1, y)\Lambda_\nu(yy'x)\right)\mathcal{G}_2(y' x_2) = \delta(x_1 - x_2) \quad (49)$$

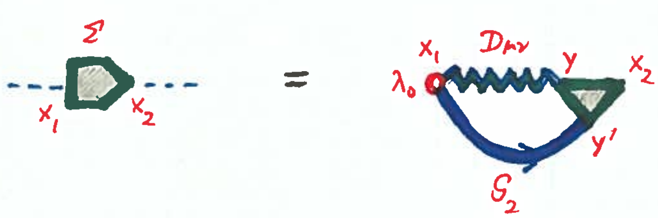
which we rewrite as

$$\boxed{\begin{aligned} &\int d^4y \left[(i\gamma^\mu \partial/\partial x_1^\mu - m)\,\delta(x_1 - y) + \Sigma(x_1, y) \right] \mathcal{G}_2(y, x_2) = \delta(x_1 - x_2) \\ &\Sigma(x_1 x_2) = \lambda^0_\mu \int d^4y \int d^4y' \, \mathcal{D}^{\mu\nu}(x_1, y)\,\mathcal{G}_2(x_1, y')\Lambda_\nu(yy'x_2) \end{aligned}} \quad (50)$$

and we can interpret these diagrammatically as



(51)



(52)

Now, by looking at the graphs in eqtns (47), (51), and (52), we see that we have a coupled pair of eqtns for $\mathcal{D}_{\mu\nu}(x_1 x_2)$ and $\mathcal{G}_2(x_1 x_2)$, in terms of a vertex part $\Lambda_\mu(yy'x)$. We can of course now go on to determine an eqtn for $\Lambda_\mu(yy'x)$ in terms of higher correlation functions – we will not do this here, since we are going to look at this thing in more detail for the very similar electron-phonon problem below.

Let's recap here. Starting from a Schwinger-Dyson eqtn of motion for Z (cf, eg., (4), or (33), or (34)), we functionally differentiate repeatedly to get eqtns of motion for the various correlators – these eqtns are all coupled.

(ii) DYSON AND BETHE-SALPETER EQUATIONS

: The most useful of the coupled integrodifferential eqtns that one deals with in either non-relativistic many-body problems, or relativistic QFT, are the eqtns for $\mathcal{G}_2$ (or $\mathcal{D}$, for bosons) and for $\mathcal{G}_4$ for fermions; when one has a fermion-boson coupling, one also deals with the composite propagator $\mathcal{G}_3^{21}[\bar{\psi},\psi,A]$, where A is the bosonic field, or with its associated vertex $\Lambda(\psi,\bar{\psi},A)$.

In the preceding discussion we have already seen a concrete example of this, for the eqtn (47), or its equivalents in (44), (46), and (48), are just Dyson eqtns for the photon propagator $\mathcal{D}_{\mu\nu}(x,x_2)$, and eqtns (50)-(52) give a Dyson eqtn for $\mathcal{G}_2(x,x_2)$.

What I would like to do now is look a little more at the structure of these low-order correlation function eqtns of motion, by focussing on one particular theory. This will be the 4-fermion theory with the Hamiltonian in (21); and we will simplify it further, when useful, to the toy model:

$$\mathcal{H}_{Toy} = \sum_{j=1}^{N} \frac{-\hbar^2\nabla_j^2}{2m} + V_0 \sum_{i<j}\sum \delta(r_i - r_j) \qquad (53)$$

This model will give us something to play with. However let's start with a few more general observations:

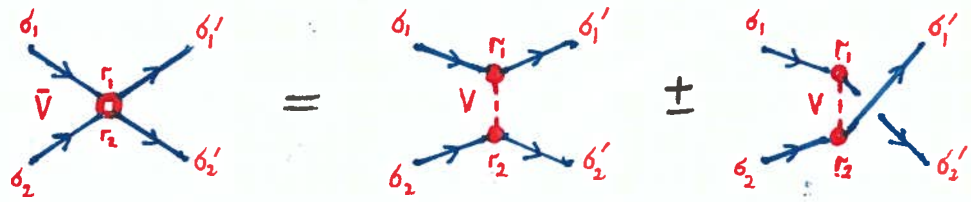
B-S EQTN for NON-RELATIVISTIC ψ^4 THEORY

: Suppose we go back to the non-relativistic theory of interacting fermions, having the Hamiltonian (21). Now we have already seen that the 2-point propagator $\mathcal{G}_2(x,x_2)$ can be written for this theory in the form given in (27). However this result can be rewritten in a way which turns out to be much more useful in some cases – this was first done by Landau in 1959, in his microscopic theory of Fermi liquids.

Let's start by introducing some useful functions connected with the 4-point interaction vertex. We start with the bare interaction $V(r_1-r_2)$ in (21) (or in q-space, $V(q)$). To take care of exchange between the 2 fermions in the interaction, we define the SYMMETRIZED VERTEX $\bar{V}$ (with $\pm$ for fermions/bosons):

$$\begin{aligned}\bar{V}_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(r_1r_2;t_1t_2) &= V(r_1-r_2;t_1-t_2)\,[\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} \pm \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}] \\ V_{\sigma_1\sigma_2\sigma_1'\sigma_2'}(\underline{q},w) &= V(\underline{q},w)\,[\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'} \pm \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}]\end{aligned} \qquad (54)$$

where we assume here that the bare vertex $V(r_1-r_2;t_1-t_2)$ has no dependence on spin (ie., we have no spin-orbit coupling, or any other dependence on spin); here σ_1 and σ_2 are the incoming spin indices, and σ_1', σ_2' the outgoing indices. In diagrammatic terms we have



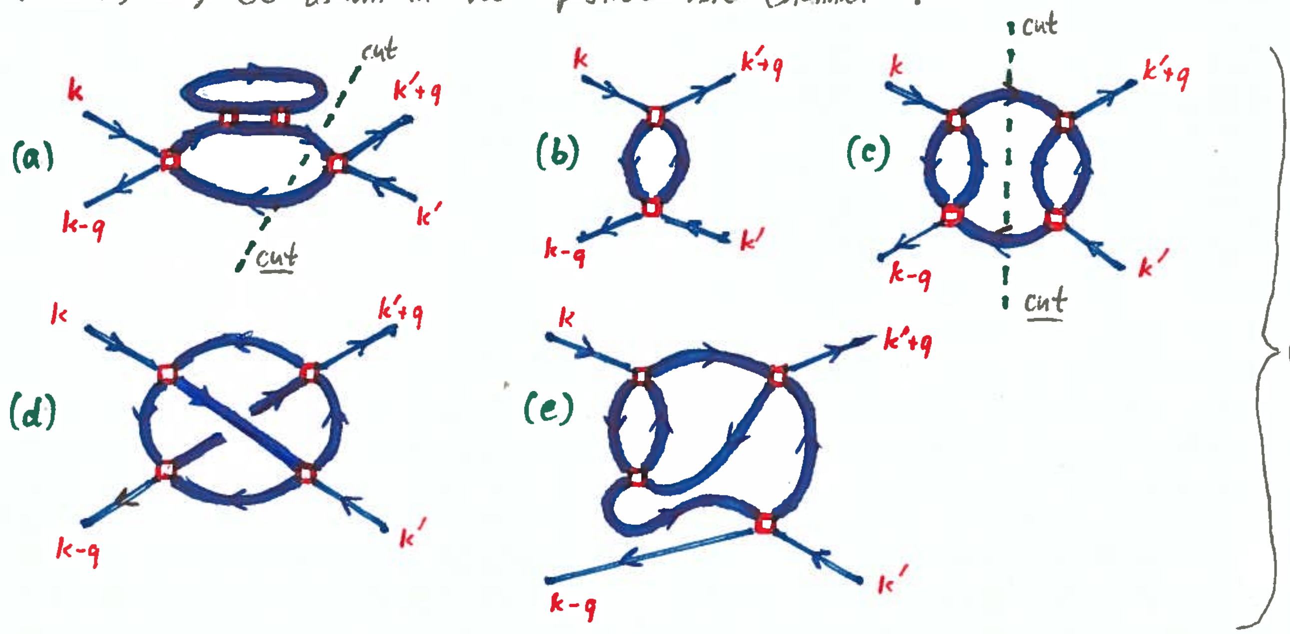
(55)

where as in (54), we have + for bosons, − for fermions. Note that for the toy model δ-function interaction in (53), we have

$$\left.\begin{array}{l} V(r_1 r_2 ; t_1 t_2) \longrightarrow V_0\, \delta(r_1 - r_2)\, \delta(t_1 - t_2) \\ V(\underset{\sim}{q}, w) \longrightarrow V_0 \end{array}\right\} \text{Toy model} \qquad (56)$$

with a corresponding simplification in $\bar{V}$.

The other vertex we want to consider is the IRREDUCIBLE 4-point vertex $\mathcal{I}(1,2,3,4)$. To see what this means, lets consider some of the diagrams that will contribute to the complete 4-point vertex. Here are a few; they are drawn in the "particle-hole channel":



(57)

The particle-hole channel is the one for which the incoming state $|k\rangle$ is a particle (it is above the Fermi surface), and for which the incoming state $|k'\rangle$ is a hole (it is below the Fermi surface). Because the charge, momentum, spin, and energy are opposite for the hole to what they would be to the particle, but switch under reversal of time and direction, we show them traveling backwards.

Now consider the "cuts" being made in these graphs at a SPECIFIC

TIME (which would be a proper time in a relativistic theory). The cuts should be vertical, since time is flowing to the right, but in (a) I have tilted the cut, for clarity; you should imagine that it marks a specific time.

Now the purpose of these cuts is to show which of the 5 graphs in (57) is "particle-hole" reducible. What this means is that at some point along the time flow of the graph, it can be cut into 2 pieces by cutting through a particle & hole line, ie., the 2 parts of the graph are joined at this time solely by a particle-hole pair. Thus in (57) we have

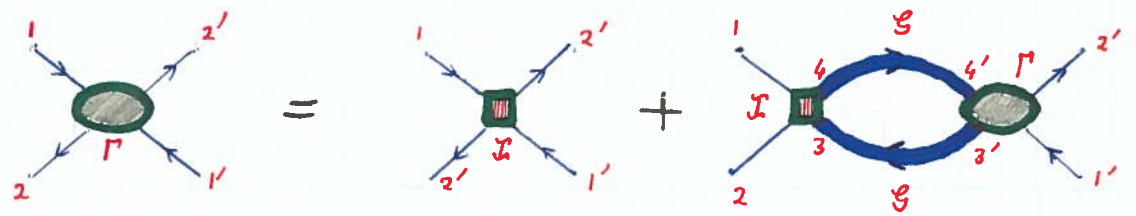
$$\left.\begin{array}{ll} (a),\,(c) & \text{reducible} \\ (b),\,(d),\,(e) & \text{irreducible} \end{array}\right\} \text{ in particle-hole channel.}$$

The irreducible 4-point vertex $I_4(1,2;34)$ is then the set of all graphs for the total 4-point vertex $\Gamma_4[12,34]$ which CANNOT be separated by cutting a particle-hole pair of lines at any point along the graph.

It then immediately follows that we can write an integral equation for Γ_4 in terms of I_4:

$$\Gamma_4(1,1';2,2') = I_4(1,1';2,2') + \int d3\,d3' \int d4\,d4'\, I_4(13;24)\, G_2(44')\, G_2(3'3)\, \Gamma_4(4'1';3'2') \qquad (58)$$

taking the graphical form:

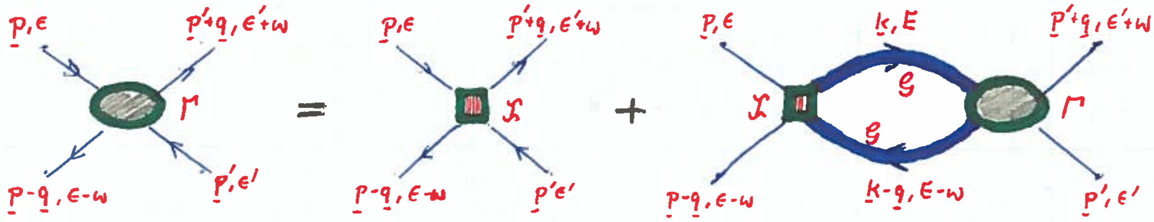


(59)

The labelling for the graph is revealing in a translationally invariant system, where momentum is conserved. Then we have

$$\Gamma_{\underline{p}\underline{p}'\underline{q}}(\epsilon\epsilon'\omega) = I_{\underline{p}\underline{p}'\underline{q}}(\epsilon\epsilon'\omega) + \sum_{\underline{k}} \int \frac{dE}{2\pi}\, I_{\underline{p}\underline{k}\underline{q}}(\epsilon,E;\omega)\, G_{\underline{k}}(E)\, G_{\underline{k}-\underline{q}}(E)\, \Gamma_{\underline{k}\underline{p}'\underline{q}}(E,\epsilon';\omega) \qquad (60)$$

with the labelling as shown:



(61)

from which see that the 4-point vertex Γ_4 in the particle-hole channel is made up by iterating the irreducible vertex $\mathcal{I}_4$, connected by particle-hole free propagators $\mathcal{G}_2\mathcal{G}_2$, i.e., we have

$$\Gamma_4 = \mathcal{I}_4 + \mathcal{I}_4 \mathcal{G}_2 \mathcal{G}_2 \mathcal{I}_4 + \mathcal{I}_4 \mathcal{G}_2 \mathcal{G}_2 \mathcal{I}_4 \mathcal{G}_2 \mathcal{G}_2 \mathcal{I}_4 + \cdots \qquad (62)$$

describing the repeated scattering of particle-hole pairs, described by $\mathcal{G}_2\mathcal{G}_2$, off each other via the irreducible vertex $\mathcal{I}_4$.

Of course all the difficulty of the many-body problem (and likewise of the relativistic QFT) is that unless there is some small parameter in the theory, we do not know the relationship between the "bare interaction" V and the irreducible interaction $\mathcal{I}_4$. However this situation, apparently hopeless, was transformed by the Landau theory of Fermi liquids, which has shaped our understanding of strong correlations in QFT ever since. Landau made the following observations (or at least these observations were implicit in his theory and subsequent developments of it):

(i) There is no particular reason for working with $\bar{V}(q)$ as opposed to $\mathcal{I}_4(pp'q;\epsilon\epsilon'\omega)$, at least from a fundamental point of view. Since all effective Hamiltonians are just that – models that assume some cut-off, and certain restrictions on the Hilbert space in which we work. The only difference, for a given physical system, between $V(q)$ and $\mathcal{I}(pp'q;\epsilon\epsilon'\omega)$ is that $V(q)$ is an interaction in a theory with a very large UV cut-off, so that it looks static in any low-energy regime (it includes fast processes, and only very slow processes are not included in it).

(ii) In principle, at low energies, we ought to be able to work with a derived quantity like $\mathcal{I}_4$, which can in principle be connected to experiment

(iii) All of the interesting energy dependence in (60) comes, at low energy, from the particle-hole pair of lines in (60).

It would take us too far afield to go into the details of Landau's Fermi liquid theory (FLT); and to fully appreciate its power, we need to have understood some important ideas about renormalization. But we can get some idea of what is involved, and see a little how (iii) comes about, by looking at the form of $\mathcal{G}\mathcal{G}$. Thus we consider the function

$$\left.\begin{aligned} R(p,q;\epsilon,\omega) &= \mathcal{G}_2(p,\epsilon)\,\mathcal{G}_2(p-q,\epsilon-\omega) \\ &= \frac{1}{\epsilon - \epsilon_p^0 - \Sigma(p,\epsilon)}\;\frac{1}{\epsilon-\omega-\epsilon^0_{p-q} - \Sigma(p-q,\epsilon-\omega)} \end{aligned}\right\} \qquad (63)$$

where of course we do not know the form of the self-energy $\Sigma(p,\epsilon)$ unless we have solved the problem completely. However, as Landau observed, when the energy of the particles is low, a remarkable simplification occurs.

Consider what we would get first if there were no interactions in the system. Then we would have, using the usual diagram rules for non-interacting non-relativistic fermions, that

$$\left.\begin{aligned} R_0(\underline{p},\underline{q};\epsilon,\omega) &= G_0(p,\epsilon)\,G_0(p-q,\epsilon-\omega) \\ &= -\left[\frac{1-f_p}{\epsilon-\epsilon_p^0+i\delta}+\frac{f_p}{\epsilon-\epsilon_p^0-i\delta}\right]\left[\frac{1-f_{p-q}}{\epsilon-\omega-\epsilon^0_{p-q}+i\delta}+\frac{f_{p-q}}{\epsilon-\omega-\epsilon^0_{p-q}-i\delta}\right]\end{aligned}\right\} \quad (64)$$

in the $T=0$ formalism, where $f_p \equiv f(\epsilon_p^0-\mu) = (e^{\beta(\epsilon_p^0-\mu)}+1)^{-1}$ is the Fermi function, and ϵ_p^0 is the "bare" energy dispersion; this result is also the appropriate limit obtained by starting from the Matsubara finite-T expression

$$\left.\begin{aligned} R_0(\underline{p},\underline{q};\epsilon_n,\omega_m) &= \mathcal{G}_0(p,i\epsilon_n)\,\mathcal{G}_0(\underline{p}-\underline{q};i(\epsilon_n-\omega_m)) \\ &= \frac{1}{i\epsilon_n-\epsilon_p^0}\;\frac{1}{i(\epsilon_n-\omega_m)-\epsilon^0_{p-q}}\end{aligned}\right\} \quad (65)$$

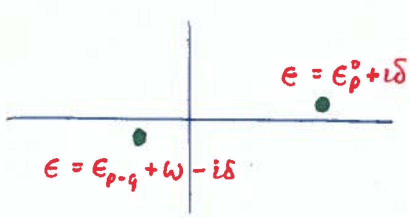
and then taking the limit as $T\to 0$, and making the analytic continuation down to the real axis; and here, as usual

$$\left.\begin{aligned} \epsilon_n &= (2n+1)\pi/\beta \qquad \text{(fermionic)} \\ \omega_m &= 2n\pi/\beta \qquad \text{(bosonic)}\end{aligned}\right\} \quad (66)$$

Now in the B-S eqtn, which is basically a kind of Dyson eqtn for the exact 4-point vertex in the particle hole channel, we see from (60)-(62) that we have deal with integrals of the general form

$$J_F = \sum_p \int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\,\mathcal{R}(\underline{p},\underline{q},\epsilon;\omega)\,\mathcal{F}(\epsilon) = \sum_{\underline{p}}\sum_n \mathcal{R}(p,q;i\epsilon_n;\omega)\,\mathcal{F}(i\epsilon_n) \qquad (67)$$

in which, from (64), we can see that as we approach the Fermi energy (so that both ϵ and $\epsilon-\omega$ are small), $R_0(p,q;\epsilon,\omega)$ is singular but $\mathcal{F}(\epsilon)$ is not. This is because $\mathcal{F}(\epsilon)$ is irreducible in the particle-hole channel, and the singularity in question is a "pinch singularity". The point is that eqtn (64) contains a product of 4 terms, but two of these have their poles on the same side (above or below) of the real axis, whereas the other two have their singularities on opposite sides; see left for one example. But then this means that we can simply



write $R_0(p,q;z,\omega)$ in the form (extending the argument to the complex energy plane, so $\epsilon \to z$):

$$R_0(p,q;z;\omega) = -\left[\frac{1-f_p}{z-\epsilon_p^\circ+i\delta}\,\frac{f_{p-q}}{z-\omega-\epsilon_{p-q}^\circ-i\delta} + \frac{f_p}{z-\epsilon_p^\circ-i\delta}\,\frac{1-f_{p-q}}{z-\omega-\epsilon_{p-q}^\circ+i\delta}\right] \quad (68)$$

since the other 2 terms will give zero in any contour integration. In the finite-T formalism we just get

$$R_0(p,q;z,\omega) = \frac{1}{\beta}\,\frac{1}{z-\epsilon_p^\circ}\,\frac{1}{z-\omega-\epsilon_{p-q}^\circ} \qquad (\text{finite } T) \quad (69)$$

Now lets go back to the integral in (67). As we noted above, $\mathcal{F}(z)$ is not supposed to have any singularities in z when $|p|$ is close to k_F, and $|q| \ll k_F$. Let's do the calculation using the finite T formalism (the zero-T calculation is trivial because $f_p = \theta(k_F-|p|)$ and $1-f_p = \theta(|p|-k_F)$). We then have, using the diagram rules discussed in the Appendix on calculation of diagrams, that

$$\frac{1}{\beta}\sum_{p,n} R_0(p,q;i\epsilon_n;\omega) = \pi_0(q,\omega) = -\sum_p \frac{f_p - f_{p-q}}{\epsilon_p^\circ-\epsilon_{p-q}^\circ-\omega-i\delta} \quad (70)$$

and that the integral $J_f^0 = \frac{1}{\beta}\sum_n R_0(p,q;i\epsilon_n;\omega)\,\mathcal{F}(i\epsilon_n)$ is then:

$$\left.\begin{aligned}\frac{1}{\beta}\sum_p\sum_n R_0(p,q;i\epsilon_n;\omega)\,\mathcal{F}(i\epsilon_n) &= \sum_p \frac{-1}{\epsilon_p^\circ-\epsilon_{p-q}^\circ-\omega-i\delta}\left(f_p\mathcal{F}(\epsilon_p^\circ) - f_{p-q}\mathcal{F}(\epsilon_{p-q}^\circ-\omega)\right)\\ &\sim -\sum_p \frac{f_p-f_{p-q}}{\epsilon_p^\circ-\epsilon_{p-q}^\circ-\omega-i\delta}\,\mathcal{F}(\epsilon_p^\circ)\end{aligned}\right\} \quad (71)$$

Now lets go back to the full integral J_F in (67). What form are we to use for the full Green functions here? It was the key insight of Landau that we could write this, again in the low-energy regime

$$\left.\begin{aligned}J_F &\sim \frac{1}{\beta}\sum_n\sum_p R(p,q;i\epsilon_n;\omega)\,\mathcal{F}(i\epsilon_n)\\ &= \frac{1}{\beta}\sum_n\sum_p\left[\frac{Z_p}{i\epsilon_n-\varepsilon_p}\,\frac{Z_{p-q}}{i\epsilon_n-\omega-\varepsilon_{p-q}} + \phi_{pq}(i\epsilon_n;\omega)\right]\mathcal{F}(i\epsilon_n)\end{aligned}\right\} \quad (72)$$

where we identify

$$\left.\begin{aligned}Z_k &: \text{"quasiparticle renormalization"}\\ \varepsilon_p &: \text{"quasiparticle energy"}\end{aligned}\right\} \quad (73)$$

and where it is assumed that $\phi_{pq}(i\epsilon_n, \omega)$ also has no singularities when $|\underline{q}| \ll k_F$ and $(|\underline{p}| - k_F) \ll k_F$.

How did Landau get this form? We will see this below once we have looked at Dyson's eqtn. But now let's use it - we immediately get

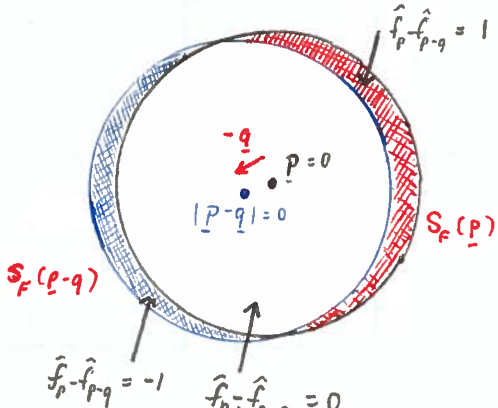
$$J_F \sim -\sum_p \left[\mathcal{Z}_p^2 \frac{\hat{f}_p - \hat{f}_{p-q}}{\varepsilon_p - \varepsilon_{p-q} - \omega - i\delta} + \phi_{pq}(\varepsilon_p, \omega) \right] \mathcal{F}(\varepsilon_p) \qquad (74)$$

where $\hat{f}_p \equiv [\exp\{\beta(\varepsilon_p - \mu)\} + 1]^{-1}$ is the Fermi function, this time for the "quasiparticle" energy ε_p, and again, as in (71), this result becomes exact in the limit where $|\underline{q}|, (|\underline{p}| - p_F) \ll p_F$, and $\omega \ll \mu$.

Let, so to cut down on notation, just write (74) as

$$J_F = \sum_p [\mathcal{P}_{\underline{p}}(q,\omega) + \phi_{\underline{p}}(q,\omega)] \mathcal{F}(\varepsilon_p) \qquad (75)$$

and now come to the next crucial observation made by Landau, viz, that we can just do a simple geometric analysis to further simplify things.



The figure at left shows how the function

$$\hat{f}_p - f_{p-q} \xrightarrow[T=0]{} \theta(p_F - |\underline{p}|) - \theta(p_F - |\underline{p} - \underline{q}|) \qquad (76)$$

varies in momentum space, for a small displacement $\underline{q}$; the Fermi surface $S_F(\underline{p})$ is defined as the locus of all points in k-space where $|\underline{p}| = p_F$. The regions shown in red have $\hat{f}_p - \hat{f}_{p-q} = 1$, and the region in blue has $\hat{f}_p - \hat{f}_{p-q} = -1$.

Now define a function

$$\underline{V}_p^* = \frac{\partial \varepsilon_p}{\partial \underline{p}} \approx V_F \frac{\underline{p}}{|\underline{p}|} \simeq \underline{p}/m_p^* \qquad (77)$$

for quasiparticles near the Fermi energy; this is the "quasiparticle velocity", and m_p^* is the quasiparticle effective mass — note that (i) we have linearized the dispersion relation near the Fermi surface, and (ii) that $\underline{V}_p^*$ and m^* are <u>not</u> the same as those quantities we would calculate for the bare particles (ie, $V_p^* \neq V_p^0 = \partial \varepsilon_p / \partial \underline{p}$, and $m_p^* \neq m_0$, the mass of the original fermion in the Hamiltonian).

It should now be clear that we can write

$$\mathcal{P}_{\underline{p}}(q,\omega) \longrightarrow -\frac{\mathcal{Z}_p^2}{V_F} \frac{\underline{V}_p^* \cdot \underline{q}}{\underline{V}_p^* \cdot \underline{q} - \omega} \delta(\varepsilon_p - \mu) \delta(|\underline{p}| - p_F) \qquad (78)$$

in the long-wavelength, low-energy limit we are discussing here (ie., $(|\underline{p}| - p_F), |\underline{q}| \ll k_F$, and $|\epsilon_p - \mu|, \omega \ll \epsilon_F$). The factor $\underline{V}_p^* \cdot q = V_F \cos\theta_{pq}$, where θ_{pq} is the angle between p and q.

With this in hand we can now give a formal solution to the Bethe-Salpeter eqtn., as follows. To do this quickly we treat quantities like Γ, $\mathcal{I}$, $\mathcal{P}$, etc, as matrices (which they are, in the space of their respective variables); then we have:

$$\Gamma = \mathcal{I} + \mathcal{I}(\mathcal{P} + \phi)\Gamma \qquad (79)$$

for the BS eqtn. (60); let us define the quantity Γ_0 by

$$\Gamma_0 = \mathcal{I} + \mathcal{I}\phi\Gamma_0 \qquad (80)$$

so that Γ_0 has no singularity in the particle-hole channel as $q, \omega \to 0$, unlike the function $\mathcal{P}$, which does (cf. eqtn. (78)). Now we have

$$\Gamma = \Gamma_0 + \Gamma_0 \mathcal{P} \Gamma \qquad (81)$$

or, translating back into the original variables, we have, from (81), that

$$\begin{aligned} \Gamma_{\underline{p}\underline{p}'}(\underline{q},\omega) &= \Gamma_0(\underline{p},\underline{p}') + \sum_{\underline{k}} \Gamma_0(\underline{p},\underline{k})\, z_k^2 \frac{f_k - f_{k-q}}{\omega - (\epsilon_{\underline{k}} - \epsilon_{\underline{k}-q}) + i\delta} \Gamma_{\underline{k}p'}(\underline{q},\omega) \\ &= \Gamma_0(\underline{p},\underline{p}') + \frac{z_F^2}{V_F}\sum_{\underline{k}} \delta(\epsilon_{\underline{k}} - \mu)\,\delta(|k| - k_F)\, \Gamma_0(\underline{p},\underline{k}) \frac{\underline{V}_k^* \cdot \underline{q}}{\omega - \underline{V}_k^* \cdot \underline{q}} \Gamma_{\underline{k}p'}(\underline{q},\omega) \end{aligned} \qquad (82)$$

where

$$\Gamma_0(p,p') = \mathcal{I}_{pp'} + \sum_k \mathcal{I}_{pk}\, \phi_{\underline{k}}(\epsilon_{\underline{k}})\, \Gamma_0(\underline{k},p') \qquad (83)$$

Finally, in a key step, Landau realized that one could rewrite this microscopic eqtn. in a way which united it with a pair of quantities defined as

$$\left.\begin{aligned} f_{pp'} &= z_p^2\, \Gamma_0(\underline{p},\underline{p}') \\ t_{pp'}(\underline{q},\omega) &= z_p^2\, \Gamma_{pp'}(\underline{q},\omega) \end{aligned}\right\} \qquad (84)$$

so that we can write (82) as

$$t_{pp'}(\underline{q},\omega) = f_{pp'} + \sum_k f_{p\underline{k}} \frac{f_k - f_{k-q}}{\omega - (\epsilon_{\underline{k}} - \epsilon_{\underline{k}-q}) + i\delta} t_{kp'}(\underline{q},\omega) \qquad (85)$$

(86)

Although we do not have time to go into it here, the quantities $f_{pp'}$ are the interaction between 2 quasiparticles having momentum p and p', near the Fermi surface (see a little more on this below). The eqtn. (85) (and thus the eqtns (82)) have the form of a "T-matrix eqtn.", ie., an eqtn of form

$$\hat{T} = \hat{V} + \hat{V} \frac{1}{E - \mathcal{H}_0 + i\delta} \hat{T} \qquad (87)$$

familiar from 1-particle QM. The difference here is that we are dealing with RENORMALIZED quantities for an N-particle system, describing not bare particles but renormalized quasiparticles.

DYSON EQTN for NON-RELATIVISTIC ψ^4 THEORY

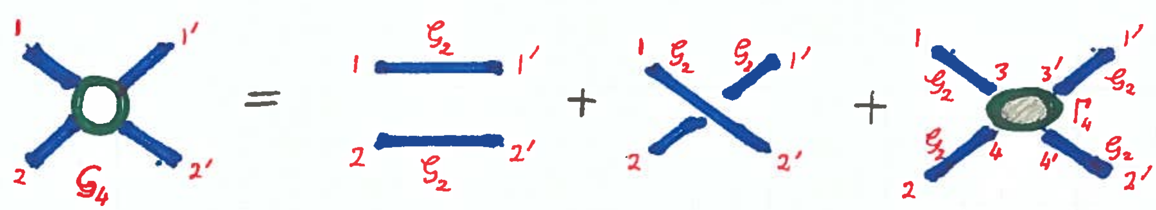
: The above development leaves a key question in the air – what is the quasiparticle energy, and how does it relate to the energy ϵ_p^0 in the original Hamiltonian? Or, what is the same thing, what allows us to go from the form (63) for the particle-hole propagator, in terms of ϵ_p^0 and $\Sigma_p(\epsilon)$, to the form (72), written in terms of a quasiparticle renormalization Z_p and a quasiparticle energy ε_p?

To answer this, let's go back to eqtn. (27) for $\mathcal{G}_2(x_1 x_2)$, and rewrite it in 2 different ways. First, using what we know about the BS eqtn., we will rewrite (27) in terms of the 4-point vertex. We shall then convert this to an expression in terms of the quasiparticle self-energy.

Consider first the result in (27), in which $\mathcal{G}_2$ is written in terms of $\mathcal{G}_4$. Notice that first, by the definition of the 4-point vertex Γ_4, we have that (for fermions):

$$\begin{aligned} \mathcal{G}_4(1,2;1'2') = {} & \mathcal{G}_2(1,1')\mathcal{G}_2(2,2') - \mathcal{G}_2(1,2')\mathcal{G}_2(2,1') \\ & + \sum_{3,3',4,4'} \mathcal{G}_2(1,3)\mathcal{G}_2(2,4)\Gamma_4(3,4;3'4')\mathcal{G}_2(3',1')\mathcal{G}_2(4'2') \end{aligned} \qquad (88)$$

which is shown diagrammatically as



(89)

which can be derived for this theory using the methods in section B.1. The physical meaning is obvious – the total correlator $\mathcal{G}_4$ results from the sum of the processes where 2 quasiparticles independently, plus the processes where they scatter via Γ_4. Note that we have external legs on Γ_4; the 2 quasiparticles must first propagate before they scatter off each other.

However we can now write Γ_4 in terms of the BS eqtn, and then substitute back into (21). This produces the following result for $\mathcal{G}_2(1,2)$:

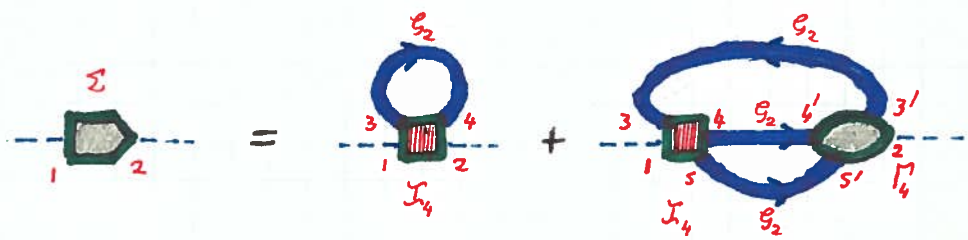
$$\mathcal{G}_2(1,2) = G_0(1,2) + \int d3\,d4\; G_0(1,3)\,\Sigma(3,4)\,\mathcal{G}_2(4,2) \tag{90}$$

which is just Dyson's eqtn, with the following result for Σ:

$$\begin{aligned}\Sigma(1,2) &= \Sigma_{HF}(1,2) + \Delta\Sigma(1,2)\\ \Sigma_{HF}(1,2) &= \int d3\,d4\; \mathcal{I}_4(1,2;3,4)\,\mathcal{G}_2(3,4)\\ \Delta\Sigma(1,2) &= \int d3\,d3'\,d4\,d4'\,d5\,d5'\; \mathcal{I}_4(1354)\,\mathcal{G}_2(4,4')\,\mathcal{G}_2(55')\,\mathcal{G}_2(3'3)\,\Gamma_4(55'3'2)\end{aligned} \tag{91}$$

and these eqtns are represented diagrammatically as follows:

(92)



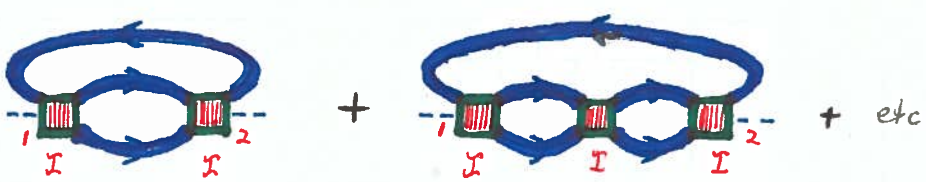
(93)

Note that the bare interaction has been absorbed here into $\mathcal{I}_4$, to get these results. The term $\Sigma_{HF}(1,2)$ is known as the "Hartree-Fock" term, and we note that all these results are exact. Obviously we can iterate the result in (93), to get, schematically, the result

$$\Sigma = \Sigma_{HF} + \mathcal{I}_4\mathcal{G}\mathcal{G}\mathcal{G}\mathcal{I}_4 + \mathcal{I}_4\mathcal{G}\mathcal{G}\mathcal{I}_4\mathcal{G}\mathcal{G}\mathcal{I}_4\mathcal{G} + \cdots \tag{94}$$

or, diagrammatically, we have

$\Delta\Sigma(1,2) =$



(95)

Clearly, if we want we can write a result for the self-energy Σ involving an iteration of $\mathcal{I}$.

Now these formal results are of some real practical use when one tries to make approximation to the exact results – we will see this briefly below. Before doing so, lets briefly see how we can understand results for $\mathcal{G}_2(1,2)$ and $\Sigma(1,2)$ near the Fermi surface.

First, lets compare the 2 basic expression we have for $\mathcal{G}_2(\underline{p},\epsilon)$, viz., in terms of the self-energy and the quasiparticle energy:

$$\mathcal{G}_2(\underline{p},\epsilon) = \frac{1}{\epsilon - \epsilon_p^0 - \Sigma(\underline{p},\epsilon)} \xrightarrow[\epsilon\to\mu]{p\to p_F} \frac{Z_F}{\epsilon - \mu - V_F^*(|\underline{p}| - p_F) + i\delta_\epsilon} \tag{96}$$

where $\delta_\epsilon \equiv \delta \,\text{sign}(\epsilon - \mu)$, and, as we saw, $V_F^* = p_F/m^*$, with m^* taken at the Fermi surface. Now the first version of $\mathcal{G}_2(\underline{p},\epsilon)$ will have a pole when

$$\epsilon - \epsilon_p^0 - \Sigma_p(\epsilon) = 0 \tag{97}$$

which, since

$$\begin{aligned} \frac{d\epsilon}{dp} &= \frac{d\epsilon_p^0}{dp} + \frac{\partial \Sigma_p(\epsilon)}{\partial p} + \frac{\partial \Sigma_p(\epsilon)}{\partial \epsilon}\frac{d\epsilon}{dp} \\ &= \frac{p}{m}\left(1 + \frac{m}{p}\frac{\partial \Sigma_p(\epsilon)}{\partial p}\right)\frac{1}{1 - \partial\Sigma_p(\epsilon)/\partial\epsilon} \end{aligned} \tag{98}$$

implies that we have a density of states $\quad N^*(\epsilon) = \dfrac{d\epsilon}{dp} = \dfrac{p}{m_p^*}$ (99)

and that

$$m_p^* = m\,\frac{1 - \dfrac{\partial \Sigma_p(\epsilon)}{\partial \epsilon}}{1 + \dfrac{m}{p}\dfrac{\partial \Sigma_p(\epsilon)}{\partial p}} \tag{100}$$

Now, as we will show immediately below, one has (in 3 dimensions) that

$$\begin{aligned} \text{Im}\,\Sigma_p(\epsilon) &\xrightarrow[\epsilon\to\mu]{p\to p_F} C\,|\epsilon_p - \mu|^2 \qquad (3d) \\ \text{Re}\,\Sigma_p(\epsilon) &\xrightarrow[\epsilon\to\mu]{p\to p_F} (1 - m^*/m)(\epsilon - \mu) \end{aligned} \tag{101}$$

Now, let us write the spectral representation of $\mathcal{G}_2(\underline{p},\epsilon)$ as

$$\mathcal{G}_2(\underline{p},\epsilon) = \int_{-\infty}^{\infty}\frac{d\epsilon'}{\pi}\,\frac{A_p(\epsilon')}{(\epsilon - \mu) - \epsilon' + i\delta_{\epsilon'}} \tag{102}$$

where as usual, $\quad \text{Im}\,\mathcal{G}_2(\underline{p},\epsilon) = -A_p(\epsilon)\,\text{sign}(\epsilon - \mu)$ (103)

and also as usual we write $\quad \dfrac{1}{\epsilon + i\delta_\epsilon} \equiv \mathcal{P}\dfrac{1}{\epsilon} - i\pi\delta(\epsilon)\,\text{sign}\,\epsilon$ (104)

so that we can also write (now writing $\Gamma_p(\epsilon) = -A_p(\epsilon)\,\mathrm{sgn}(\epsilon-\mu)$) that

$$\mathcal{G}_2(p,\epsilon) = \frac{1}{(\epsilon-\epsilon_p^0-\Delta_p(\epsilon))^2+\Gamma_p^2(\epsilon)}\left[\Delta_p(\epsilon) - i\Gamma_p(\epsilon)\right] \tag{105}$$

Now, as we approach the Fermi surface, $\Gamma_p(\epsilon) \longrightarrow \delta_\epsilon$, and we can write, using (100), (101), and (105), that

$$\mathcal{G}_2(p,\epsilon) = \frac{Z_p(\epsilon)}{\epsilon-\mu-v_F^*(|p|-k_F)+i\delta_\epsilon} \tag{106}$$

where the "wave-function renormalization" $Z_p(\epsilon)$ is

$$Z_p(\epsilon) = \frac{1}{1-\partial\Sigma_p(\epsilon)/\partial\epsilon} \xrightarrow[\substack{|p|\to p_F \\ \epsilon\to\mu}]{} Z_F \tag{107}$$

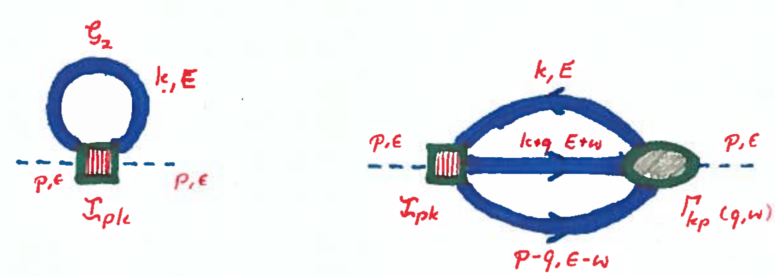
Now let us verify the low-energy forms of $\mathrm{Re}\,\Sigma_p(\epsilon) = \Delta_p(\epsilon)$ and of $\mathrm{Im}\,\Sigma_p(\epsilon) = \Gamma_p(\epsilon)$ that we quoted above. We wish to do this quite generally, without making any diagrammatic approximation, and so we call upon our results in (91) - (95) above. Now the easiest way to calculate $\Sigma_p(\epsilon)$ in general is to start by calculating $\mathrm{Im}\,\Sigma_p(\epsilon)$ using the Landau-Cutkowsky rules described in the Appendix on "how to calculate diagrams", and then from this find $\mathrm{Re}\,\Sigma_p(\epsilon)$ using the dispersion relation quoted in (102).

According to the Landau-Cutkowsky rules, the imaginary part of $\Sigma_p(\epsilon)$, with $\Sigma_p(\epsilon)$ given by (93), is actually rather straightforward. Writing (93) in momentum-energy space, we have (ignoring the non-singular dependence of $\mathcal{I}$ on q,ω):

$$\Sigma_{HF}(p,\epsilon) = \sum_k \int\frac{dE}{2\pi}\,\mathcal{I}_{pk}(\epsilon,E)\,\mathcal{G}_2(k,E)$$

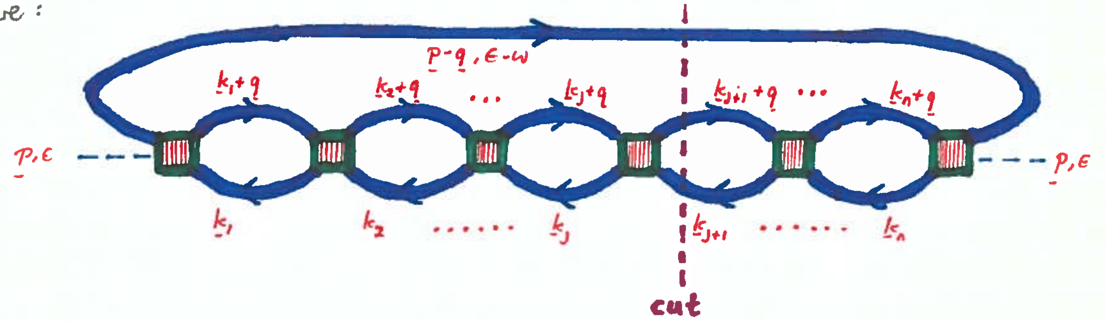
$$\Delta\Sigma(p,\epsilon) = \sum_{k,q}\int\frac{dE}{2\pi}\int\frac{d\omega}{2\pi}\,\mathcal{I}_{pk}(\epsilon,E)\,\mathcal{G}_2(p-q,\epsilon-\omega)\,\mathcal{G}_2(k+q,E+\omega)\,\mathcal{G}_2(k,E)\,\Gamma_{kp}(E,\epsilon;q,\omega) \tag{108}$$

which diagrammatically are just:



(109)

so that if we now look at an arbitrary Landau-Cutkowsky cut, we see that, because (109) allows us to iterate $\mathcal{I}_q$ arbitrarily many times in the particle-hole channel, we get "reduced graphs" for $\mathrm{Im}\,\Sigma_p(\epsilon)$ of the general form shown here:



(110)

where we see that for a graph involving Γ_4 to n-th order, we can cut through a pair of particle-hole lines at any point along the chain. By iterating on both sides of the cut, it is then obvious that we can now write

$$\left.\begin{aligned} \mathrm{Im}\,\Sigma_{HF}(p,E+i\delta) &= \sum_k \int \frac{d\epsilon}{2\pi} \left(\Gamma_{pk}(\epsilon,E)\, \mathrm{Im}\, G_2(k,\epsilon) + G_2(k,\epsilon)\, \mathrm{Im}\, \Gamma_{pk}(\epsilon,E) \right) \\ &\xrightarrow[p,k\to S_F]{} \sum_k \int \frac{d\epsilon}{2\pi}\, \Gamma_{pk}(\epsilon)\, \mathrm{Im}\, G_2(k,\epsilon) \sim 0 . \end{aligned}\right\} \quad (111)$$

where we get exactly zero if particle-hole symmetry is assumed; and

$$\mathrm{Im}\,\Delta\Sigma_p(\epsilon+i\delta) = -\pi \sum_{k,q} \int \frac{dz_1}{2\pi}\frac{dz_2}{2\pi}\frac{dz_3}{2\pi} \frac{f(z_1)f(z_2)f(z_3)}{f(\epsilon)}\, \delta(\epsilon+(z_1-z_2-z_3)) \times \left|\Gamma_{pk}(q,\omega)\right|^2 A_{p-q}(z_1)\, A_{k+q}(z_2)\, A_k(z_3) \qquad (112)$$

where we ignore any dependence of $\Gamma_{pk}(\epsilon,E;q\omega)$ on ϵ and E, assuming we are near S_F, and that this dependence is slow. Now, the intermediate states in this integral are, if the are near the Fermi surface, going to have small imaginary parts; so if we make the approximation

$$A_k(\epsilon) \sim -2\pi Z_k\, \delta(\epsilon - \varepsilon_k) \qquad (113)$$

where now the quasiparticle energy is defined as

$$\varepsilon_k = \epsilon_k^0 + \mathrm{Re}\,\Sigma_k(\varepsilon_k) \qquad (114)$$

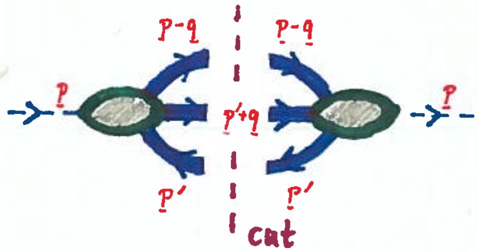
(a self-consistent eqtn.), then (112) becomes

$$\mathrm{Im}\,\Delta\Sigma_p(\varepsilon_p+i\delta) = -\pi \sum_{p'}\sum_{q} \left|\Gamma_{pp'}(q,\omega)\right|^2 \tilde{f}_{p'} (1-\tilde{f}_{p-q})(1-\tilde{f}_{p'+q})\, \delta(\varepsilon_p+\varepsilon_{p'}-\varepsilon_{p-q}-\varepsilon_{p'+q}) \qquad (115)$$

where now we go to the pole in $G_2(p,\epsilon)$, with $\epsilon \to \varepsilon_p$, which satisfies eqtn. (114). From this result it is then obvious that we can write the total self-energy for the QUASIPARTICLE in the form

$$\Sigma_p(\varepsilon_p + i\delta) = \frac{1}{Z_p} \sum_{\underline{p}'} \sum_{\underline{q}} |t_{pp}(\underline{q},\omega)|^2 \; \frac{\check{f}_p (1-\check{f}_{p-q})(1-\hat{f}_{p'+q})}{\varepsilon_p + \varepsilon_{p'} - \varepsilon_{p-q} - \varepsilon_{p'+q} + i\delta} \qquad (116)$$

where we divide by Z_p because the pole in $\mathcal{G}_2$ is proportional to Z_p, and we wish to have a properly normalized Σ. The diagrammatic portrayal of these results is summed up in the Landau - Cutkosky graph:

$\Im m\, \Sigma_p(\varepsilon_p)$  (117)

and we see that the δ-fn in (115) and (116) simply reflects the conservation of quasiparticle energy.

Now we have everything we need to calculate $\Im m\, \Sigma_p(\varepsilon)$ explicitly. In fact the zero - T integration is relatively straightforward, because the integral

$$\sum_{\underline{p}'} \sum_{\underline{q}} \theta(p_F - |\underline{p}|)\,\theta(|\underline{p}-\underline{q}| - p_F)\,\theta(|\underline{p}'+\underline{q}| - p_F)\,\delta(\varepsilon_p + \varepsilon_{p'} - \varepsilon_{p-q} - \varepsilon_{p'+q}) \propto (|p| - p_F)^2 \qquad (118)$$

as one can see using geometric considerations. The generalization to finite T involves a little more algebra, but one eventually gets

$$\left.\begin{aligned} \Im m\, \Sigma_p(\varepsilon_p) &= \langle |t_{pp'}(\underline{q},0)| \rangle \left(|\varepsilon_p - \mu|^2 + \pi^2 k_B^2 T^2 \right) \\ &\quad + O\left(|\varepsilon_p - \mu|^3 \right) \end{aligned}\right\} \qquad (119)$$

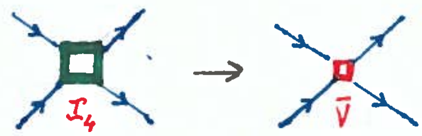
where $\langle \cdots \rangle$ refers to a specific angular average around the Fermi surface; we need not go into the details here.

RANDOM PHASE APPROXIMATION for FERMI LIQUIDS :

For those who like to see the calculations proceed in a simpler fashion, and dislike all the clutter that comes from the integrations over $\underline{p}$ and $\underline{p}'$ (which are basically just angular integrations) as well as the renormalizations, there is an extremely simple approximation that one can look at, called the RPA (for "Random Phase Approximation"; the historical origin of the name is not important here). The idea is very simple, and is summarized as follows:

(i) Let $\mathcal{I}_{\underline{p}\underline{p}'}(\epsilon,\epsilon';\underline{q},\omega) \longrightarrow \bar{V}(\underline{q})$

i.e.,



(120)

(ii) In all internal lines in the diagram, we make the replacement

$$\mathcal{G}_2(p,\epsilon) \longrightarrow G_0(p,\epsilon) \tag{121}$$

Since the final result we get for $\mathcal{G}_2(p,\epsilon)$ is incompatible with (121), we see that this RPA is internally inconsistent. Nevertheless it leads to interesting results, as we will now see.

From these assumptions we immediately get a result for $\mathcal{G}_2^{RPA}(\underline{p},\epsilon)$, given in the coordinate representation by the usual

$$\mathcal{G}_2(1,2) = G_0(1,2) + \int d3d3'd4\; G_0(1,3')\bar{V}(34)\Sigma^{RPA}(3,4)\,\mathcal{G}_2(4,2) \tag{122}$$

where $\Sigma^{RPA} = \Sigma_{HF}^{RPA} + \Delta\Sigma^{RPA}$, and we have

$$\Sigma_{HF}^{RPA}(1,2) = \int d3\; V(1,3)\,G_0(3,2)\,\delta(3-2) \tag{123}$$

$$\Delta\Sigma^{RPA}(1,2) = \int d3\,d4\; \bar{V}(1,3)\,\Pi^{RPA}(3,4)\,G_0(1,2)\,\bar{V}(4,2) \tag{124}$$

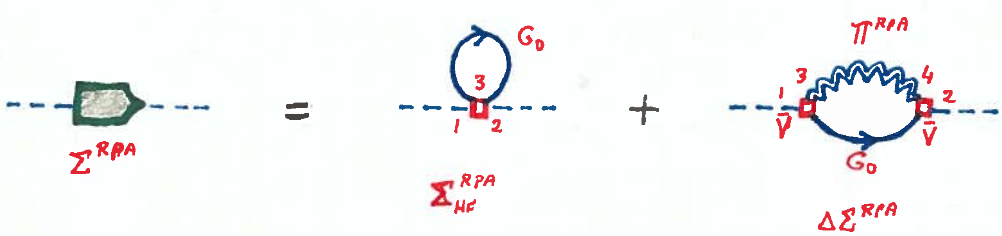
and in momentum representation by

$$\mathcal{G}_2(p,\epsilon) = \frac{G_0(\underline{p},\epsilon)}{1 - G_0(\underline{p},\epsilon)\,\Sigma_p(\epsilon)} \tag{125}$$

with self-energies:

$$\Sigma_{HF}^{RPA}(\underline{p},\epsilon) = \sum_{\underline{q}} \bar{V}(\underline{q})\,G_0(\underline{p}-\underline{q},\epsilon)$$
$$\Delta\Sigma^{RPA}(\underline{p},\epsilon) = \int\frac{d\omega}{2\pi}\sum_{\underline{q}} |\bar{V}(\underline{q})|^2\,G_0(\underline{p}-\underline{q},\epsilon-\omega)\,\Pi_{\underline{q}}^{RPA}(\omega) \tag{126}$$

and we represent this diagrammatically by



(127)

where the "polarization propagator" $\Pi^{RPA}(\underline{q},\omega)$ or $\Pi^{RPA}(1,2)$ satisfy the integral eqtns:

$$\Pi^{RPA}(1,2) = \pi_0(1,2) + \int d3\, d4\, \pi_0(1,3)\bar{V}(3,4)\Pi^{RPA}(4,2) \tag{128}$$

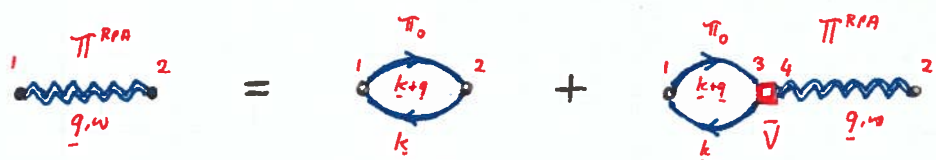
and.

$$\begin{aligned} \Pi^{RPA}(\underline{q},\omega) &= \pi_0(\underline{q},\omega) + \pi_0(\underline{q},\omega)\bar{V}(q)\Pi^{RPA}(\underline{q},\omega) \\ &= \frac{\pi_0(\underline{q},\omega)}{1-\bar{V}(q)\pi_0(\underline{q},\omega)} \end{aligned} \tag{129}$$

where the "bare polarization part", or "polarization bubble" is just

$$\begin{aligned} \pi_0(\underline{q},\omega) &= \sum_{\underline{p}} \int \frac{d\epsilon}{2\pi} G_0(p+q,\epsilon+\omega)\, G_0(p,\epsilon) \\ &\equiv \sum_{p} \int \frac{d\epsilon}{2\pi} R_0(p,q;\epsilon,\omega) \end{aligned} \tag{130}$$

where R_0 is the function introduced earlier (eqtns (63) & (65)); the generalization to finite T is obvious. Diagrammatically:



(131)

Now we can rewrite $\Sigma^{RPA}(p,\epsilon)$ in several ways. The first is obvious from what we've already done; we have

$$\Sigma^{RPA}(\underline{p},\epsilon) = \Sigma^{RPA}_{HF}(\underline{p},\epsilon) + \sum_{q}\int\frac{d\omega}{2\pi}|\bar{V}(q)|^2 G_0(\underline{p}-\underline{q},\epsilon-\omega)\frac{\pi_0(\underline{q},\omega)}{1-\bar{V}(q)\pi_0(\underline{q},\omega)} \tag{132}$$

in which we simply iterate the bubbles. Another way to write $\Sigma^{RPA}(1,2)$ is seen by referring back to the expression for $\Sigma(p,\epsilon)$ in eqtns. (92) - (95). Substituting for I_4 and G_2 in these expressions according to the RPA prescription given in (120) and (121), we get

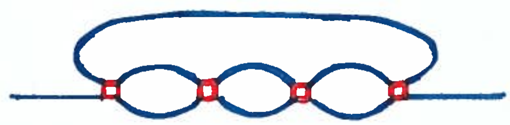
$$\begin{aligned} \Delta\Sigma^{RPA}(1,2) &= \int d1' \int d3\, d3'\, d4\, \bar{V}(1,1')\, G_0(1',3)\, G_0(1',3')\, G_0(1',4)\, \Gamma_{RPA}(2;3,3',4) \\ \Delta\Sigma^{RPA}(\underline{p},\epsilon) &= \sum_{p'}\sum_{q}\int\frac{d\epsilon'}{2\pi}\int\frac{d\omega}{2\pi}\bar{V}(\underline{q})\, G_0(\underline{p}-\underline{q},\epsilon-\omega)\, G_0(\underline{p}'+\underline{q},\epsilon'+\omega)\, G_0(\underline{p}'\epsilon')\, \Gamma^{RPA}_{pp'}(\epsilon\epsilon';\underline{q},\omega) \end{aligned} \tag{133}$$

where the RPA expression for $\Gamma^{RPA}_{pp'q}(\epsilon\epsilon';\omega)$ is extremely simple;

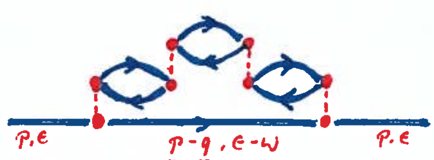
$$\Gamma_{pp'}(\epsilon,\epsilon';q\omega) \quad = \quad |\bar{V}(q)|^2 \; \Pi^{RPA}(q,\omega) \; + \; \bar{V}(q) \tag{134}$$

where we see now that the new 4-point vertex is an entirely isotropic function of p, p', and independent of energy – in other words, all it has left is the crucial singular dependence on (q,ω). It is for this reason that it is quite popular – it captures a key feature of Fermi liquid theory, but is otherwise trivial. However it is also internally inconsistent – the internal lines include no self-energy corrections.

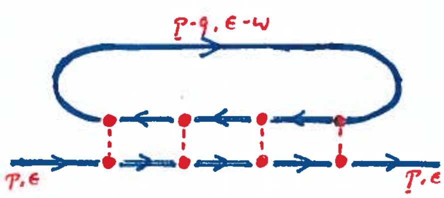
One thing that is not so obvious from what we've done so far, but is clear if we unpack the symmetrized vertices we've been using, is that all these "bubble sums" are actually not just sums of bubbles. To see this, consider the graph


(135)

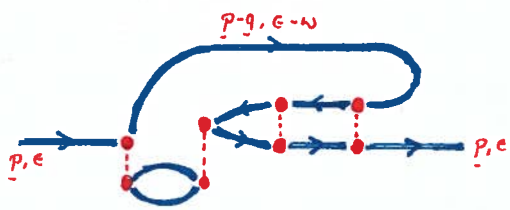
and notice immediately that 2 ways of reading this in terms of unsymmetrized graphs are



"bubble sum" (136)



"ladder sum" (137)



mixed "ladder-bubble" (138)

Even the Hartree-Fock term resolves itself into 2 separate terms: the Hartree term

$$\Sigma^{RPA}_{HF}(p,\epsilon) \;=\; V(0)\sum_k \int \frac{d\epsilon}{2\pi} G_0(k,\epsilon) \;-\; \sum_q \int \frac{d\omega}{2\pi} V(q)\, G_0(p-q,\epsilon-\omega) \tag{139}$$

which describes the averaged effect of the background fermions, on a single fermion, and the "Fock" term which describes the averaged effect of the exchange of this fermion with any of the other fermions. – see (139).

In detailed theories of Fermi liquids, ranging from the electron liquid in metals to the neutral Fermi liquid theory describing liquid ^3He at low T, one selects out certain diagrams or diagram classes as being more important. This kind of application is beyond the scope of these lectures.

Finally, let us note that the whole theory developed here is equally useful for scattering experiments and the general theory of the atomic nucleus, and in high-energy physics. (the Bethe-Salpeter eqtn was originally developed in the context of atomic & nuclear physics). Various elaborations are also important in astrophysical theory. Naturally one now develops the theory in relativistic form, but this leads to only trivial modifications. The key question to ask in all such modifications is – what is the energy scale in comparison to the Fermi energy? Thus, in scattering experiments, the density is almost zero, and the Fermi sea is non-existent – the theory is then much simpler. However in nuclei, or in white dwarfs & neutron stars, the Fermi energy may actually be the largest energy – in this case the theory is quite similar to that we have just discussed.

B.5.3. DYSON & BETHE-SALPETER EQTNS: QED and the ELECTRON-PHONON SYSTEM

The similarity between the electron-phonon interaction and the relativistic electron-photon interaction in QED means that we can develop the two theories in parallel. In what follows we will develop the Dyson & Bethe-Salpeter eqtns in detail for both of these systems, and compare with the results we have just seen for Fermi liquids. We will also, for the electron-phonon system, see what happens when we also add in electron-electron interactions – this leads to very important insights.

B.5.3 (a) B-S EQUATIONS for Q.E.D.

The original development of the Bethe-Salpeter eqtn. in QED in the 1950's (beginning with the work of Bethe & Salpeter in 1951) is of great historical significance - it led to many important applications in atomic physics, in nuclear physics (Bethe & others) and astrophysics (Salpeter & others), which served to confirm and cement the theory as a blueprint for all quantum field theories. As we will also see, it later led to serious questions about the validity of perturbation expansions, when pursued too far.

In what follows, we pursue a fairly limited goal. Using the Schwinger-Dyson eqtns derived earlier for QED, we first derive a Bethe-Salpeter eqtn. for the 4-point electron vertex, i.e., for the 2-particle propagator. This tells us how a pair of electrons (or an electron-positron pair) will propagate. We then derive a similar Bethe-Salpeter eqtn. for the 3-point vertex, which in QED describes how an electron interacts with photons. The key feature we will discover is that, buried in the perturbation expansion is the possibility of a non-perturbative phenomenon, viz., the formation of bound states.

(i) B-S EQTN for $\mathcal{G}_4(x_1x_2;x_1'x_2')$

: Suppose we go back to the eqtn. of motion (49) or (50), for the fermion propagator in QED. This is written in the form of an integro-differential eqtn. for $\mathcal{G}_2(x-x')$; we can think of (49) as a modification of the free particle Dirac eqtn - which is just a simple differential eqtn like the Schrodinger eqtn - to include all the effects of photons. In the same way, eqtns (47) and (48) for the photon propagator are equivalent to a modification of Maxwell's eqtns, to include the effect of electron-positron pair excitation, and higher processes, on ordinary electrodynamics.

Now from these eqtns we notice that, just as in ordinary QM, there are different ways to set up the eqtns of motion - they can be differential eqtns., or integro-differential eqtns., or integral eqtns, either for the propagator or for some wave-function. This diversity becomes much larger when we are dealing with 2 coupled fields, and when we are looking at higher correlation functions - to look at a pair of particles, we need to look at $\mathcal{G}_4$ rather than $\mathcal{G}_2$. Thus the theory for pair propagation can be written in many different, but equivalent ways.

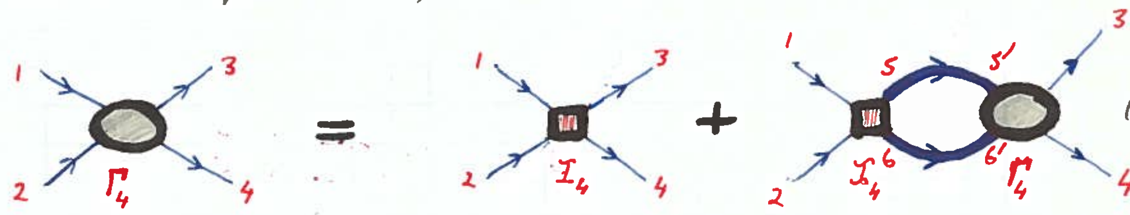
In what follows we will derive 2 forms of the Bethe-Salpeter eqtn. for $\mathcal{G}_4(x_1x_2x_1'x_2')$, and for the related 4-point vertex $\Gamma_4(x_1x_2;x_1'x_2')$. Each of these forms is interesting, and their derivations are instructive. Then, in (ii) below, we will derive an analogous result for the 3-point vertex Λ_3.

B-S EQTN in INTEGRAL FORM

: We have seen already how intuitively clear is the integral eqtn. for the 4-point vertex for the interacting Fermi liquid - compare eqtns. (59)-(62). It is then intuitively obvious that we must be able to write down very similar eqtns for the QED system. Let's just articulate why this is. In our discussion of the non-relativistic Fermi liquid, we introduced a phenomenological short-range irreducible vertex I_4, which depended in some unknown way on a more microscopic interaction $V(\underline{q})$, considered to act instantaneously.

However it is physically obvious that $V(q)$ itself is an approximation to some even more microscopic interaction, which, since it involves electrons and ions, must in a more microscopic theory be described by QED. The interaction between 2 electrons is now mediated by photons, indeed, by the renormalized photon propagator $D_{\mu\nu}(q) \equiv D_{\mu\nu}(\underline{q},\omega)$ in (47) and (48).

We see that without even troubling to evaluate the details, we can see immediately what diagrammatic form the theory will take, in our discussion in section B.3 of coupled fields. In particular, eqtn. (102) shows some low-order graphs for the 4-point vertex $\Gamma_4(1,2,3,4)$ for any theory involving the interaction between a bosonic field and a fermionic field having the same structure as either the electron-photon or the electron-phonon interaction. It the follows that we can immediately write the graphical eqtns for the Bethe-Salpeter eqtn. describing the effective electron-electron interaction in terms of an irreducible 4-point vertex, as:



(140)

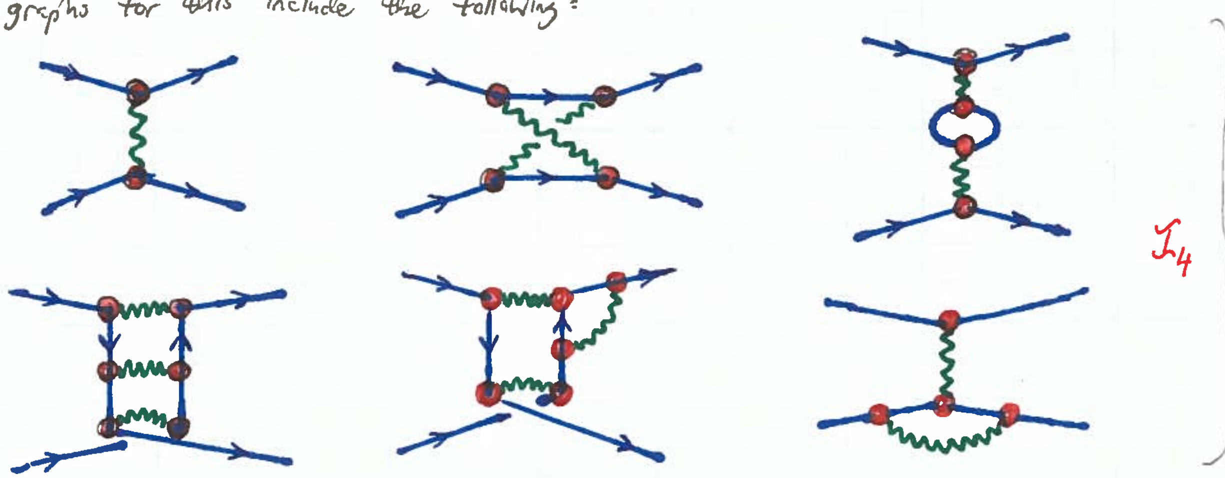
where we notice that the only difference between this eqtn. and the B-S eqtn. we developed for Fermi liquids (in eqtns. (58) and (59)) is that now we are working in the particle-particle channel (scattering of 2 electrons) instead of the particle-hole channel. Thus the mathematical form of (140) is identical to that in (58); with the labelling shown above we have

$$\Gamma_4(12;34) = I_4(12;34) + \int d5\,d5' \int d6\,d6'\, I_4(12;56)\, G(5,5')\, G(66')\, \Gamma_4(5'6';34) \qquad (141)$$

and the Schwinger-Dyson eqtn. for the system is identical to (88), which we repeat here:

$$\left.\begin{aligned} G_4(12;1'2') = {} & G_2(11')\,G_2(2,2') - G_2(12')\,G_2(2,1') \\ & + \int d3\,d3' \int d4\,d4'\, G_2(13)\,G_2(24)\,\Gamma_4(34;3'4')\,G_2(3'1')\,G_2(4'2') \end{aligned}\right\} \qquad (142)$$

Of course the key difference between this theory and the 4-fermion theory must come in the input function, ie., in the irreducible vertex part $I_4(12;34)$. The graphs for this include the following:



(143)

You should look at these and compare them with the graphs exemplifying irreducible vertex parts in the particle-hole channel. They are not the same — irreducibility in the particle-particle channel implies that one has no graphs that can be separated into two parts by cutting a pair of particle lines.

In the history of QED, the Bethe-Salpeter eqtn played an important role, since it allowed a connection with important physical problems (a good example being positronium), which allowed experimental tests of QED. Below I will give a brief discussion of one aspect of this, viz., the appearance of bound states in the B-S eqtn. First, however, lets derive the B-S eqtn. in a different form.

B-S EQTN. IN INTEGRODIFFERENTIAL FORM:

Let's now make the link with the functional formulation of QED that we set up ~~earlier~~ in section B.5.1. Let's return to the Schwinger-Dyson eqtn. (34) for the fermion fields $\psi(x)$ and $\bar{\psi}(x)$, we immediately have (here q is again the charge: $q = -e$ for electrons):

$$\left[(i\gamma^\mu \partial_\mu - m) + q\gamma^\mu \frac{\delta}{\delta J^\mu(x)} - \eta(x) \right] Z[\eta, \bar{\eta}; J] = 0 \tag{144}$$

or, differentiating with respect to $\eta(x)$, and then setting $\eta(x) = \bar{\eta}(x) = 0$, we have

$$\left[(i\gamma^\mu \partial_\mu - m) + q\gamma^\mu \frac{\delta}{\delta J^\mu(x)} \right] Z[J]\, \mathcal{G}_2(x,x'|J) = Z[J]\, \delta(x-x') \tag{145}$$

where $Z[J] \equiv Z[\eta = 0, \bar{\eta} = 0; J]$, and $\mathcal{G}_2(x,x'|J) \equiv \mathcal{G}^{\bar{\psi}\psi}(x,x'|J)$, as before. We can rewrite (145) as

$$\left[(i\gamma^\mu \partial_\mu - m) + q\gamma^\mu \left(iA_\mu(x;[J]) + \frac{\delta}{\delta J^\mu(x)} \right) \right] \mathcal{G}_2(x,x'|J) = \delta(x-x') \tag{146}$$

which is just a differential form of the Dyson eqtn. found in (49), and which we can think of as a generalization of the Dirac/Schrodinger differential eqtn. for free electrons. (NB: we have defined $A_\mu(x;[J])$ here using (38), ie., we have

$$A_\mu(x;[J]) = \frac{-i}{Z[J]} \frac{\delta}{\delta J^\mu(x)} Z[J] \equiv \frac{\delta W[J]}{\delta J^\mu(x)} \tag{147}$$

so that $A_\mu(x;[J])$ is a functional of $J^\mu(x)$).

Now let's try to re-derive the BS eqtn for 2 particles, from this functional form. We could start from (49), but it is easier to start from (145), and further differentiate it with respect to $\eta(x)$. We then get

$$\left.\begin{aligned} &\left[(i\gamma^\mu \partial_\mu - m) + q\gamma^\mu \left(iA_\mu(x;[J]) + \frac{\delta}{\delta J^\mu(x)} \right) \right] \mathcal{G}_4(x,x';y,y'|J) \\ &\quad = \left[\mathcal{G}_2(x',y'|J)\,\delta(x-y) - \mathcal{G}_2(x',y|J)\,\delta(x-y') \right] \end{aligned}\right\} \tag{148}$$

which is a purely differential eqtn.; it relates $\mathcal{G}_4$ to $\mathcal{G}_2$, and so we can think of it as another form for a Schwinger-Dyson eqtn. However it is not yet in a form which is symmetric between the 2 particles – to get it in such a form, we operate on both sides with $\mathcal{G}_2^{-1}$, to produce the differential eqtn:

$$\left(i\gamma^\mu\partial_\mu - m - \Sigma(x,y)\right)\left(i\gamma^\nu\partial'_\nu - m - \Sigma(x'y')\right)\mathcal{G}_4(xx';yy'|J)$$
$$= \Big\{\delta(x-y)\delta(x'-y') - \delta(x-y')\delta(x'-y)$$
$$+ \left(i\gamma^\mu\partial_\mu - m - \Sigma(xy)\right)\left(-iq\gamma^\nu\frac{\delta}{\delta J^\nu(x')} - \Sigma(x'y')\right)\mathcal{G}_4(xx';yy'|J)\Big\} \qquad (149)$$

where we have defined $\partial^\mu = \partial/\partial x_\mu$ and $\partial'_\nu = \partial/\partial x'^\nu$.

We can think of (149) as another (derived) form of a Schwinger-Dyson eqtn, only now it only contains $\mathcal{G}_4$; thus we are much closer to having a BS eqtn, which is an integral eqtn for Γ_4 rather than a differential eqtn. for $\mathcal{G}_4$.

The eqtn. (149) is still not in a symmetric form – the R.H.S. has a product of operators acting on the 2 separate particles that are not symmetric. The reason for this is to do with the dependence of (149) on an external source current $J(x)$, which by necessity acts on one or other of the 2 particles. Thus we can fully symmetrize (149) by adding to it the eqtn we would have if we also operated with $J(x)$ on the 2nd particle; we also then set $J=0$ to produce a eqtn independent of any source, to get:

$$\boxed{\begin{aligned}&\left(i\gamma^\mu\partial_\mu - m - \Sigma(x,y)\right)\left(i\gamma^\nu\partial'_\nu - m - \Sigma(x'y')\right)\mathcal{G}_4(xx';yy'|J)\Big|_{J=0}\\ &\quad= \Big\{\delta(x-y)\delta(x'-y') - \delta(x-y')\delta(x'-y)\\ &\qquad - \Big[\left(i\gamma^\mu\partial_\mu - m - \Sigma(xy)\right)\left(iq\gamma^\nu\frac{\delta}{\delta J^\nu(x')} - \Sigma(x'y')\right)\\ &\qquad\quad + \left(iq\gamma^\mu\frac{\delta}{\delta J^\mu(x)} - \Sigma(xy)\right)\left(i\gamma^\nu\partial'_\nu - m - \Sigma(x'y')\right)\Big]\mathcal{G}_4(xx';yy'|J)\Big|_{J=0}\Big\}\end{aligned}}$$

(150)

Now this is, by any standards, an unwieldy eqtn! However we will not try to use it for anything – I have derived it to show you that one can actually set up a differential eqtn. for $\mathcal{G}_4$ that can be used, if one wishes, to derive non-perturbative results for the 2-particle Green function.

Our original task, however, was to make the connection to the BS eqtn. We can do this in gory detail, but it is simpler to see how it works by developing a symbolic notation. Thus we define

$$\underline{u}_1 \equiv (x,y) \qquad \underline{u}_2 = (x',y') \qquad \underline{z} = (z,z') \qquad (151)$$

for different coordinate pairs, and the operators

$$\left.\begin{aligned} \hat{L}_1(\underline{U}_1) &= i\gamma^\mu\partial_\mu - m - \Sigma(x,y) \\ \hat{L}_1(\underline{U}_2) &= i\gamma^\mu\partial'_\mu - m - \Sigma(x'y') \\ L_{12}(\underline{U}_1,\underline{U}_2) &= \hat{L}_1(\underline{U}_1)L_1(\underline{U}_2) + V_{12}(\underline{U}_1,\underline{U}_2) \end{aligned}\right\} \quad (152)$$

where we see that $\hat{L}_1$ is just the inverse operator of $\hat{\mathcal{G}}_2$, ie., symbolically we have $\hat{L}_1\hat{\mathcal{G}}_2 = \hat{I}$; and where we can rewrite (150) as

$$\hat{L}_1\hat{L}_1\,\mathcal{G}_4(U_1,U_2) = \hat{L}_1\hat{L}_1\,\mathcal{G}_2(\underline{U}_1)\mathcal{G}_2(\underline{U}_2) + \hat{V}_{12}\,\mathcal{G}_4(\underline{U}_1\underline{U}_2) \quad (153)$$

from which the definition of $\hat{V}_{12}$ is obvious from comparison with (150). We can write this in even more abbreviated form as

$$\hat{L}_{12}\,\mathcal{G}_4(\underline{U}_1,\underline{U}_2) = \hat{L}_1\hat{L}_1\,\mathcal{G}_2(\underline{U}_1)\mathcal{G}_2(\underline{U}_2) \quad (154)$$

Now let's rewrite this in a slightly different way:

$$\left.\begin{aligned} L_1(\underline{U}_1)L_1(\underline{U}_2)\,\mathcal{G}_4(\underline{U}_1\underline{U}_2) &= L_1(\underline{U}_1)L_1(\underline{U}_2)\,\mathcal{G}_2(\underline{U}_1)\mathcal{G}_2(\underline{U}_2) \\ &\quad + \int d\underline{Z}\; I_4(\underline{U}_1,Z)\,\mathcal{G}(\underline{Z},U_2) \end{aligned}\right\} \quad (155)$$

or, written out in full

$$\begin{aligned} \hat{L}_1(x,y)L_1(x'y')\,\mathcal{G}_4(xx',yy') = \Big\{&\big[\hat{L}_1(xy)\,\mathcal{G}_2(xy)\,L_1(x'y')\,\mathcal{G}_2(x'y') \\ &- L_1(xy')\,\mathcal{G}_2(xy')L_1(x'y)\,\mathcal{G}_2(x'y)\big] \\ &+ \int dz\,dz'\; I_4(xx';zz')\,\mathcal{G}_4(zz';yy')\Big\} \end{aligned} \quad (156)$$

The identification of the quantity I_4 in this equation comes with the comparison of the structure of (156) with that of the BS eqtn – to see this look at eqtns (89) and (86).

(ii) B-S EQTN for $\Lambda(x_1x_2;y)$:

It is important to see how the singular structure we saw in studying the BS eqtn. for Fermi liquids, and which we will also see presently in the BS eqtn for QED, also exists in the 3-point vertex. To see this we now develop an analogous BS eqtn. for $\Lambda_3(x,x';y)$, for QED.

Since we have already seen the derivation for BS eqtns for Γ_4 done in several different ways, I will not go into much detail here, but simply derive the eqtn. graphically – yet again this will show the power of diagrammatic methods. We are interested in two functions, viz., the 3-point propagator $\mathcal{G}_3^{\bar\psi\psi A}(x_1x_2x_3)$, and the associated 3-point vertex $\Lambda^{\bar\psi\psi A}(x_1x_2x_3)$. Let us simplify things by assuming we deal with a Lorentz-invariant system (ie., there is no background Fermi sea to fix a reference frame), so that we can then do a 4-dimensional Fourier transform, according to

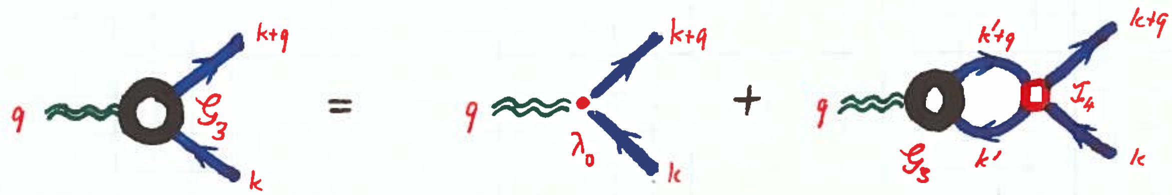
$$\mathcal{G}_3(k,q) = \int d^4x_1 \int d^4x_2 \int d^4x_3 \; \mathcal{G}_3(x_1,x_2,x_3)\, e^{i(k+q)\cdot(x_1-x_3)}\, e^{-ik\cdot(x_2-x_3)} \qquad (157)$$

and likewise for $\Lambda(k,q)$.

Then it should be fairly clear by now that we can write the following integral eqtn. for $\mathcal{G}_3(k,q)$:

$$\mathcal{G}_3(k,q) = \mathcal{G}_2(k+q)\,\mathcal{G}_2(k)\Big[\lambda_0 + \sum_{k'} \mathcal{G}_3(k',q)\,\mathcal{I}_4(k,k';q)\Big] \qquad (158)$$

which has the diagrammatic interpretation shown in (159):



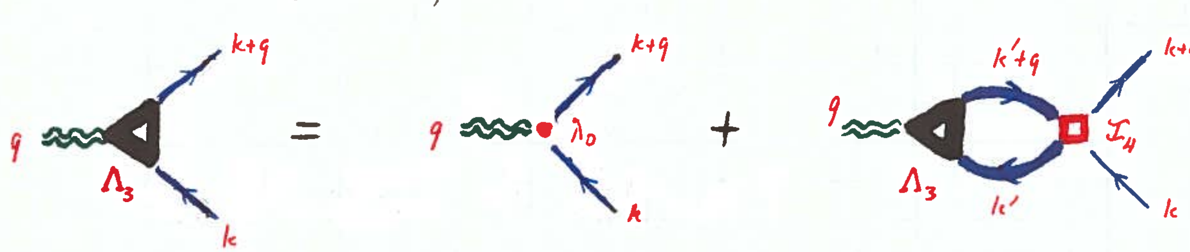
(159)

In these eqtns, the "bare coupling" λ_0 is just that appropriate to QED, ie., $\lambda_0 = iq$ (for the moment I am suppressing all γ-matrices and 4-momentum indices, to keep the clutter to a minimum).

In the same way we can write an integral eqtn. for $\Lambda(k,q)$, as

$$\Lambda_3(k,q) = \lambda_0 + \sum_{k'} \Lambda_3(k',q)\,\mathcal{G}_2(k'+q)\,\mathcal{G}_2(k')\,\mathcal{I}_4(k,k';q) \qquad (160)$$

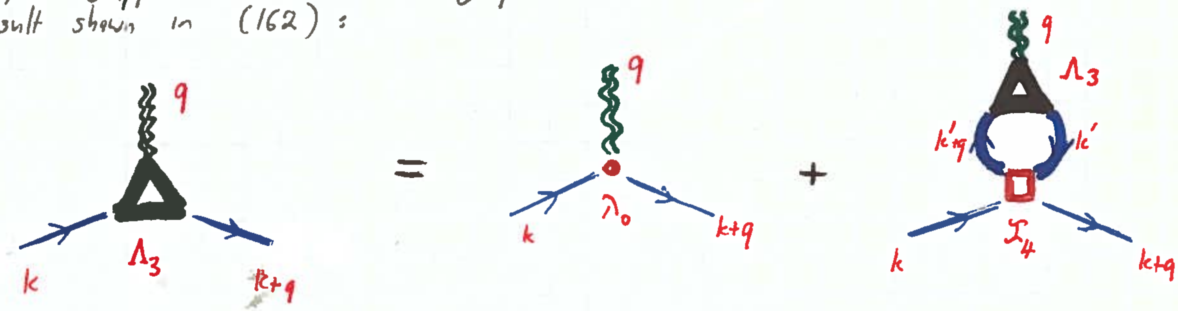
which we show diagrammatically as (161):



(161)

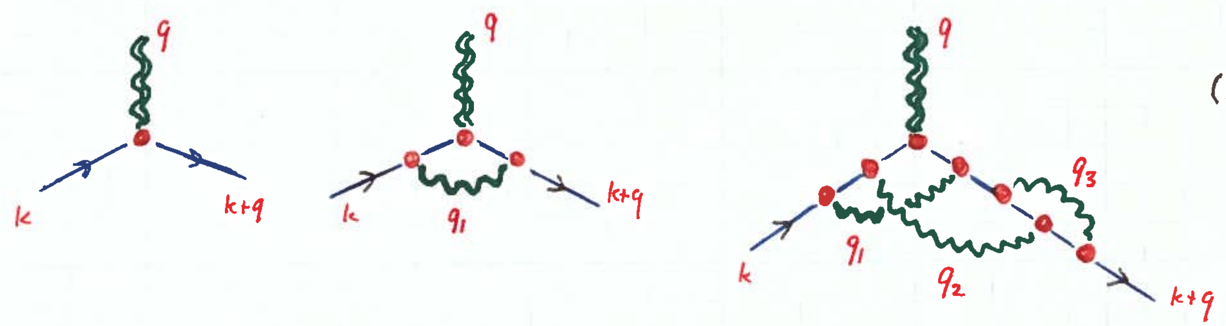
The physical interpretation of these graphs is as follows. We can imagine a photon coming in from the left, and then disintegrating into an electron-positron pair. This pair can either proceed on its way, or rescatter off each other in an arbitrary way – described by $\mathcal{I}_4$ – an arbitrary number of times, with independent propagation of the pair in between the scattering processes. Note that some of the graphs for $\mathcal{I}_4$ were shown in (143) above — this will allow you to construct some graphs for Λ_3 and $\mathcal{G}_3$.

We can also interpret these results for $\mathcal{G}_3$ and Λ_3 in a different way. Suppose we rotate the graphs in (161) on their side, to get the result shown in (162):



(162)

We see that we are now free to interpret this relation as a description of the scattering of an electron off a photon. The electron can either scatter once, or during the entire scattering processes, interactions of a retarded nature can occur between electron states before it scatters off the external photon, and electron states after this scattering event. Moreover, these retarded processes are themselves mediated by photons. If we show a few graphs this becomes clear:



(163)

What (163) shows us is that the complete interaction between an external photon and an electron must also include "vertex corrections" to the bare vertex λ_0, caused by photon exchange occuring in the vicinity of the bare scattering event. In these graphs I only show bare photons, but obviously these photons themselves can be "renormalized", ie, they will include all sorts of internal processes, of the kind shown in eqtn (47) and (48).

To be complete, let us now insert all the 4-momentum indices and γ-matrices into these eqtns. This is simple; eqtn (160) becomes

$$\Lambda_3^\mu(k,q) = \lambda_0^\mu + \sum_{k'} \Lambda_3^\mu(k'q)\, \mathcal{G}_2(k'+q)\, \mathcal{G}_2(k')\, \mathcal{I}_4(kk'q) \qquad (164)$$

with, as before,

$$\lambda_0^\mu = ie\gamma^\mu \qquad (165)$$

The other eqtns can be fixed up in the same way.

(iii) BOUND STATES IN Q.E.D. : POSITRONIUM

: It is obvious physically that the BS eqtn. for a electron-positron pair must be able to describe the "bound state" sector of this system, ie., the states in which the effective photon-mediated interaction between them produces a bound pair. To extract this physics, let's start from the BS eqtn in the integral form given in (141).

Now, as with any integral eqtn which corresponds to some eigenvalue problem, the eigenstates will show up as poles of the integral eqtn; we have already seen this in section A, when studying the analytic properties of both $G(2,1)$ and of the scattering T-matrix. However, the poles arise because of the repeated scattering of the electron-positron pair — they cannot be in I_4, because this term does not contain this scattering. The role of I_4 is to produce an attractive interaction; but to get the bound states, we need the singular structure coming from the product $\mathcal{G}_2\mathcal{G}_2$. The divergences in $\mathcal{G}_2\mathcal{G}_2$ then create the poles.

Once we have realized this, our job becomes a lot easier, because as we approach the poles, we can make the approximation

$$\Gamma_4(12;34) \sim \int d5\, d5' \int d6\, d6'\, I_4(12;56)\, \mathcal{G}_2(5,5')\, \mathcal{G}_2(6,6')\, \Gamma_4(5'6';34) \qquad (166)$$

since the inhomogeneous term $I_4(1234)$ remains finite. To proceed further let's assume Lorentz invariance and a surrounding vacuum; this allows us to Fourier transform.

Now the last time we did this Fourier transform was in going from (58) to (60). Before doing the Fourier transform again, let's look a little more at the structure of the BS eqtn., and see the different ways in which we can carry out this Fourier transform. The point is that there are 3 independent 4-momenta in the vertices Γ_4 and I_4, and we can combine them in various ways. There are 3 common ways of doing this, depending on which of the 3 momenta is small — these are as follows:

(a) <u>Direct or "Cooper" channel</u>: The singular behaviour in the $\mathcal{G}_2\mathcal{G}_2$ intermediate states comes when $|p+p'| \to 0$; we have the BS eqtn:

$$\Gamma_4(pp';q) = I_4(pp';q) + \sum_k I_4(pp';k-p)\, \mathcal{G}_2(k)\, \mathcal{G}_2(p+p'-k)\, \Gamma_4(p+p'-k;pp') \qquad (167)$$

with a pinch singularity in $\mathcal{G}_2\mathcal{G}_2$ as $|p+p'| \to 0$.

(b) <u>Crossed or "Peierls" channel</u>: Now we pick out singular behaviour when the difference $|p_1-p_2| \to 0$; this yields the BS eqtn:

$$\Gamma_4(pp';q) = I_4(pp';q) + \sum_k I_4(p;k;k-p+q)\, \mathcal{G}_2(k)\, \mathcal{G}_2(p-p'+q+k)\, \Gamma_4(p+p'+q+k;pp') \qquad (168)$$

which now has a pinch singularity in the $\mathcal{G}_2\mathcal{G}_2$ intermediate state as $|p-p'| \to 0$.

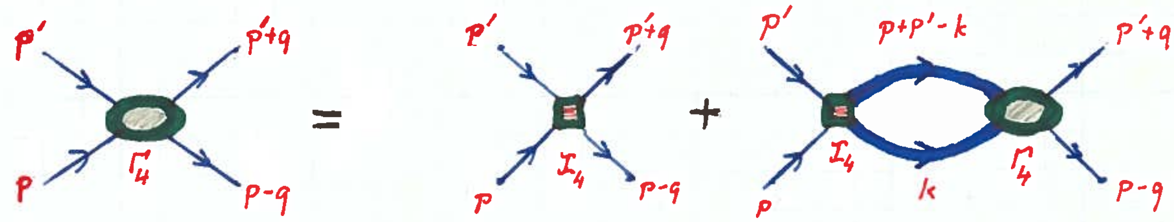
(c) <u>Particle-Hole or "Zero sound" or "Landau" channel</u> : This is the case we already looked at before, in discussing Fermi liquids — the singular behaviour comes when the momentum transfer $q \to 0$, to give the BS eqtn:

$$\Gamma_4(pp';q) = I_4(pp';q) + \sum_k I_4(pk;q)\,\mathcal{G}_2(k)\,\mathcal{G}_2(k+q)\,\Gamma_4(k;p'q) \qquad (169)$$

with the pinch singularity as $q \to 0$. The only difference between this eqtn. and (60) is that here we are relativistic, and we have no Fermi sea.

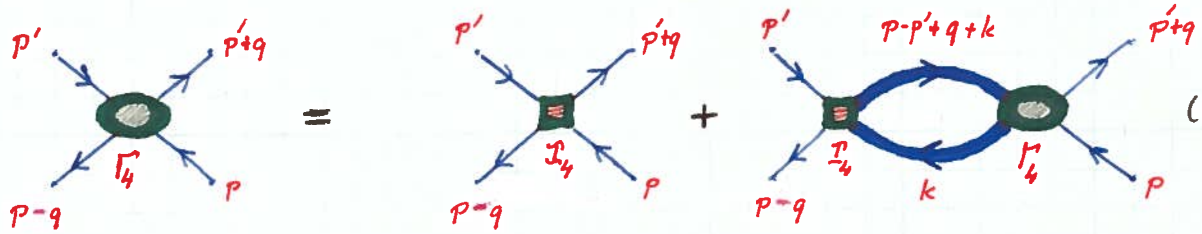
It is helpful to see these eqtns. written diagrammatically, in the figure:

(a) Direct / Cooper channel:



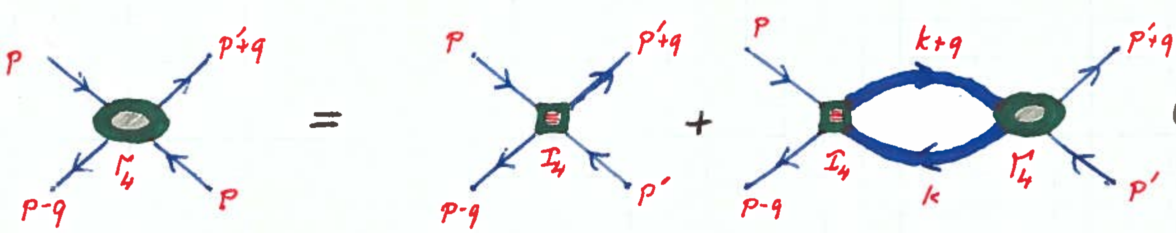
(170)

(b) Crossed / Peierls channel:



(171)

(C) Particle-Hole / Zero Sound / Landau channel



(172)

A few comments on these eqtns are useful. The direct channel is the one we are looking at when we are interested in the scattering of 2 particles. The name "Cooper channel" comes from its importance in the theory of superconductivity, in the presence of a background Fermi sea — it leads to the Cooper instability. The Peierls channel, unlike the Cooper channel which is singular when the sum of the 4-momenta is zero, is instead singular when their DIFFERENCE $p-p'$ goes to zero. It got its name when Peierls showed its importance for structural instabilities in solids. Finally, the Landau or "zero sound" channel comes from Fermi liquid theory, as we have already seen above; it is singular when the momentum transfer $q \to 0$.

Note that in all 3 cases, we deal with the same process, viz., 2 particles with incoming momenta p, p', scattering to outgoing states $|p-q\rangle$, $|p'+q\rangle$, with momentum transfer q. The differences arise in the intermediate states; the diagrams are drawn so that the intermediate states leading to poles/singularities in $\Gamma_4(pp';q)$ are separated from all other non-singular processes, which then contribute to the relevant irreducible vertices I_4. Thus, in (167), I_4 is irreducible in the Cooper channel; in (168), I_4 is irreducible in the Peierls channel; and in (169), it is irreducible in the Landau channel.

For notational completeness, we should specify which irreducible part we are dealing with in any of these eqtns; thus, eg., we should write (167) as

$$\Gamma_4 = I_4^C + \sum I_4^C (\mathcal{G}_2 \mathcal{G}_2)^C \Gamma_4 \qquad (173)$$

but normally we won't do this, since it will be obvious from the context.

Let us now return to the positronium problem. Physically it will be obvious that bound states of the electron-positron pair will result if the 2 particles move together in space, interacting repeatedly via photons. We are then led to look at the Cooper channel. From the diagram in (170), we immediately read off the expression

$$\Gamma_4(pp';q) = I_4(pp') + \sum_k I_4(p,k)\, \mathcal{G}_2(k)\, \mathcal{G}_2(k-(p+p'))\, \Gamma_4(k, p+p'-k; q+k-p) \qquad (174)$$

where we have simplified the notation a little; since $I_4(pp')$ has no important dependence on the momentum transfer, we suppress its dependence on this variable.

Now we wish to focus on the poles of this expression; around these poles, Γ_4 diverges, because of the singularity in $\mathcal{G}_2\mathcal{G}_2$, but I_4 remains finite; we therefore have

$$\Gamma_4(p,p') \sim \sum_k I_4(pk)\, \mathcal{G}_2(k)\, \mathcal{G}_2(k-(p+p'))\, \Gamma_4(k, p+p'-k) \qquad (175)$$

where we have also now dropped the dependence of Γ_4 on the momentum transfer q, since this is also unimportant near the poles. We can further simplify (175) by defining

$$\Psi(pp') = \mathcal{G}_2(p)\, \Gamma_4(pp')\, \mathcal{G}_2(p') \qquad (176)$$

which will turn out to be a kind of "pair wave-function" for the particle pair. We can then write the eigenvalue eqtn. (176) as

$$\Psi_{pp'} = \mathcal{G}_2(p)\left[\sum_k \mathcal{I}_4(p,k)\,\Psi_{k,k-(p+p')}\right]\mathcal{G}_2(p') \qquad (177)$$

To proceed further in the analysis we need to say more about the forms of both $\mathcal{G}_2(p)$ and $\mathcal{I}_4(p,p')$. In the discussion of Fermi liquids we relied heavily on the existence of the Fermi surface, and used the fact that all the interesting physics happened in the vicinity of this surface. In the present case we are working in a vacuum, and this changes things significantly.

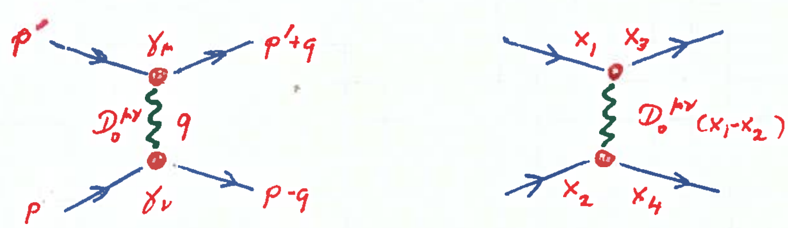
There are actually 2 interesting limiting cases here that are illuminating. The first is the low-energy limit, where the problem reduces to a non-relativistic one. The second is the very high-energy regime, where we can use an "eikonal" technique – this will be discussed in the chapter on non-perturbative methods. Here we focus on the non-relativistic regime.

Notice first that because the coupling constant $\alpha \sim 1/137$ in QED is small, we can at low E simply use the lowest order graph for $\mathcal{I}_4(p,p')$, ie., we write (now letting the charge $q = e$):

$$\mathcal{I}_4(p,p';q) \sim -e^2\gamma_\mu D_0^{\mu\nu}(q)\gamma_\nu \qquad (178)$$

$$\mathcal{I}_4(x_1x_2;x_3x_4) \sim -e^2\gamma_\mu D_0^{\mu\nu}(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_2-x_4) \qquad (179)$$

where the graph (which is just the lowest order graph in (143)) is labelled according to the figure shown in (180).



(180)

If we now take the non-relativistic limit of (178), we just get the Coulomb interaction:

$$\mathcal{I}_4(pp';q) \xrightarrow[v/c \ll 1]{} -e^2\gamma_0 D_0^{00}(q)\gamma_0 = V(\underline{q})\gamma^0\gamma^0 \qquad (181)$$

where

$$U(\underline{q}) = -4\pi\, e^2/|\underline{q}|^2 \qquad (182)$$

is just the unscreened Coulomb interaction. Our eigenvalue eqtn. for the wave-fn. $\Psi_{pp'}$ now reads:

$$\Psi_{pp'} = \mathcal{G}_2(p)\,\gamma_0\left[\sum_{\underline{k}}\int\frac{dk_0}{2\pi}\,V(\underline{p}-\underline{k})\,\Psi_{k,k-(p+p')}\right]\gamma_0\,\mathcal{G}_2(p') \qquad (183)$$

where the internal momentum transfer (a 3-momentum transfer) now appears in $V(\underline{p}-\underline{k})$; we have separated out the 3-momentum integral $\sum_{\underline{k}} = \int d^3k/(2\pi)^3$ from the energy integral $\int dk_0/2\pi$, in this non-relativistic regime.

In this non-relativistic regime we can also assume that $\mathcal{G}_2$ propagators take the free-particle form, with the (renormalised) physical mass m, ie we have

$$\left.\begin{aligned} \mathcal{G}_2(p) \;\sim\; \frac{i}{\gamma^\mu p_\mu - m + i\delta} &\equiv \frac{\gamma_0 p^0 - \underline{\gamma}\cdot\underline{p} + m}{(p_0+m)(p_0+m) - |\underline{p}|^2 + i\delta} \\ &\xrightarrow[|p| \ll m]{} \tfrac{1}{2}(1+\gamma_0)\,\frac{1}{p_0\; m - |\underline{p}|^2/2m + i\delta} \end{aligned}\right\} \qquad (184)$$

where in the last form we explicitly use the assumption that $v \ll c$.

There are various ways we can label the momenta in this calculation. To make it clear we are looking at electron-positron scattering, we can also write, in the centre of mass frame where $\underline{p}+\underline{p}' = 0$, our variables as

$$\left.\begin{aligned} p &= (\underline{p}, \epsilon) \equiv (\underline{p}, m+\varepsilon) \\ p' &= (-\underline{p}, \epsilon') \equiv (-\underline{p}, m+\varepsilon') \end{aligned}\right\} \qquad (185)$$

so that

$$\left.\begin{aligned} p+p' &= (0, 2m+(\varepsilon+\varepsilon')) \equiv (0, 2m+E) \\ p-p' &= (2\underline{p}, \varepsilon-\varepsilon') \equiv (2\underline{p}, 2\omega) \end{aligned}\right\} \qquad (186)$$

so that the "centre of mass" variable $p+p'$ has zero 3-momentum, and energy E above the lowest possible energy $2m$.

We can also simplify the factor $R_0(p,p') = \mathcal{G}_2(p)\,\gamma_0^2\,\mathcal{G}_2(p')$, using these variables, as

$$\left.\begin{aligned} R_0(p,p') &= \frac{1}{p_0\;m - |\underline{p}|^2/2m + i\delta}\;\frac{1}{p_0' - m - |\underline{p}'|^2/2m + i\delta} \\ &= \frac{1}{\varepsilon - |\underline{p}|^2/2m + i\delta}\;\frac{1}{\varepsilon' - |\underline{p}|^2/2m + i\delta} \\ &= -\,\frac{1}{\omega + \frac{1}{2}(E - |\underline{p}|^2/m) + i\delta}\;\frac{1}{\omega - \frac{1}{2}(E - |\underline{p}|^2/m) - i\delta} \end{aligned}\right\} \qquad (187)$$

so that $R_0(p,p') \rightarrow R_0(\underline{p}, E; \omega)$, a function only of the relative momentum $\underline{p}$, the c.o.m. energy E, and the relative energy ω.

In the same variables our bound-state eqtn. (184) now simplifies to:

$$\Psi_{\underline{p}}(E;\omega) = R_{\underline{p}}(E;\omega) \sum_{\underline{k}} \int \frac{dk_0}{2\pi} V(\underline{p}-\underline{k}) \Psi_{\underline{k}}(E;k_0) \qquad (188)$$

which is our BS eqtn for the bound states in the non-relativistic regime. We now observe that in these eqtns, we can integrate out the extra frequency in the wave-function $\Psi_{\underline{p}}(E;\omega)$, ie., write

$$\psi_{\underline{p}}(E) = \int \frac{d\omega}{2\pi} \Psi_{\underline{p}}(E;\omega) \qquad (189)$$

and likewise for $\Psi_{\underline{k}}(E;k_0)$; integrating over ω on both sides of (188), closing the integral in the upper half-plane, and also doing the integral over k_0, we get

$$|\underline{P}|^2/2m \; \psi_{\underline{p}}(E) + \sum_{\underline{k}} V(\underline{p}-\underline{k}) \psi_{\underline{k}}(E) = E \, \psi_{\underline{p}}(E) \qquad (190)$$

which is a simple Schrodinger eqtn for positronium; if we Fourier transform back to real space we just have

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\underline{r}) \right] \psi(\underline{r}, E) = E \, \psi(\underline{r}, E) \qquad (191)$$

What is of course interesting in QED is to go beyond this non-relativistic result. This is done by (i) moving away from the limit $V/c \ll 1$, and (ii) calculating the higher diagrams for the scattering function $\mathcal{I}(pp'/q)$. Doing such calculations, and comparing them with experiment, was an important activity in the 1950's and early 1960's. In the case of positronium, these corrections cause a "Lamb shift" of the energy levels, which can be measured to better than 10^{-12} accuracy. Another way of testing QED is to measure the electronic g-factor. The g-factor characterizes the interaction between photons and electrons, and is thus calculated by summing diagrams like those in (163) for the 3-point electron-photon vertex shown in (162).

B.5.3 (b) PHONON-MEDIATED ELECTRON PAIRING

As we already know, the electron-phonon system looks quite similar to QED, because the structure of the diagrams and of the bosonic propagators involved are similar (apart from the coupling functions).

In what follows we will not look at the full electron-phonon problem, which is very complicated (because of the angular structure imposed by any crystal lattice, and because of the different phonon branches). We will instead look at the simplified model for phonons and electron-phonon coupling introduced in section B.3.5, in which we have only longitudinal acoustic phonons, coupled to electrons in a spherical Fermi sea by an isotropic coupling constant. The diagram rules are given in section B.3.5., and a few graphs are worked out in Appendix B.3.1(c).

What we will look at is a classic problem from 20th-century physics, viz., the "BCS" problem of the instability of this simple electron-phonon system to the formation of a superconducting state. The history is a little more complicated than this; the first key theoretical step was taken by Bogoliubov in 1947, who showed how to quantize a bosonic field for N interacting bosons, and how to isolate the "dangerous diagrams" which led to superfluidity and a Bose condensate; in 1950 Fröhlich then produced a simpler field theory for the electron-phonon system. In 1956 Cooper produced a very simple argument to show how electron pairing via, for example, electron-phonon coupling, could lead to a "pairing instability" — and in 1957 BCS (Bardeen, Cooper, & Schrieffer), and almost simultaneously Bogoliubov, gave a theory for the resulting superconducting state. This theory was put on a more satisfactory basis later, by Nambu & by Bogoliubov et al., in a full field-theoretical treatment. Somewhat later, Larkin & Leggett, and also Migdal, discussed the generalization to pairing interactions having their origin in other excitations (ie., not phonons), including excitations of the same fermions that undergo the instability.

(i) RENORMALIZED SELF-ENERGY & DYSON EQTN

: We have already seen how one can go from bare quantities like $G_0(p,z)$ or bare vertices to fully renormalized quantities, when discussing Fermi liquids. We can do the same in a fairly simple way for the electron-phonon problem as well. This is a good warm-up for discussing renormalization in QED, which we will do properly later on. The key difference between the 2 cases is that QED has UV divergences, and so the renormalizations are infinite.

We begin by recalling the form of the electron self-energy in lowest-order perturbation theory; this was derived at $T=0$ in section B.3.5(b), eqtn (142) et seq., and at finite T in Appendix B.3. The result is

$$\Sigma_{\underline{p}}^{(0)}(z) = \lambda_0^2 \sum_q \left[\frac{f_{p-q} + n_q}{z - \epsilon^0_{\underline{p}-\underline{q}} + \omega_q} + \frac{1 - f_{p-q} + n_q}{z - \epsilon^0_{p-q} - \omega_q} \right] \qquad (192)$$

So how does this expression compare with the self-energy for a real system? There are essentially 4 issues that one can deal with, viz.,

(i) Vertex corrections (ie., corrections to λ_0)
(ii) Dressing of internal lines (ie., corrections to ϵ^0_{p-q} and ω_q)
(iii) Anisotropy of real systems
(iv) Corrections coming from electron-electron interactions.

Without giving anything like a complete discussion, we will say something about each of these issues. The first two issues have an exact parallel in QED, and you may find it useful to look again at the earlier parts of this section B.5, notably eqtns (46)-(52) and (160)-(165).

We already briefly discussed vertex corrections to λ_0 in section B.3 (see B.3, eqtn (150)). The essential content of "Migdal's theorem" is that the higher-order correction to λ_0 given by the lowest-order vertex correction, evaluated in appendix B.3, is very small, because the phonon velocity $c_0 \ll v_F$, the electronic Fermi velocity. We shall see, in discussing the B-S integral eqtn for $\Lambda_3(k,q)$ below, that Migdal's theorem is actually misleading, because when we iterate graphs in the Cooper channel, we find a pole. However, let us ignore this for the moment, and look now at items (ii) and (iii) above.

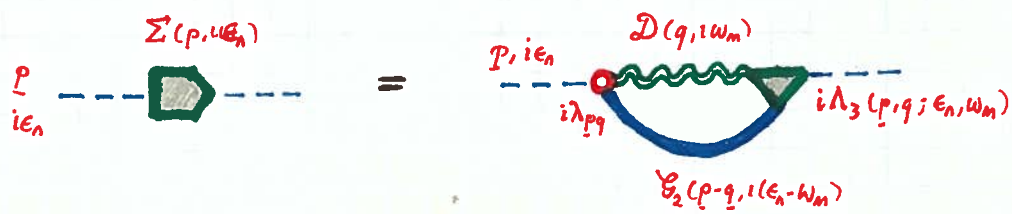
To do this, let's first assume that the <u>exact</u> forms for the fully renormalized phonon and electron lines can be written in spectral representation as

$$\left.\begin{aligned} \mathcal{G}_2(\underline{p}, z) &= \int_{-\infty}^{\infty} \frac{d\epsilon'}{\pi} \frac{A_p(\epsilon')}{z - \epsilon' + i\delta_{\epsilon'}} \\ \mathcal{D}(\underline{q}, z) &= \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{B_q(\omega')}{z - \omega' + i\delta_{\omega'}} \end{aligned}\right\} \tag{193}$$

(compare eqtn (102)), so that if we write the exact eqtn.

$$\Sigma(\underline{p}, i\epsilon_n) = -\frac{1}{\beta}\sum_m \sum_q \lambda_{\underline{pq}}\, \mathcal{G}_2(\underline{p}-\underline{q}, i(\epsilon_n - \omega_m))\, \mathcal{D}(q, i\omega_m)\, \Lambda_3(\underline{p}, \underline{q}; \epsilon_n, \omega_m) \tag{194}$$

illustrated in the figure (195):



(195)

then we can rewrite this as

$$\left.\begin{aligned} \Sigma_{\underline{p}}(z) = \sum_{\underline{q}} \lambda_{\underline{pq}} \int \frac{d\epsilon'}{2\pi} \int \frac{d\omega'}{2\pi}\, A_{\underline{p}-\underline{q}}(\epsilon')\, B_{\underline{q}}(\omega')\, \Lambda_3(\underline{p}, \underline{q}; \epsilon', \omega') \\ \times \left[\frac{f(\epsilon') + n(\omega')}{z - \epsilon' + \omega'} + \frac{1 - f(\epsilon') + n(\omega')}{z - \epsilon' - \omega'} \right] \end{aligned}\right\} \tag{196}$$

In these eqtns we have generalized the bare structureless vertex to a vertex $\lambda_{\underline{pq}}$ which depends on the electronic and phonon momenta (in a way which depends on the lattice structure); and we have defined the full 3-point vertex $\Lambda_3(p-q, \epsilon', \omega')$, in a way exactly analogous to that done for QED (see eqtns (50)-(52) of this section).

Now we can use this formal result if we add in a little physics of the kind we have already discussed. First, we already know that the sound velocity $c \ll v_F$, the Fermi velocity - this is the basis for Migdal's theorem. We shall soon establish that the only effect of all the interactions (electron-phonon, electron-electron, etc.) is to renormalize the sound velocity, without affecting the phonon propagator's form at low energy (this is physically clear: sound propagates at low frequencies at a specific velocity, with almost no attenuation).

Second, we have the experience of our arguments with Fermi liquids, from sections B.5.2(a) and B.5.2(b), showing that near the Fermi surface, the only effect of interactions is to renormalize the electrons to electron quasiparticles, and give these quasiparticles a weak decay. Does this sort of argument still work with phonons?

The argument here will proceed in 2 steps. First we will assume that we can assume a weak decay (the "quasiparticle assumption"). We will then see what we get, and verify that the assumption is self-consistent (and indeed is correct).

To do our preliminary analysis we will actually assume Migdal's theorem is correct, and ignore the possibility of a superconducting instability - this means we will replace $\Lambda_3(p,q;\epsilon',\omega')$ in (196) by λ_{pq}, and return later to a full examination of $\Lambda_3(p,q,\epsilon',\omega')$. We will also assume, what is to be verified later, that we can simply use a renormalized propagator for the phonons, and that the Fermi liquid form for G_2 is valid. We can then replace (196) by

$$\left.\begin{aligned} \Sigma_p(z) \sim \sum_q |\lambda_{pq}|^2 \int \frac{d\epsilon'}{2\pi} \int \frac{d\omega'}{2\pi}\, 2\pi Z_{p-q}\, \delta(\epsilon' - \varepsilon_{p-q})\, 2\pi \delta(\omega' - \omega_q) \\ \times \left[\frac{f(\epsilon') + n(\omega')}{z - \epsilon' + \omega'} + \frac{1 - f(\epsilon') + n(\omega')}{z - \epsilon' - \omega'} \right] \end{aligned}\right\} \quad (197)$$

where the quasiparticle renormalization for phonons is assumed to be unity (we explain this below), and ε_p and ω_q are the renormalized quasiparticle energies. We also assume that

$$Z_{p-q} \sim Z_p \sim Z_F \simeq \frac{1}{1 - \partial \Sigma_p(\epsilon)/\partial \epsilon}\bigg|_{\substack{p \to p_F \\ \epsilon \to 0}} \quad (198)$$

(compare (107), noting that we have put $\mu = 0$ here); equivalently, we have $\varepsilon_p = \epsilon_p / Z_F$.

We now see we can write

$$\Sigma_p(z) \sim \frac{1}{Z_F} \sum_q |\lambda_{pq}|^2 \left[\frac{\tilde{f}_{p-q} + \tilde{n}_q}{z - \varepsilon_{p-q} + \omega_q} + \frac{1 - \tilde{f}_{p-q} + \tilde{n}_q}{z - \varepsilon_{p-q} - \omega_q} \right] \quad (199)$$

Now in the case of the electron-phonon interaction there is a standard way of rewriting this. We note first that the phonon energies range over $0 \le \omega_q \le \Theta_D$, where Θ_D is the "Debye" energy, while the momenta range over $0 \le |q| \lesssim 2k_F$. Now $\Theta_D \ll \epsilon_F$, so the quantities in (199) vary much more slowly with momentum than with energy — to a good approximation we can ignore the momentum dependence in the electronic quantities, and also factorize the energy and momentum dependence in (199). Defining $N^*(0) = d\epsilon/d|p|$ as before (cf. eqtn. (99)), we then immediately get

$$\Sigma_p(\epsilon+i\delta) \sim \int d\epsilon' \int d\omega' \, \alpha^2(\omega')F(\omega') \left[\frac{f(\epsilon') + n(\omega')}{\epsilon - \epsilon' + \omega' + i\delta} + \frac{1 - f(\epsilon') + n(\omega')}{\epsilon - \epsilon' - \omega' + i\delta} \right] \qquad (200)$$

where we have defined the "$\alpha^2 F$" factor as

$$\alpha^2(\omega')F(\omega') = N^*(0) \int \frac{d\Omega_{p-q}}{4\pi} |\lambda_{p-q}|^2 \, \delta(\omega' - \omega_q) \qquad (201)$$

and $\int d\Omega_k$ is a surface integral in momentum space, around the Fermi surface. The reason for this notation is that the energy integration in (200) now looks like a 2nd-order perturbative "Fermi Golden rule" integration, with a convolution of a "density of states" $F(\omega) = \sum_q \delta(\omega - \omega_q)$ and a "coupling constant squared" $\alpha^2(\omega)$.

The quasiparticle properties near S_F may now be extracted from (199) or (200). Thus we have

$$\left.\begin{aligned} Z_F &= 1 - \left.\frac{\partial \Sigma(\epsilon)}{\partial \epsilon}\right|_{\epsilon \to 0} \\ &\underset{\epsilon \to 0}{\longrightarrow} - \int d\epsilon' \int d\omega' \, \alpha^2(\omega)F(\omega) \, \frac{\partial}{\partial \epsilon'} \left[\frac{1}{\epsilon' + \omega' - i\delta} + \frac{1}{\epsilon' + \omega' + i\delta} \right] \end{aligned}\right\} \qquad (202)$$

where, since we go to the Fermi surface, we can drop $n(\omega')$ and let the Fermi functions assume their $T=0$ Θ-function forms. We then get

$$\left.\begin{aligned} Z_F &= \frac{m}{m^*} = 1 + \lambda \\ &= 1 + 2\int_0^\infty \frac{d\omega}{\omega} \alpha^2(\omega) F(\omega) \end{aligned}\right\} \qquad (203)$$

where we see that the effective mass increases above m, as we increase λ; in (203) we have neglected the slow dependence of m^* on momentum (compare (100)).

In real metals λ is not small; even for simple metals it is $\sim 0.2-0.5$, and for more strongly-coupled metals like Hg, $\lambda \sim O(1)$. Thus mass renormalizations can be strong.

If we now go back to the full form in (200), we can determine the imaginary part of the self-energy, and hence check the assumption of quasiparticle behaviour. From (200) we have

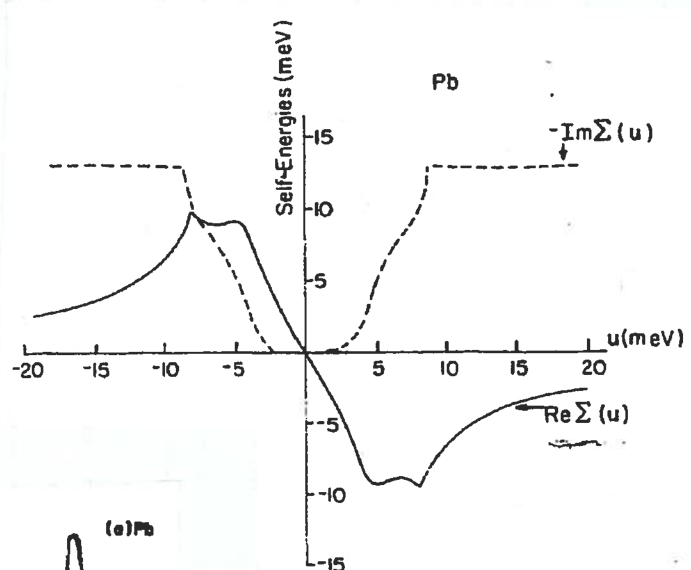
$$\mathrm{Im}\, \Sigma_p(\epsilon + i\delta) = -\pi \int d\omega' \alpha^2(\omega') F(\omega') \left[f(\epsilon+\omega') + f(\omega'-\epsilon) + 2n(\omega') \right] \quad (204)$$

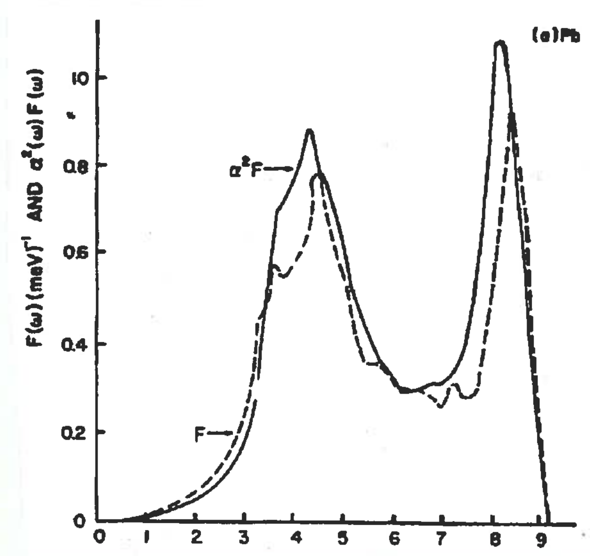
Now we can estimate this result for low ϵ and low T, such that $\epsilon/\Theta_D \ll 1$, and $kT/\Theta_D \ll 1$. Then the coupling $\alpha^2(\omega)F(\omega)$ depends essentially on the density of phonon states, and this is $\propto \omega^2/\Theta_D^2$, in the simple Debye model. Since the number of electron states with energy such that $|\epsilon| < kT$ is $\propto kT$, and likewise the number of electron states in the range $|\epsilon| < \omega$ is $\propto \omega$, we expect $\mathrm{Im}\, \Sigma_p(\epsilon) \sim \epsilon^3$ for $kT = 0$, and $\sim (kT)^3$ for $\epsilon = 0$. In general one actually finds

$$\mathrm{Im}\, \Sigma_p(\epsilon) \sim \frac{\pi}{2} \lambda \left[(\epsilon/\Theta_D)^3 + \pi^2 (kT/\Theta_D)^3 \right] \quad (205)$$

so that, at least for $\epsilon, kT \ll \Theta_D$, the quasiparticle assumption appears to be good. In the figure we show computed results for the Real and Imaginary parts of $\Sigma_p(\epsilon)$ for the well-understood metal (Pb).

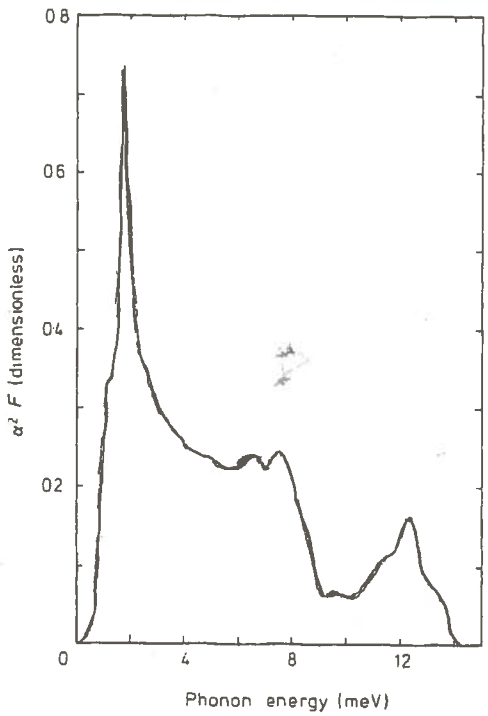
At right we see the Real & Imaginary parts of $\Sigma_{p_F}(u)$, where u is the energy in mV units ($N.B.$: $1\,\mathrm{meV} \sim 11.6\,^\circ\mathrm{K}$). The imaginary part saturates when the energy exceeds the largest phonon energy. The slope of the real part tells us the effective mass. In Pb, $\lambda \sim 1.6$





At left, the phonon spectrum for Pb; again in mV units. We show both $\alpha^2 F(\omega)$ and $F(\omega)$; $F(\omega)$ is simply the phonon density of states, whereas $\alpha^2 F(\omega)$ also contains the energy-dependent coupling constant.

The variety in the forms for $\alpha^2 F(\omega)$ in real materials arises from the different lattice structures in real solids — the phonon spectrum then becomes a strong function of direction. The sharp peaks arise from van Hove singularities in the density of states. The results can then look very different from the standard Debye form, where one assumes that



ABOVE: The function $\alpha^2 F(\omega)$ for solid Hg.

$$\alpha^2 F(\omega) \propto (\omega/\Theta_D)^2\, \theta(\Theta_D - \omega) \qquad (206)$$

which, if true, leads to the result

$$\left.\begin{aligned} \mathrm{Im}\, \Sigma(\epsilon) &\sim O(\epsilon^2) \\ \mathrm{Re}\, \Sigma(\epsilon) &\sim O(\epsilon) \end{aligned}\right\} \qquad (207)$$

which is just the standard Fermi liquid form we have already seen (cf. eqtn. (101)).

However, we can see that even in the case of Hg metal, shown in the Figure, the strong departures of $\alpha^2 F(\omega)$ from the Debye form at high ω does not affect the low-ω asymptotic form, which does obey (206); we also see this in the case of Pb. There is a good reason for this. Even in a strongly anisotropic metal, in which there will be several phonon branches, it is still the case that at low ω, the spectrum will be dominated by one branch only (the lowest one, viz., the acoustic phonon branch with the lowest velocity).

If the low-ω behaviour of this lowest phonon branch is described by a dispersion $\omega_q = c_q |q|$, where c_q is a (direction-dependent) velocity, then (206) follows immediately. How can we demonstrate that this will be true?

In the very low-ω regime one can use a hydrodynamic argument — starting from an effective Lagrangian of hydrodynamic form (derived by starting from the conservation laws applying to bulk matter, written in a long-wavelength expansion) one then derives the existence of sound modes, with velocities determined by the various bulk compressibilities of the system. But this argument gives little information on the range of validity of the phonon approximation.

A field-theoretic argument starts by considering the electron-phonon system with electron-electron interactions included along with the electron-phonon coupling. The argument, developed properly, is quite lengthy, so we only summarize it here.

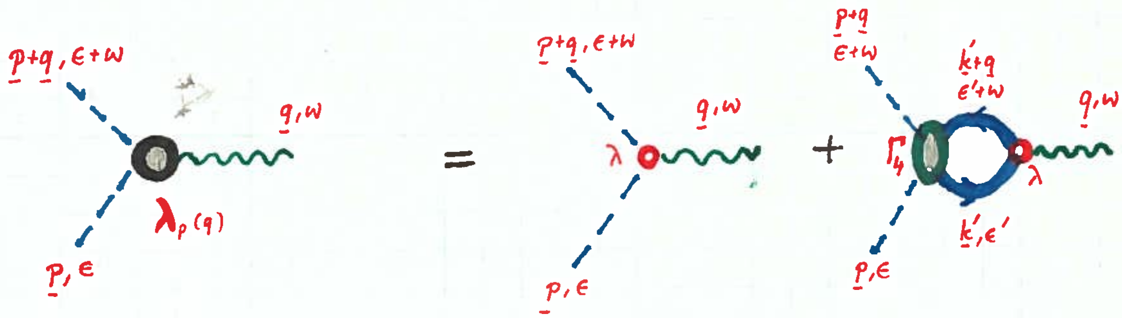
(i) The electron-phonon vertex is renormalized by the electron-electron interactions. Thus there are 2 renormalizations of λ_{pq}; one by electron-electron interactions, and the other by higher vertex corrections from the phonons themselves. In general, to take account of 2 interactions simultaneously like this is a difficult problem; but things simplify here because the energy scales in the

2 interactions are very different. As we will see properly in Chapters 6 and 7, this means that the renormalization caused by the 2 interactions are largely independent.

The purely electronic renormalization of the electron-phonon vertex is described by the simple result:

$$\bar{\lambda}_p(q) = \lambda_{pq} + \sum_{k} \int \frac{d\epsilon'}{2\pi} \lambda_{kq} \mathcal{G}_2(k+q)\, \mathcal{G}_2(k)\, \Gamma_4(k,p;q) \qquad (208)$$

which we show diagrammatically as (209).



(209)

Now this equation depends on what we choose for Γ_4. The simplest form we can take is the RPA form for the electron-electron interaction, according to which we write the "bubble sum" (compare (134)):

$$\Gamma_4(pp';q,\omega) = \frac{V_q}{1 - V_q \Pi(q,\omega)} \equiv \frac{V_q}{\epsilon(q,\omega)} \qquad (210)$$

where $\epsilon(q,\omega)$ is the dielectric function. If we substitute this into (208), and assume that $\lambda_{pq} \to \lambda_q$, ie, that the bare phonon vertex has no important dependence on the fermion energy-momentum, then (208) becomes

$$\bar{\lambda}(q,\omega) = \lambda_q \left[1 + \frac{\Pi(q,\omega) V_q}{1 - V_q \Pi(q,\omega)} \right] = \frac{\lambda_q}{\epsilon(q,\omega)} \qquad (211)$$

We apply the same approximation to the renormalized phonon propagator, ie., write

$$\left.\begin{aligned} \mathcal{D}(q,\omega) &= D_0(q,\omega) - D_0(q,\omega)\, |\bar{\lambda}(q,\omega)|^2\, \Pi(q,\omega)\, \mathcal{D}(q,\omega) \\ &= \frac{D_0(q,\omega)}{1 + \frac{\lambda_q^2 \Pi(q,\omega)}{\epsilon^2(q,\omega)} D_0(q,\omega)} \end{aligned}\right\} \qquad (212)$$

Now at this point we could continue with a detailed analysis of this expression, with emphasis on the detailed properties of $\epsilon(q,\omega)$ and $\Pi(q,\omega)$. However there are really only 2 points we wish to establish, viz.,

(a) The new poles of this expression give a dispersion relation ω_q which is also linear in $|q|$ for $|q| \ll k_F$

(b) The damping of these renormalised phonons is very small, again for $|q| \ll k_F$.

In the case of point (a), it is possible to do this via lengthy calculations for real systems. However such derivations miss the point – the general result should not depend on details of the interactions, and/or of $\epsilon(q,\omega)$ and $\Pi(q,\omega)$. In reality the linearity depends on fundamental conservation laws in the system, a topic we will begin to address in the next section on Ward identities.

In the case of point (b), the key result to be extracted from (212) is that the source of the phonon damping is to be found in the decay of phonons, into particle-hole pairs, as embodied in $\Pi(q,\omega)$ and $\epsilon(q,\omega)$. The rate at which this can happen is actually very, as can be seen in any direct calculation, simply because phonon energies are so low compared to electronic energies; for $\omega \ll \epsilon_F$, we have a very small phonon linewidth.

These results justify the use of the renormalised ω_q in eqtns (197)-(201). For a deeper discussion of such issues, we must wait for Ch. 7.

(ii) ELECTRON-PHONON VERTEX & B-S EQUATION

: Our analysis here is in many ways the same as that given for QED; so we can take some short cuts. First we will look at the B-S eqtns for $\Lambda_3(p,q)$, and then at the B-S eqtn. for $\Gamma_4(pp';q)$ in the Cooper channel. This will give us the Cooper instability to superconductivity, which we will derive in a simple RPA.

Without derivation (we have already seen it for QED), we write down the B-S eqtns. for $\Lambda_3(p,q)$ and $\Gamma_4(pp'q)$:

$$\Lambda_3(\underline{p},\epsilon;\underline{q},\omega) = \lambda_{pq} + \sum_{\underline{k}}\int\frac{d\epsilon'}{2\pi}\Lambda_3(\underline{k},\epsilon';\underline{q},\omega)\,\mathcal{G}_2(\underline{k}+\underline{q},\epsilon'+\omega)\,\mathcal{G}_2(\underline{k},\epsilon')\,\mathcal{I}_4(\underline{p},\underline{k},\underline{q};\epsilon\epsilon'\omega) \quad (213)$$

(cf. eqtns. (160) and (161)), and, for Γ_4 in the Cooper channel:

$$\Gamma_{pp'q}(\epsilon\epsilon';\omega) = \mathcal{I}_{pp'q}(\epsilon,\epsilon';\omega) + \sum_{\underline{k}}\int\frac{dE}{2\pi}\,\mathcal{I}_{pp';k-p}(\epsilon,\epsilon';E-\epsilon)$$
$$\times\,\mathcal{G}_2(\underline{k},E)\,\mathcal{G}_2(p+p'-k,\epsilon+\epsilon'-E)\,\Gamma_{\underline{p}+\underline{p}'-\underline{k},\underline{p}\underline{p}'}(\epsilon+\epsilon'-E;\epsilon,\epsilon') \quad (214)$$

an eqtn previously depicted in (170).

In what follows we will simplify the analysis of these eqtns by making the same RPA as we did for the positronium problem. This is done for two reasons: first, it allows us to simplify the calculations; and second, it brings out very clearly the role of multiple-scattering processes in the Cooper channel. It will also make it easy to compare things with the positronium calculation we did earlier.

We will look at the RPA applied to the calculation of $\Lambda_3(p,q)$, to $\Gamma_4(pp'q)$, and to $\Sigma(p)$ (ie., to the 3-point, 4-point, and 2-point vertices). We write these in a way which emphasizes the Cooper channel:

$$\Sigma(p) = \sum_Q \mathcal{G}_2(Q-p)\,\Gamma_4(p, Q-p\,;\,Q) \tag{215}$$

$$\Gamma_4(p, Q-p\,;\,Q) = \lambda_Q + \sum_k \lambda_Q\, \mathcal{G}_2(k)\,\mathcal{G}_2(p-k)\,\Gamma_4(k, Q-k\,;\,p-q-k) \tag{216}$$

$$\Lambda_3(p,q) = \lambda_q + \lambda_q \sum_Q \mathcal{G}_2(Q-p)\,\mathcal{G}_2(Q-(p+q))\,\Gamma_4(p, Q-p\,;\,q) \tag{217}$$

and then apply the RPA to the 4-point vertex; we make the substitution

$$\lambda_q \rightarrow -V_0\,\theta(\Theta_D - |\epsilon_p^0|) \qquad \text{(attractive)} \tag{218}$$

the "BCS approximation", which has the effect of assuming a constant attractive interaction up to the Debye energy; the Θ-function will simply cut off the integrals over internal phonons at the Debye energy. We then have

$$\Gamma_4(p, Q-p\,;\,Q) \longrightarrow \Gamma_{RPA}(Q) = \frac{-V_0}{1 + V_0 \tilde{\chi}_0(Q)}\,\theta(\Theta_D - |\epsilon_p^0|) \tag{219}$$

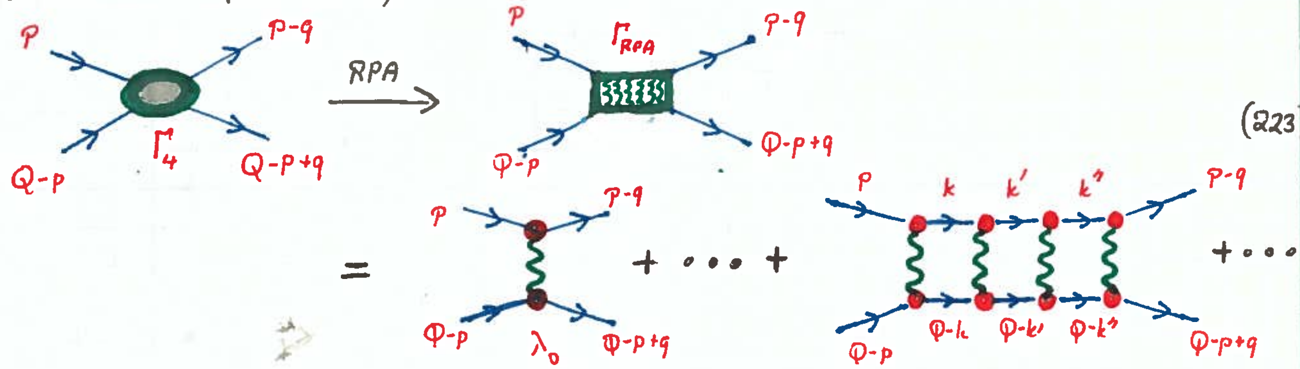
$$\Lambda_3(p,q) \longrightarrow \Lambda_{RPA}(q) = \frac{-V_0}{1 + V_0 \tilde{\chi}_0(q)}\,\theta(\Theta_D - |\epsilon_p^0|) \tag{220}$$

$$\Sigma(p) \longrightarrow \Sigma^{RPA}(p) = \sum_Q G_0(Q-p)\,\frac{-V_0\,\theta(\Theta_D - |\epsilon_p^0|)}{1 + V_0 \tilde{\chi}_0(Q)} \tag{221}$$

where the "Cooper bubble" $\chi_0(q)$ (which includes the energy cutoff) is

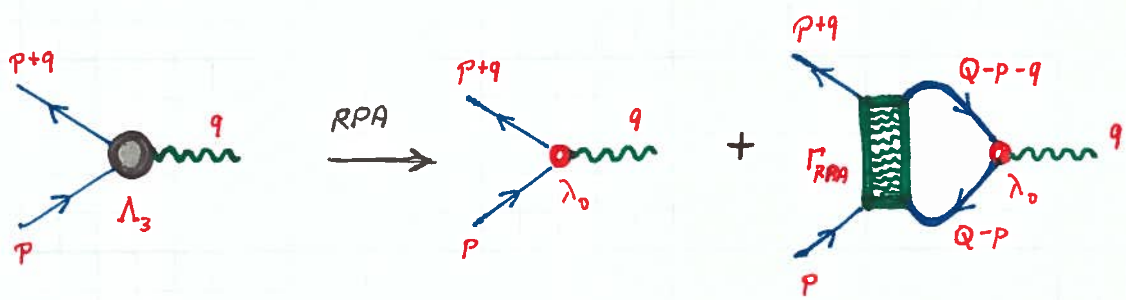
$$\chi_0(\underline{q},\omega) = \sum_{\underline{k}} \int \frac{d\epsilon}{2\pi}\, G_0(\underline{k},\epsilon)\, G_0(\underline{q}-\underline{k}, \omega-\epsilon)\,\theta(\Theta_D - |\epsilon_k^0|) \tag{222}$$

which differs from the particle-hole bubble $\pi_0(q,\omega) = \sum_k \int \frac{d\epsilon}{2\pi} G_0(k,\epsilon)\, G_0(k+q, \epsilon+\omega)$, in that the Cooper bubble involves a particle-pair rather than a particle-hole pair. The physical meaning of these eqtns is made clearer by their diagrammatic representations; in the Figure we show the "ladder sum" for the 4-point vertex, and the other 2 vertices in terms of this ladder sum. The ladder sum (223) contains the Cooper singularity.

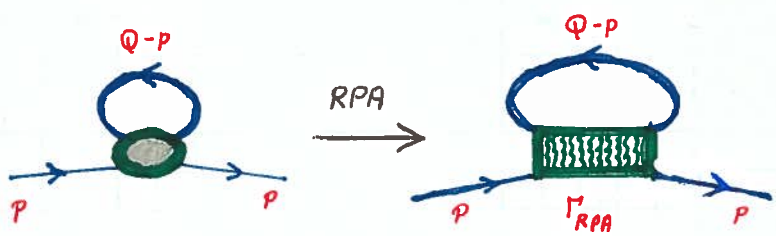


(223)

The same singularity then appears, integrated, in the 2-point and 3-point vertices of (224) and (225).



(224)



(225)

The existence of this Cooper singularity invalidates the Migdal theorem. Let us now see how strong the singularity is. At this point it is useful & illuminating to switch to a Matsubara formulation, in terms of which the Cooper bubble now becomes

$$\tilde{\chi}_0(\underline{q}, i\omega_m) = -\frac{1}{\beta}\sum_n \sum_{\underline{p}} \frac{1}{i(\omega_m - \epsilon_n) - \epsilon^0_{\underline{q}-\underline{p}}} \; \frac{1}{i\epsilon_n - \epsilon^0_{\underline{p}}} \; \theta(\theta_D - |\epsilon^0_p|) \qquad (226)$$

with poles at $i\epsilon_n = i\omega_m - \epsilon^0_{q-p}$ and $i\epsilon_n = \epsilon^0_p$. Noting that if we have particle-hole symmetry, $\epsilon^0_{q-p} = \epsilon^0_{p-q}$ (we are still measuring energies from the chemical potential, ie., putting $\mu = 0$), we get after contour integrating that

$$\left.\begin{aligned} \tilde{\chi}(\underline{q}, i\omega_m) &= \sum_{\underline{p}} \theta(\Theta_D - |\epsilon^\circ_{\underline{p}}|) \frac{1 - f_{\underline{p}-\underline{q}} - f_{\underline{p}}}{i\omega_m - (\epsilon^\circ_{\underline{p}-\underline{q}} + \epsilon^\circ_{\underline{p}})} \\ &\equiv \sum_{\underline{p}} \theta(\Theta_D - |\epsilon^\circ_{\underline{p}}|) \frac{\tanh(\beta\epsilon^\circ_{\underline{p}}/2) + \tanh(\beta\epsilon^\circ_{\underline{p}-\underline{q}}/2)}{i\omega_m - (\epsilon^\circ_{\underline{p}-\underline{q}} + \epsilon^\circ_{\underline{p}})} \end{aligned}\right\} \quad (227)$$

Now this is still a fairly complicated form. What we are interested in, for application to any real system, is where the poles in eqtns (219)-(221) are going to arise; and we are interested in poles when $i\omega_m \to 0$, ie., in the static limit. Thus we wish to find solution of

$$1 + V_0 \hat{\chi}_0(\underline{q}, 0) = 1 - V_0 \sum_{\underline{p}} \theta(\Theta_D - |\epsilon^\circ_{\underline{p}}|) \frac{\tanh\left(\frac{\beta\epsilon^\circ_{\underline{p}}}{2}\right) + \tanh\left(\frac{\beta\epsilon^\circ_{\underline{p}-\underline{q}}}{2}\right)}{\epsilon^\circ_{\underline{p}-\underline{q}} + \epsilon^\circ_{\underline{p}}} = 0 \quad (228)$$

If we look at the momentum integrals here, it becomes apparent that the integrand attains a maximum at $\underline{q} = 0$, so we then get, as our eqtn. for the poles, that

$$1 - V_0 \sum_{\underline{p}} \theta(\Theta_D - |\epsilon^\circ_{\underline{p}}|) \frac{\tanh(\beta\epsilon^\circ_{\underline{p}}/2)}{2\epsilon^\circ_{\underline{p}}} = 0 \quad (229)$$

which is one form of the BCS instability criterion. The integration over $\underline{p}$ is independent of angle, so doing the angular integrations then leaves us with an integral over energy near the Fermi energy, and we get (integrating by parts)

$$N(0)\left\{ \log(\tfrac{1}{2}\beta\Theta_D) - \int_0^{\frac{1}{2}\beta\Theta_D} dx \, \ln x \, \mathrm{sech}^2 x \right\} = 1 \quad (230)$$

which is solved by letting the upper limit in the integral $\to \infty$; we then have (NB: $\log\gamma \sim 0.577$ is Euler's constant):

$$V_0 N_0 \log\left(\frac{2\gamma\beta\Theta_D}{\pi}\right) = 1 \quad (231)$$

or, letting $\beta^{-1} = kT_c$, the highest temperature at which this can be obeyed, we get the BCS transition temperature

$$T_c \sim 1.14\, \Theta_D\, e^{-1/N(0)V_0} \quad (232)$$

which is the famous result of Cooper.

Actually Cooper derived this result differently, in a way which is more like our derivation for positronium. He imagined a single pair of electrons, and took into account the effect of the Fermi sea by introducing a restriction on the scattering states available to the pair. Thus he wrote down a

Schrodinger eqtn for the pair wave function $\Psi(p_1,p_2)$, where p_1 and p_2 are the 2 momenta, and then, defining $P = \frac{1}{2}(p_1+p_2)$ and $p = p_1 - p_2$, he went to the c.o.m. frame $P = 0$, and wrote

$$\left[\left(\frac{p^2}{2m} - E\right)\theta(|p| - p_F)\right]\Psi_p \;+\; \sum_{p'} V_{pp'}\,\theta(|p'| - p_F)\,\Psi_{p'} \;=\; 0 \qquad (233)$$

thereby enforcing the Pauli restriction on all states. Using again the form in (218) for the interaction, we then get a separable integral eqtn; again, we have no angular dependence in the integral, and we now get.

$$(\epsilon_p^0 - E)\Psi_p - V_0 N(0)\int_0^{\Theta_D} d\epsilon_{p'}^0\,\Psi_{p'} \;=\; 0 \qquad (234)$$

Going now through the same manoeuvres as we did for the positronium problem (integrating over ϵ_p^0, and factoring out the integrals over Ψ_p, $\Psi_{p'}$), we then get the result:

$$V_0 N_0 \int_0^{\Theta_D} \frac{d\epsilon_{p'}^0}{\epsilon_{p'}^0 - E} \;-\; 1 \;=\; 0 \qquad (235)$$

which now gives a solution for a binding / bound state energy

$$E_b \sim -\Theta_D\, e^{-1/N_0 V_0} \qquad (236)$$

Notice the key difference with the positronium problem arises with the background Fermi sea, which restricts the intermediate states allowed in the scattering. The equivalence between the self-consistent integral eqtn for Ψ_p and the RPA solution simply reflects the "multiple scattering" nature of the Schrodinger eqtn when written in integral form.

In real superconductors, just as with positronium, one takes account of higher-order terms, vertex corrections, etc. Unlike positronium, one cannot expand in the dimensionless coupling $N(0)V_0$. However one can use another small parameter, V_0/ϵ_F (or Θ_D/ϵ_F) to get very accurate theoretical results.

Finally, a methodological point. Migdal's theorem, and all the diagrammatic expansions we have been using, are perturbative results. And yet, by summing to infinite order a specific class of diagrams which lead to a singularity, we have found a NON-PERTURBATIVE result. The energy E_b in (263) has the form

$$E_b(\bar{V}_0) \sim \text{const}\; e^{-1/\bar{V}_0} \qquad (237)$$

where $\bar{V}_0 = N(0)V_0$ is the dimensionless coupling. This is not perturbative - indeed $E_b(\bar{V}_0)$ has an essential singularity as $V_0 \to 0$! In the next chapter we give a much more thorough treatment of non-perturbative methods in QFT.

B.5.4 : WARD IDENTITIES & CONSERVATION LAWS

In classical physics we are used to the idea that there will usually be a set of conserved quantities, also called "constants of motion", or "integrals of motion"; and these are often associated with simple symmetries in the system (although not always, as workers in celestial mechanics are well aware).

In quantum mechanics the same is true, and things work out fairly simply in 1-particle QM. However for an N-body non-relativistic system, or for a quantum field, things are not always so simple. One can have "anomalies", in which classical symmetries are lost; and there is always the possibility of spontaneous symmetry-breaking. In non-Abelian theories things become quite complicated.

In this section we will look at both the general form of the set of equations known as "Ward identities", and also at a number of examples of these; the examples are drawn from both relativistic and non-relativistic QFT. We shall see how it is that certain simple approximations (eg., the RPA) acquire respectability just because they obey Ward identities, which are a way of enforcing the conservation laws and/or symmetries existing in the system. Finally, we will briefly look at anomalies, and understand how they arise.

In what follows I will follow an "anti-historic" order of development. Ward identities were first described for QED in a short note by Ward in 1950, and generalized by Fradkin (1955) and Takahashi (1957). Here we will simply apply the Schwinger-Dyson eqtns to derive them in a more general way, and then look at the application to theories like QED, and the non-relativistic electron fluid. I will not give the most general discussion possible, which would be rather lengthy.

B.5.4 (a) DERIVATION OF WARD IDENTITIES

We begin by giving the general idea, and, to see how it works out in practise, we use the general technique to derive Ward identities for 3 canonical theories, viz., scalar QFT, QED, and the non-relativistic interacting Fermi liquid.

We will begin here with the path integral / functional formulation of QFT; the derivation then closely parallels the derivation of the Schwinger-Dyson eqtns themselves.

(i) SCALAR FIELD THEORY

: We will consider here the simplest version of this, with a single scalar field $\phi(x)$; the generalization to an N-component scalar field $\underline{\phi}(x) \equiv \phi_\alpha(x)$ is then easily accomplished. The action will have the form

$$
\left.\begin{aligned}
S &= \int d^4x \left\{ \tfrac{1}{2}\left[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2\right] - V(\phi)\right\} \\
&\equiv \int d^4x \left\{ \tfrac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \mathcal{P}(\phi)\right\}
\end{aligned}\right\} \quad (238)
$$

where $V(\phi) = \sum_{n=3}^{\infty} g_n/_{n!}\, \phi^n(x)$, and $\mathcal{P}(\phi) = \sum_{n=2}^{\infty} (g_n/n!)\,\phi^n(x)$. Let us first recall how we derive the Schwinger-Dyson eqtns for this general form in eqtn. (238). Functional differentiation of this gives

$$\delta S = \left[\frac{\delta L}{\delta\phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi)}\right]\delta\phi \tag{239}$$

which in terms of the Noether current $j^\mu = \partial L/\partial(\partial_\mu\phi)$ discussed in section B.4 (cf. eqtns (111) - (119) of this section) is just

$$\frac{\delta S}{\delta\phi(x)}\,\delta\phi(x) = \delta L - \partial_\mu j^\mu(x)\,\delta\phi(x) \tag{240}$$

and we note in passing that setting this to zero gives the classical eqtn. of motion, viz.,

$$\left.\begin{aligned} \frac{\delta S}{\delta\phi(x)} &= -\left[\partial^2\phi + P'(\phi)\right] \\ &\equiv -\left[(\partial^2+m^2)\phi + V'(\phi)\right] = 0 \qquad (\text{classical}) \end{aligned}\right\} \tag{241}$$

If instead we functionally differentiate the normalized generating functional $Z[J]$, which we recall is

$$Z[J] = \frac{1}{Z[0]}\int \mathcal{D}\phi\, e^{\left(\frac{i}{\hbar}S[\phi] + \int J\phi\right)} \tag{242}$$

then

$$\delta Z[J] = \frac{1}{Z[0]}\int \mathcal{D}\phi\, e^{\left(\frac{i}{\hbar}S[\phi] + \int J\phi\right)}\left[\frac{\delta S}{\delta\phi} + J(x)\right]\delta\phi(x) = 0 \tag{243}$$

which we argue is identically zero because by making the "change of variable" $\phi(x) \to \phi(x) + \delta\phi(x)$ (actually a "change of function", since we deal here with functional integrals), we have not changed the integration measure $\mathcal{D}\phi(x)$.

Now, starting from (243) it is a simple matter to rederive the SD eqtns., by repeatedly functionally differentiating w.r.t. $J(x)$, and then setting $J(x) = 0$. Thus, e.g., the first functional differentiation gives

$$\left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J=0} = \frac{1}{Z[0]}\int \mathcal{D}\phi(x')\, e^{\frac{i}{\hbar}S[\phi]}\left[\phi(x)\frac{\delta S}{\delta\phi(x')} + i\hbar\,\delta(x-x')\right] \tag{244}$$

and if we now set this to zero (arguing that $\delta\phi(x')$ is arbitrary in (243), so that the functional integrand must be zero), then we have

$$\langle 0|\hat{T}\left\{\phi(x)\frac{\delta S}{\delta\phi(x')}\right\}|0\rangle + i\hbar\,\delta(x-x') = 0 \tag{245}$$

or, from (238), that

$$\partial_x^2 G_2(x,x') + \langle 0|T\{\phi(x')\,P'(\phi(x))\}|0\rangle = -i\hbar\,\delta(x-x') \tag{246}$$

to be compared to eqtn. (11) of this section (where have switched x and x' in going from (245) to (246)). If we now repeatedly functionally differentiate $Z[J]$ in (242) w.r.t. $J(x)$, and then set $J(x)$ to zero, we get, instead of (243), the result

$$\delta^{n-1} Z[J]\Big|_{J=0} = \frac{1}{Z[0]} \int D\phi \, e^{i/\hbar S[\phi]} \left[\frac{\delta S}{\delta \phi(x)} \prod_{j=1}^{n-1} \phi_j(x_j') + i\hbar \sum_{j=1}^{n-1} \sum_{\ell \neq j}^{n-1} \delta(x-x_j')\, \phi(x_\ell') \right] \delta\phi(x) \quad (247)$$

and if we again assume that this result is valid independently of the form of $\delta\phi(x)$, as we did to get (245), we now find that

$$\langle 0 | \hat{T} \left\{ \frac{\delta S}{\delta \phi(x)} \prod_{j=1}^{n-1} \phi(x_j') \right\} | 0 \rangle = -i\hbar \sum_{j=1}^{n-1} \sum_{\ell \neq j}^{n-1} \delta(x-x_j') \langle 0 | \hat{T} \{ \phi(x_\ell) \} | 0 \rangle \quad (248)$$

which we can rewrite using (238) as:

$$\left. \begin{aligned} \partial_x^2 G_n(x; x_1' \cdots x_{n-1}') + \langle 0 | \hat{T} \{ P'(\phi(x)) \prod_{j=1}^{n-1} \phi(x_j') \} | 0 \rangle \\ + i\hbar \sum_{j=1}^{n-1} \delta(x-x_j)\, G_{n-2}(x_1' \cdots x_{j-1}'; x_{j+1}' \cdots x_{n-1}') = 0 \end{aligned} \right\} \quad (249)$$

which reduces, in the case of ϕ^4 theory, to (11). The Fourier transformed version of this is also useful; we have

$$\left. \begin{aligned} q^2 G_n(q; k_1 \cdots k_{n-1}) - \langle 0 | \hat{T} \{ P'(\phi_q) \prod_{j=1}^{n-1} \phi_{k_j} \} | 0 \rangle \\ - i \sum_{j=1}^{n-1} \delta(q + \sum_{\ell=1}^{n-1} k_\ell)\, G_{n-2}(k_1 \cdots k_{j-1}; k_{j+1} \cdots k_{n-1}) \end{aligned} \right\} \quad (250)$$

where $P'(\phi_q) \equiv dP(\phi_q)/d\phi_q$.

So far so good - we have given a thorough discussion of the SD eqtns. for a scalar field. But now we notice that using (240) we can also rewrite things in terms of the Noether current. There are 2 cases of interest here, viz.,

(a) In (240), we can assume that $\delta L = 0$ under the variation $\delta\phi(x)$, so that we simply have

$$\delta S / \delta\phi(x) = - \partial_\mu j^\mu(x) \quad (251)$$

(b) Alternatively, whilst still assuming $\delta S = 0$ under the variation $\delta\phi(x)$ (again, because we assume $D\phi$ is invariant), we allow $\delta L = \partial_\mu I^\mu(x)$, which is a total 4-divergence; assuming appropriate boundary conditions for $\delta\phi(x)$, we then redefine $j^\mu(x)$ as

$$j^\mu(x) = \frac{\partial L}{\partial(\partial_\mu \phi(x))} - I^\mu(x) \quad (252)$$

Now, in either of these cases, we can rewrite $\delta S/\delta\phi$ in (247) in terms of $j^\mu(x)$, using (251); if we then assume that $\delta^{n-1} Z[J] = 0$, we get one form of the Ward identities:

$$\boxed{\partial_\mu \langle 0|\hat{T}\{ j^\mu(x) \prod_{j=1}^{n-1} \phi(x_j')\}|0\rangle - i\hbar \sum_{j=1}^{n-1} \sum_{\ell\neq j}^{n-1} \delta(x-x_j') \langle 0|\hat{T}\{\phi(x_\ell')\}|0\rangle = 0} \qquad (253)$$

or, if we Fourier transform to momentum space

$$\boxed{q_\mu \langle 0|\hat{T}\{ j^\mu(q) \prod_{j=1}^{n-1} \phi_{k_j}\}|0\rangle - \sum_{j=1}^{n-1} \sum_{\ell\neq j}^{n-1} \delta(q+k_\ell) \langle 0|\hat{T}\{\phi_{k_\ell}\}|0\rangle = 0} \qquad (254)$$

Now these eqtns, as with the SD eqtns., are a set of coupled eqtns which connect the different correlators (and, as we will see, one can write similar eqtns for the vertex parts). We have no space here to explore all their consequences; instead we will look at certain special cases.

The first thing to understand is why eqtns. like (253) or (254) have anything to do with the conservation laws of the QFT. Let's recall how this works in simple classical mechanics or in 1-particle QM. In the classical mechanics of particles we have invariance of the Lagrangian and the action under time and space translations (modulo the addition of a total time derivative), along with invariance under rotations; this leads to the conservation laws

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\right) = \dot{\underline{P}} = 0 \qquad \text{(momentum)} \qquad (255)$$

$$\frac{d}{dt}\left[\dot{\underline{q}}\frac{\partial L}{\partial \dot{\underline{q}}} - L\right] = \dot{E} = 0 \qquad \text{(energy)} \qquad (256)$$

and a similar result for angular momentum conservation, coming from the result $\partial L/\partial\hat{\underline{\Theta}} = 0$, where $\hat{\Theta}$ is some rotation. Now for a scalar QFT, the quantities corresponding to $\underline{P}$ and E are as follows

$$\underline{P} \iff j^\mu(x) \equiv \frac{\partial L}{\partial(\partial_\mu \phi(x))} \qquad (257)$$

$$E \iff T^{\mu\nu}(x) \equiv \partial^\mu\phi(x)\frac{\partial L}{\partial(\partial_\nu\phi(x))} - \eta^{\mu\nu}(x) L \qquad (258)$$

where in (258) we introduce the metric tensor $\eta^{\mu\nu}(x)$; for curved spacetime this just becomes the general metric tensor $g^{\mu\nu}(x)$. We already know that j^μ is conserved in scalar field theory, as expressed by the conservation law

$$\partial_\mu j^\mu(x) = 0 \qquad (259)$$

The conservation law for the "stress tensor", or "energy-momentum tensor" $T^{\mu\nu}(x)$, is then expressed as

$$\partial_\mu T^{\mu\nu}(x) = 0 \tag{260}$$

For the scalar field the stress tensor components take the following form:

$$\left.\begin{aligned} T^{00}(x) &= \tfrac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) + \mathcal{P}(\phi) \\ &= \tfrac{1}{2}[\pi^2(x) + (\underline{\nabla}\phi)^2] + \tfrac{1}{2}m^2\phi^2 + V(\phi) \end{aligned}\right\} \tag{261}$$

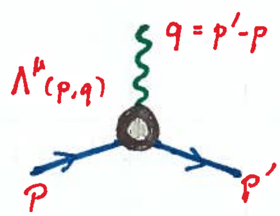
where $\pi(x) = \partial_0\phi(x)$ is the canonical momentum (so that $[\pi(x), \phi(x)] = -i\hbar$). Thus eqtn (261) shows that $T^{00}(x)$ is the energy density - it is nothing in fact but the Hamiltonian density.

$$\left.\begin{aligned} T^{0j}(x) &= \partial^0\phi\,\partial^j\phi \equiv \pi(x)\nabla^j\phi(x) \\ T^{ij}(x) &= \partial^i\phi(x)\,\partial^j\phi(x) \end{aligned}\right\} \tag{262}$$

giving respectively the energy/momentum currents, and the stress tensor (note that $T^{jj}(x)$ is a pressure, whereas $T^{ij}(x)$, with $i \neq j$, is a stress). The result (260) then gives conservation laws for all of these quantities.

The question we then wish to address is - how are these conservation laws connected to the Ward identities? To answer this, we begin by looking at a simple example.

EXAMPLE : 3-POINT VERTEX in ϕ^4 THEORY : Almost the simplest vertex we can look at in scalar field theory is the 3-point vertex which describes the interaction of a field excitation with a field current J^μ. We can look at either the Green function $G_3(p,q)$ or the vertex $\Lambda(p,q)$; the latter is the most useful, but the Ward identities describe the former. The diagram shows $\Lambda(p,q)$; the diagram for $G_3(p,q)$ will have fully renormalized external legs.



From the Ward identity in (254) we have the result

$$\left.\begin{aligned} q_\mu G_3^\mu(p,q) &\equiv (p'_\mu - p_\mu)G_3^\mu(p,q) \\ & \quad G_2(p+q) - G(p) \equiv i\hbar\left(\Delta_F(p+q) - \Delta_F(p)\right) \end{aligned}\right\} \tag{263}$$

Note that this relationship is <u>exact</u>; multiplying both sides by $G_2^{-1}(p+q)G_2^{-1}(p)$, we get the relation for $\Lambda(p,q)$:

$$q_\mu \Lambda_3^\mu(p,q) = G_2^{-1}(p+q) - G_2^{-1}(p) \tag{264}$$

which is again exact - all possible internal processes are included in the circle shown.

The result in (264) may or may not seem all that dramatic (but notice how strikingly simple it is, given that it is an exact relationship, incorporating graphs to all order!). If it does not seem all that striking to you, you should realize that:

- A demonstration of (263) and (264), even for such a simple Ward identity connecting Λ_3 and G_2, is much more complicated in the canonical theory; using field commutators, etc. For higher-order Ward identities it becomes almost prohibitively complex;

- We have yet to see the real power of Ward identities, which comes when we look systematically at renormalization theory — this will come in Ch. 7.

(ii) QUANTUM ELECTRODYNAMICS

: Back in section B.5.1 (c) I gave a fairly quick discussion of the SD eqtns for QED; the aim there was to quickly arrive at a set of consistent eqtns relating $\Pi^{\mu\nu}(q)$ and $D^{\mu\nu}(q)$, $\mathcal{G}_2(p)$ and $\Sigma(p)$, and $\Lambda_3^\mu(p,q)$, λ_5^μ, and $\Gamma_4^\mu(p,p';q)$. We ignored the fact that with 3 different fields, and external sources, the complexity of the SD eqtns when all 3 external sources are kept finite is rather intimidating.

In what follows we will continue to keep things simple; even though there are 3 Noether currents, and hence a plethora of possible Ward identities, we will focus on just 2 simple ones; viz., the Ward identity relating Λ_3 to $\mathcal{G}_2$, for the electron, and the photon Ward identity for $D^{\mu\nu}(q)$ which allows us to prove the photon is massless.

Before we begin, let us go back to eqtn (36) in section B.5.1(c), and rewrite it in a form which is amenable for the derivation of Ward identities. Instead of an eqtn for $J^\mu(x)$, we write (33) as an eqtn. for $\partial^\mu J_\mu$, in the form:

$$\left\{ i\left[\eta_{\mu\nu}\partial^2 - (1-1/\alpha)\partial_\mu\partial_\nu\right]\partial^\mu \frac{\delta}{\delta J^\nu} - e\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right) - \partial^\mu J_\mu \right\} \mathcal{Z}[A^\mu,\bar\eta,\eta] = 0 \tag{265}$$

or, in terms of $W = -i\hbar \ln \mathcal{Z}$, we have:

$$\left\{ \left[\eta_{\mu\nu}\partial^2 - (1-1/\alpha)\partial_\mu\partial_\nu\right]\partial^\mu \frac{\delta}{\delta J^\nu} + ie\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right)\right\} W[A^\mu,\bar\eta,\eta] = -\partial^\mu J_\mu(x) \tag{266}$$

where we have assumed a charge $q=-e$ (so as not to confuse things with the 4-momentum q). These 2 eqtns are alternative forms of the SD eqtn (36), but now in a form more suitable for the derivation of Ward identities. Note that in (265), and from now on in this section, we are putting $\hbar = 1$.

Finally, we can put the SD eqtns into a form involving the 2-point vertex $\Gamma_2(x,x')$, the inverse of the electronic propagator $\mathcal{G}_2(x,x')$ (compare eqtn. (41)). Substituting the indentities in (38) into (266), we have, using (37) for our definition of the vertex generating function $\Gamma[A^\mu,\bar\psi,\psi]$, and multiplying by $\eta^{\mu\nu}$, we then find that

$$\boxed{\frac{1}{\alpha}\partial^2\partial^\nu A_\nu(x) - \left[\partial_\nu \frac{\delta\Gamma}{\delta A_\nu(x)} + ie\left(\bar\psi\frac{\delta\Gamma}{\delta\bar\psi(x)} - \psi\frac{\delta\Gamma}{\delta\psi(x)}\right)\right] = 0} \qquad (267)$$

which can be used as the starting point to generate lots of different Ward identities; it is in a form where one may differentiate with respect to the fields $\bar\psi, \psi$, and A, as much as one likes, to get to the desired vertices.

Note that Γ is not in general gauge invariant; but one can rewrite (267) in terms of a gauge-invariant function $\tilde\Gamma$, defined by

$$\tilde\Gamma[A_\mu - \partial_\mu\delta\theta, \bar\psi(1-ie\delta\theta), \psi(1+ie\delta\theta)] = \tilde\Gamma[A_\mu,\bar\psi,\psi] \qquad (268)$$

to which the solution is

$$\tilde\Gamma = \Gamma - \frac{1}{2\alpha}\int d^4x\,(\partial_\nu A^\nu)^2 \qquad (269)$$

and this function satisfies the simpler relation

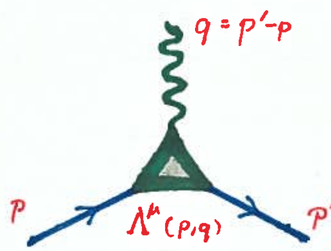
$$\boxed{\left[\partial_\nu\frac{\delta}{\delta A_\nu} + ie\left(\bar\psi\frac{\delta}{\delta\bar\psi} - \psi\frac{\delta}{\delta\psi}\right)\right]\tilde\Gamma[A,\bar\psi,\psi] = 0} \qquad (270)$$

in place of (267)

Starting from either (267) or (270), we can now proceed to derive some very useful results.

EXAMPLE 1 : PHOTON-FERMION 3-POINT VERTEX :

This vertex has great physical significance, because it describes the fully renormalized interaction between a photon and an electron. Depending on what is q (in particular, the ratio $\omega/|q|$), this vertex can describe the physical scattering amplitude, (for $\omega/|q| \to 0$), or the electron energy shift caused by the photon (for $|q|/\omega \to 0$), or something between two.



The Ward identity for this vertex requires that we functionally differentiate (267) or (270) twice, with respect to ψ and $\bar\psi$, in order to get the 3rd functional derivative $\Gamma_3 \equiv e\Lambda_3$; we then set $\bar\psi = \psi = A^\mu = 0$. This then gives the result (cf eqtn. (45))

$$\left.\begin{aligned} e\,\partial_x^\mu \Lambda_\mu(x;y,y') &= \partial_x^\mu \left.\frac{\delta^3\Gamma}{\delta A^\mu(x)\,\delta\bar\psi(y)\,\delta\psi(y')}\right|_{A,\bar\psi,\psi=0} \\ &= ie\left[\frac{\delta^2\Gamma}{\delta\bar\psi(y)\,\delta\psi(y')}\left(\delta(x-y') - \delta(x-y)\right)\right]\Big|_{A,\bar\psi,\psi=0}\end{aligned}\right\} \quad (271)$$

However, this is just a real space version of the same kind of Ward identity as we found in (264), because

$$\mathcal{G}_2^{-1}(x,x') = \left.\frac{\delta^2\Gamma}{\delta\bar{\psi}(x)\,\delta\psi(x')}\right|_{A,\bar{\psi},\psi=0} \tag{272}$$

by definition, ie., (271) reads

$$\partial_x^\mu \Lambda_\mu(x;y,y') = i\,\mathcal{G}_2^{-1}(y,y')\left[\delta(x-y') - \delta(x-y)\right] \tag{273}$$

which has the classic form of a Ward identity. Fourier transforming, we get (putting $\hbar$ back):

$$\boxed{q^\mu \Lambda_\mu(p,q) = i\hbar\left[\mathcal{G}_2^{-1}(p+q) - \mathcal{G}_2^{-1}(p)\right]} \tag{274}$$

or, written out in full

$$q^\mu \Lambda_\mu(p,q) = \gamma_\mu q^\mu - \Sigma(p+q) + \Sigma(p) \tag{275}$$

This is clearly a very useful (and certainly not immediately obvious) relationship between Λ_3 and Σ. Notice that if we take the limit $q \to 0$ in these relations, we have

$$\Lambda_\mu(p,0) = \frac{\partial}{\partial p^\mu}\mathcal{G}_2^{-1}(p) = \gamma_\mu - \frac{\partial\Sigma(p)}{\partial p^\mu} \tag{276}$$

which was the original result derived by Ward.

To give a more complete physical understanding of this result, I will now rederive it in a slightly different way, which brings out the connection of all of this with gauge invariance. To do this, we apply the gauge transformation already used in (268), in differential form, viz.,

$$\left.\begin{aligned}
A^\mu(x) &\to A^\mu(x) + \partial^\mu_\mu\Theta(x) &&\sim A^\mu(x) - \partial^\mu\delta\Theta(x)\\
\bar{\psi}(x) &\to \bar{\psi}(x)\,e^{-ie\Theta(x)} &&\sim \bar{\psi}\,(1 - ie\,\delta\Theta(x))\\
\psi(x) &\to \psi(x)\,e^{ie\Theta(x)} &&\sim \psi\,(1 + ie\,\delta\Theta(x))
\end{aligned}\right\} \tag{277}$$

so that the 2-point fermion propagator changes as $\mathcal{G}_2 \to \mathcal{G}_2 + \delta\mathcal{G}_2$, where

$$\delta\mathcal{G}_2(x,x') = ie\,\mathcal{G}_2(x,x')\left[\delta\Theta(x) - \delta\Theta(x')\right]. \tag{278}$$

If we now Fourier transform this to

$$\mathcal{G}_2(p,p') = \iint d^4x\, d^4x'\, e^{i(px - p'x')}\,\mathcal{G}_2(x,x') \tag{279}$$

we find a change for $\mathcal{G}_2$ given in momentum space as

$$\delta \mathcal{G}_2(p,p') = ie\,\delta\theta(p'-p)\left[\mathcal{G}_2(p) - \mathcal{G}_2(p')\right] \qquad (280)$$

Now the point is, of course, that this gauge change also changes A^μ; we have $\delta A_\mu(x) = -\partial_\mu \delta\theta(x)$, or in momentum space, that $\delta A_\mu(q) = iq_\mu \delta\theta(q)$, so we can write

$$q^\mu \delta\mathcal{G}_2(p+q,p) = e\,\delta A^\mu(q)\left[\mathcal{G}_2(p) - \mathcal{G}_2(p+q)\right] \qquad (281)$$

Now one observes that the change $\delta\mathcal{G}_2$ brought about by δA is actually just a definition of G_3; we have (with $q = p'-p$), that:

$$\delta\mathcal{G}_2(p,p') = e\,G_3^\mu(p,p')\,\delta A_\mu(p-p') \qquad (282)$$

since G_3^μ is just the total amplitude for a process involving a small photon perturbation (incoming line δA^μ) acting on an electron (incoming p, outgoing p'). Thus we are back to where we were in discussing (263) for scalar fields; we now have

$$\delta\mathcal{G}(p,q) = e\left(\mathcal{G}_2(p+q)\,\Lambda^\mu(p,q)\,\mathcal{G}_2(p)\right)\delta A_\mu(q) \qquad (283)$$

which just gives us, using (281), the Ward identity (274) again.

EXAMPLE 2: PHOTON PROPAGATOR:

In our previous discussion of the phonon propagator, it was shown, by a combination of physical and formal arguments, that the phonon spectrum $\omega(q)$ remained linear in $|q|$, for small $|q| \ll k_F$, even after renormalization.

Let us now ask the same question of photons – how is the photon propagator modified by the coupling to fermions? We are looking here for an exact answer – perturbation theory is not good enough.

This question differs from the phonon problem in 2 respects. First, we do not have a background Fermi sea to deal with (QED for a finite density of fermions – as in a star, or a metal – is very different of course; photons disperse and move at a velocity $< c$). Second, we have a gauge symmetry here that is different from that in a solid, or any other non-superfluid condensed medium.

To analyze this question we are looking for the Ward identity governing $\mathcal{D}_{\mu\nu}(q)$, which is the inverse of Γ_2^{AA}, ie.,

$$\mathcal{D}_\mu^{\ \alpha}(q)\,\Gamma_{\alpha\nu}^{AA}(q) = \eta_{\mu\nu} \qquad (284)$$

where
$$\Gamma_{\mu\nu}^{AA}(x,x') \equiv \left.\frac{\delta\Gamma}{\delta A^\mu(x)\,\delta A^\nu(x')}\right|_{A,\bar\psi,\psi=0} \qquad (285)$$

was previously defined in (43). Now it is a simple matter, starting again from (267) or (270), to get a result for $\Gamma_2^{\mu AA}$; differentiating with respect to $A_\mu(x)$, and setting $A^\mu, \bar\psi$, and $\psi = 0$, we have from (267) that

$$\frac{1}{\alpha}\partial^2\partial^\mu\delta(x-x') = \partial^\nu \frac{\delta\Gamma}{\delta A_\mu(x)\,\delta A_\nu(x')}\bigg|_{A,\bar\psi,\psi=0} \tag{286}$$

or, in momentum space, that

$$\frac{1}{\alpha} q^\nu q^2 = q^\mu D^{-1}_{\mu\nu}(q) \tag{287}$$

This almost uniquely determines the form of $D^{-1}_{\mu\nu}(q)$; since it must be symmetric, one can write it in terms of one undetermined function $d(q)$, a symmetric function of q (ie, a polynomial in q^2), as

$$D^{\mu\nu}(q) = \frac{d(q)}{q^2}\left[\eta^{\mu\nu} - \hat q^\mu \hat q^\nu\right] + \frac{\alpha}{q^2}\hat q^\mu \hat q^\nu \tag{288}$$

where $\hat q^\mu = q^\mu/q$. However we can also write it more transparently in the form

$$D^{\mu\nu}(q) = \frac{1}{q^2}\left(\eta^{\mu\nu} - \hat q^\mu \hat q^\nu\right)\left(1 - \Pi(q)\right) - \alpha/_{q^2}\, \hat q^\mu \hat q^\nu \tag{289}$$

where $\Pi(q) = 1 - d^{-1}(q)$. Consider now the function

$$\Pi^{\mu\nu}(q) = \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right)\Pi(q) \tag{290}$$

We see from (287) that

$$q_\mu \Pi^{\mu\nu}(q) = 0 \tag{291}$$

and moreover, we can equate this function with the polarization part, defined by the usual relation

$$D^{\mu\nu}(q) = D_0^{\mu\nu}(q) + D_0^{\mu\alpha}(q)\,\Pi_{\alpha\beta}(q)\,D^{\beta\nu}(q) \tag{292}$$

The forms in (289) and (290) tell us something very important, viz., that the photon, even fully renormalized, has no mass – if it did, there would have to be a term $\sim \eta^{\mu\nu} m^2$ on the R.H.S. of (290). Thus the condition (291) on $\Pi^{\mu\nu}(q)$, which is a Ward identity, is equivalent to the restriction of masslessness on the photon. There are other ways of getting this conclusion; for example, one can functionally differentiate $\Pi_{\mu\nu}(q)$ with respect to an external photon (a photon which would interact with an internal electron line in $\Pi_{\mu\nu}$); this simply brings us back to the 3-point photon-electron vertex, which describes the interaction. This sort of manoeuvre, and other ways of deriving the masslessness of the photon, are discussed in many texts.

(iii) FERMI LIQUIDS

: Our final example is the non-relativistic interacting fermion system, already discussed in section B.5.2. We shall find very similar relations as we have just done for the relativistic scalar & vector fields; but this time we will show how they can be used to determine relations between the renormalization constants for different quantities. This can also be done for the relativistic theories we have just discussed - it is one of the main uses of Ward identities - but this discussion is more complex because the renormalizations are infinite, and the integrals do not separate into space and time components.

We begin with exactly the same manoeuvre as we just employed to study $\Lambda^\mu_3(p,q)$, the photon-fermion vertex. However what we do now is a little more analogous to the scalar field theory. Suppose we apply an external field to the fermion system, so that we now have a time-dependent Hamiltonian

$$\hat{\mathcal{H}}(\underline{r},t) = \hat{\mathcal{H}}_{FL} + \delta\hat{V}(\underline{r},t) \qquad (293)$$

where $\mathcal{H}_{FL}$ is just the Hamiltonian for interacting fermions given in (21), and $\delta V(\underline{r},t)$ is some small perturbation. In QFT language, the perturbation is written

$$\delta V(t) = \int d^3r\, \psi^+(\underline{r},t)\, \delta\hat{V}(\underline{r},t)\, \psi(\underline{r},t) \qquad (294)$$

where we ignore spin (I comment on this below).

Now the topic of Ward identities and conservation laws is sufficiently important that we are going to look, in greater or lesser detail, at two different ways to think about how the system responds to the perturbation in (294). A 3rd way would be start from the SD eqtns for a non-relativistic fermion system, but this would be reinventing things we have already done several times.

We will be interested first in deriving the results using a non-relativistic version of the gauge transformation performed above; and then we will rederive the results (up to a point) using a standard response function analysis. We begin by recalling how to write correlation functions in path integral language. In particular we found in part A that in a zero-temperature system, we can write the correlation between two physical quantities $A(t)$ and $B(t)$ in the 2 equivalent forms:

(i) In operator language, we have

$$\left.\begin{aligned} \chi_{AB}(t,t') &= \langle 0|\hat{A}(t)\,\hat{B}(t')|0\rangle \\ &= \sum_n A_{0n} B_{n0}\, e^{i/\hbar (E_n - E_0)(t-t')} \end{aligned}\right\} \qquad (295)$$

where the energies E_n are the exact eigenenergies of the full N-body system, and $A_{0n} = \langle 0|\hat{A}|n\rangle$, where $|0\rangle$ and $|n\rangle$ are exact eigenstates of this system - clearly we do not know these states or energies.

(ii) In path integral language, generalizing from the discussion given for a simple quantum system, we can ask how to calculate the correlation between 2 operators $A(x)$ and $B(x)$ acting on a field $\psi(x)$ (which might be any kind of field). Then we have

$$\chi_{AB}(x,x') = \int \mathcal{D}\psi(x)\, e^{i/\hbar S[\psi]} A(\psi(x))\, B(\psi(x')) \qquad (296)$$

where $A(\psi)$ and $B(\psi)$ are the classical expression for the quantities A and B in terms of the field operators. Introducing two currents J_A and J_B which couple to these functions, we write a generating functional

$$Z[J_A, J_B] = \int \mathcal{D}\psi\, e^{i/\hbar \left(S[\psi] + \int d^4x\, [J_A(x) A(x) + J_B(x) B(x)]\right)} \qquad (297)$$

where we write $A(x) \equiv A(\psi(x))$ for short; then we have

$$\chi_{AB}(x,x') = (-i\hbar)^2 \left. \frac{\delta^2 Z[J_A,J_B]}{\delta J_A(x)\, \delta J_B(x')} \right|_{J_A, J_B = 0} \qquad (298)$$

Now in relativistic systems we typically start from an expression like (298), whereas in non-relativistic systems it is more traditional to start from an expression like (295). What we are going to see is the relationship between the two.

GAUGE TRANSFORMATION ON $\mathcal{G}_2(p,\epsilon)$:

Previously (see eqtn (277)) we performed a QED gauge transformation, on both fermions and photons. In a 4-fermion theory we only have non-relativistic fermions, and the relevant gauge transformation is just

$$\left.\begin{aligned} \psi^+(r,t) &\to \psi^+ e^{-i\theta(r,t)} \sim \psi^+(r,t)\,[1 - i\delta\theta(r,t)] \\ \psi(r,t) &\to \psi\, e^{i\theta(r,t)} \sim \psi(r,t)\,[1 + i\delta\theta(r,t)] \end{aligned}\right\} \qquad (299)$$

where again the last result is under an infinitesimal transformation. Now suppose the fermions are described as usual by the Schrodinger eqtn; then we have

$$\left.\begin{aligned} \hat{T} = \frac{-\hbar^2}{2m}\nabla^2 &\to \frac{-\hbar^2}{2m}\left(\nabla + i\nabla\theta(r,t)\right)^2 \\ i\hbar\partial_t &\to i\hbar\left(\partial_t + i\partial_t\theta(r,t)\right) \end{aligned}\right\} \qquad (300)$$

so that the Schrodinger eqtn for the i-th particle transforms according to:

$$\left.\begin{aligned} &\int d^3r\, \psi_i^+(r,t)\,[\hat{\mathcal{H}}_{FL} - i\hbar\partial_t]\,\psi_i(r,t) \\ &= \int d^3r\, \psi_i^+(r,t)\left[\left(\frac{-\hbar^2}{2m} + \sum_{j\neq i}\int d^3r'\, \psi_j^+(r',t)\, V_{ij}\, \psi_j(r',t)\right) - i\hbar\partial_t\right]\psi_i(r,t) = 0 \\ &\to \int d^3r\, \psi_i^+(r,t)\,[\mathcal{H}_{FL} - i\hbar\partial_t]\,\psi_i(r,t) + \delta V_i(t) = 0 \end{aligned}\right\} \qquad (301)$$

where the "perturbation" $\delta V(r,t)$ takes the form

$$\delta V_i(t) = -\int d^3r\, \psi_i^*(\underline{r},t)\left[\frac{i\hbar}{2m}(\underline{\nabla}\delta\theta\cdot\underline{\nabla} + \nabla^2\delta\theta) + \hbar\partial_t\delta\theta\right]\psi_i(\underline{r},t) \qquad (302)$$

In other words, the perturbation is like an external "gauge field", which is not of course gauge-invariant, but rather an artefact of the gauge transformation. Because the system is isotropic and translationally invariant, the perturbation is better written in energy-momentum space, as (switching now to field operators $\psi(r,t)$, $\psi^+(\underline{r},t)$, and so dropping the particle index):

$$\delta\hat{V} = \int\frac{d\epsilon}{2\pi}\int\frac{d\epsilon'}{2\pi}\sum_{\underline{p}}\sum_{\underline{p}'}\psi_p^+(\epsilon)e^{i(\underline{p}\cdot\underline{r}-\epsilon t)}\,\delta V_{\underline{p}\underline{p}'}(\epsilon,\epsilon')\psi_{p'}(\epsilon')e^{-i(\underline{p}'\cdot\underline{r}-\epsilon' t)} \qquad (303)$$

where

$$\delta V_{\underline{p}\underline{p}'}(\epsilon,\epsilon') = \int d^3r\int dt\, e^{-i(\underline{p}\cdot\underline{r}-\epsilon t)}\,\delta V(\underline{r},t)\,e^{i(\underline{p}'\cdot\underline{r}-\epsilon' t)} \qquad (304)$$

and in the same way we write

$$\delta\theta(\underline{q},\omega) = \int d^3r\int dt\, e^{i(\underline{q}\cdot\underline{r}-\omega t)}\,\delta\theta(\underline{r},t) \qquad (305)$$

Now changing units so that $\underline{p}$ becomes a momentum rather than a wave-number, and ϵ an energy rather than a frequency, we have from (302) that

$$\delta V_{pp'}(\epsilon,\epsilon') = -i\delta\theta(\underline{q},\omega)\left[\frac{\underline{p}+\underline{p}'}{2m}\cdot\underline{q} - \omega\right]\delta(\underline{p}-\underline{p}'+\underline{q})\,\delta(\epsilon-\epsilon'+\omega) \qquad (306)$$

Again, we see that this perturbation is an artefact of the gauge transformation, since it is proportional to $\delta\theta(\underline{q},\omega)$.

However, we still expect this artificial perturbation to have an effect on the fermion propagator $\mathcal{G}_2$, which is also not invariant under these phase shifts. As before (cf. eqtn. (278)) we have

$$\delta\mathcal{G}_2(\underline{r}t;\underline{r}'t') = \mathcal{G}_2(\underline{r}-\underline{r}';t-t')\left[\delta\theta(\underline{r},t) - \delta\theta(\underline{r}',t)\right] \qquad (307)$$

or, Fourier transforming according to

$$\mathcal{G}_2(\underline{p},\underline{p}';\epsilon,\epsilon') = \int d^3r\int d^3r'\int dt\int dt'\, e^{-i(\underline{p}\cdot\underline{r}-\underline{p}'\cdot\underline{r}')}\,e^{i(\epsilon t-\epsilon' t')}\,\mathcal{G}_2(\underline{r},\underline{r}';t,t') \qquad (308)$$

we have

$$\delta\mathcal{G}_2(\underline{p},\underline{p}';\epsilon,\epsilon') = -i\delta\theta_{p'-p}(\epsilon'-\epsilon)\left[\mathcal{G}_2(\underline{p},\epsilon) - \mathcal{G}_2(\underline{p}',\epsilon')\right] \qquad (309)$$

(cf. eqtns. (279) and (280)), which we write as

$$\delta\mathcal{G}_2(\underline{p},\underline{q};\epsilon,\omega) = -i\delta\theta(\underline{q},\omega)\left[\mathcal{G}_2(\underline{p},\epsilon) - \mathcal{G}_2(\underline{p}+\underline{q},\epsilon+\omega)\right] \qquad (310)$$

Now at first glance this is not terribly useful, since for a set of N fermions there is no obvious physical meaning to the gauge transformation in (299)-(300); we are just shifting the phase, and, unlike QED (where the phase change $\delta\Theta(r,t)$ is proportional to the change in the photon field, $\delta A(r,t)$), there is no connection to any physical variable*

However, we can in fact link this to physical quantities, again using the remarkable insights provided by the Landau Fermi Liquid theory (FLT). Recall that the fundamental quantity appearing in that theory was the 4-point vertex $\Gamma_4(p,p';q)$, which in the low-energy regime took the form given in eqtns. (82)-(85). There we saw that the general interaction $t_{pp'}(q,\omega)$ between quasiparticles could be written in terms of the function $f_{pp'}$, and that this function was related to the vertex $\Gamma_0(p,p')$, which was the limiting form of $\Gamma_4(p,p';q)$ in the case where $|\underline{p}|, |\underline{p}'| \sim p_F$, and where $|q| \ll q_F$, $|\omega| \ll \epsilon_F$, and $|q|/\omega \to 0$.

It then becomes clear that we need to write an expression for $\delta\mathcal{G}_2(p,p')$ in terms of the 4-point vertex in Bethe-Salpeter form, as we have already done for the QED system (cf. eqtns. (160), (161), and (164)) and the electron-phonon system (eqtns. (208) and (209)). All that changes here is the vertex connecting the system to the external field δV; thus we immediately have

$$\left.\begin{aligned}\delta\mathcal{G}_2(p,p') = \;& \mathcal{G}_2(p)\,\delta V(p,p')\,\mathcal{G}_2(p') \\ &+ \mathcal{G}_2(p)\mathcal{G}_2(p')\sum_{kk'}\delta V(k,k')\,\mathcal{G}_2(k)\mathcal{G}_2(k')\,\Gamma_4(kk';pp')\,\delta(k+p-k'-p')\end{aligned}\right\} \quad (311)$$

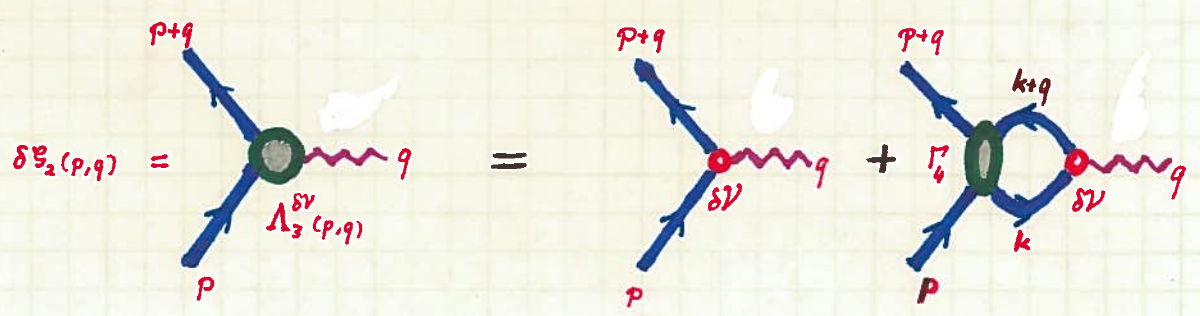
or, using the notation we've used previously, and incorporating energy-momentum conservation, we have

$$\delta\mathcal{G}_2(p,p+q) = \mathcal{G}_2(p)\mathcal{G}_2(p+q)\Big[\delta V(p,p+q) + \sum_k \delta V(k,k+q)\,\mathcal{G}_2(k)\mathcal{G}_2(k+q)\,\Gamma_4(k,p;q)\Big] \quad (312)$$

However this change $\delta\mathcal{G}_2(p,p+q)$ is nothing but the 3-point correlator between the field $\delta V(p,p+q)$ and the fermions, ie.,

$$\delta\mathcal{G}_2(p,q) = G_3^{\delta V}(p,q)\,\delta V(q) \quad (313)$$

The results (312) and (313) can be depicted diagrammatically as in (314).



(314)

* Note that this is not the case if the fermion system has undergone a phase transition to a superfluid state. Then $\Theta(r,t)$ is related to the phase variable of the superfluid state - and the phase of all the fermions are locked together (spontaneous phase symmetry breaking). We discuss this in later chapters.

We see from (314) that we can also write this as a result for the 3-point vertex $\Lambda_3^{\delta V}(p,q)$, in the form

$$\Lambda_3^{\delta V}(p,q) \;=\; \delta V(p,q) \;+\; \sum_k \delta V(k,q)\, \mathcal{G}_2(k+q)\, \mathcal{G}_2(k)\, \Gamma_4(k,p;q) \qquad (315)$$

Let's now gather all these results together. From (306), (310), (312), and (315), we see that

$$\mathcal{G}_2^{-1}(p) - \mathcal{G}_2^{-1}(p+q) \;=\; (\Omega^0_{\underline{pq}} - \omega)\, \Lambda_3^{\delta V}(p,q) \qquad (316)$$

where we have defined

$$\Omega^0_{\underline{pq}} \;:\; \epsilon^0_{\underline{p+q}} - \epsilon^0_{\underline{p}} \;=\; \frac{1}{m}\left(\underline{p}\cdot\underline{q} + \tfrac{1}{2}|\underline{q}|^2\right) \qquad (317)$$

(a factor you will be familiar with from the Appendix on graphs). Eqtn. (316) is just the non-relativistic counterpart to the Ward identity that we found for QED (cf. eqtns. (264) and (274)).

We also have, using (306), (315), and (316), that

$$\mathcal{G}_2^{-1}(p+q) - \mathcal{G}_2^{-1}(p) \;\equiv\; (\omega - \Omega^0_{pq}) + \Sigma(p) - \Sigma(p+q) \qquad (318)$$

$$\mathcal{G}_2^{-1}(p+q) - \mathcal{G}_2^{-1}(p) \;=\; (\omega - \Omega^0_{\underline{pq}}) \;+\; \sum_k (\omega - \Omega^0_{\underline{kq}})\, \mathcal{G}_2(k+q)\, \mathcal{G}_2(k)\, \Gamma_4(k,p;q) \qquad (319)$$

so that

$$\Sigma_{\underline{p}}(\epsilon) - \Sigma_{\underline{p+q}}(\epsilon+\omega) \;=\; \sum_k (\omega - \Omega^0_{\underline{kq}})\, \mathcal{G}_2(k+q)\, \mathcal{G}_2(k)\, \Gamma_4(k,p;q) \qquad (320)$$

which is a general identity relating the 1-particle self-energy with the 4-point vertex. Note that this identity is valid for any value of $p = (\underline{p},\epsilon)$, $q = (\underline{q},\omega)$, and involves the integration over all $k = (\underline{k},E)$.

The identity (320) can be linked up to the usual form of Landau FLT if we go to the limit of small q, ie., $|\underline{q}| \ll k_F$, $|\omega| \ll \epsilon_F$. As we saw previously, both the integral over the energy-momentum k and the form of $\mathcal{G}_2\mathcal{G}_2\Gamma_4$ in the integrand simplify in this limit. Moreover we can drop the term $\propto |\underline{q}|^2$ in (317), and we get

$$\Sigma_{\underline{p}}(\epsilon) - \Sigma_{\underline{p+q}}(\epsilon+\omega) \;\sim\; \left(\underline{q}\cdot\frac{\partial \Sigma_{\underline{p}}(\epsilon)}{\partial |\underline{p}|} + \omega\, \frac{\partial \Sigma_{\underline{p}}(\epsilon)}{\partial \epsilon}\right) \;\sim\; \sum_k \left(\frac{\underline{k}\cdot\underline{q}}{m} - \omega\right) \mathcal{G}_2(k)\, \mathcal{G}_2(k+q)\, \Gamma_4(k,p;q) \qquad (321)$$

Again, using the guide provided to us by Landau FLT, let's consider the 2 key limits here:

(1) "ω-limit" $(|\underline{q}|/\omega \to 0)$: Then we get, from (321), that

$$\frac{\partial \Sigma_{\underline{p}}(\epsilon)}{\partial \epsilon} \;=\; -\sum_k \mathcal{G}_2(k)\, \mathcal{G}_2(k)\, \Gamma_4^0(\underline{k},\underline{p}) \qquad (322)$$

where $\Gamma^0(p,p')$ is just the quantity defined and discussed in eqtns. (80) - (84); using the definition of the Landau f-function $f_{pp'}$ in terms of $\Gamma_0(p,p')$ in (84), and the low-energy form for $\mathcal{G}_2(k+q)\mathcal{G}_2(k)$ given in eqtns: (63), (72), and (78), we can then rewrite this as

$$\left.\begin{aligned}\frac{\partial \Sigma_p(\epsilon)}{\partial \epsilon} &= -\frac{1}{v_F}\sum_{p'} \delta(\epsilon_{p'} - |\mu|)\,\delta(|p'| - k_F)\, f_{\underline{pp}'} \\ &= -N^*(0)\sum_{p'} f_{\underline{pp}'} \quad = -\sum_{p'} F_{\underline{pp}'}\end{aligned}\right\} \tag{323}$$

where $N^*(0)$ is the renormalized quasiparticle density of states.

(ii) "$\underline{k}$-limit" $(|\underline{q}|/\omega \to \infty)$: Then we get

$$\frac{\partial \Sigma_p(\epsilon)}{\partial |\underline{p}|} = \sum_k \frac{\underline{k}}{m}\, \mathcal{G}_2(k+q)\mathcal{G}_2(k)\, \Gamma_4^{\infty}(\underline{k},p') \tag{324}$$

where

$$\Gamma_4^{\infty}(\underline{p},\underline{p}') = \lim_{q\to 0}\ \lim_{|\underline{q}|/\omega\to\infty} \Gamma_{\underline{p},\underline{p}'}(\underline{q},\omega) \tag{325}$$

We can write this in a much simpler form by defining a function $a_{\underline{pp}'}$, which is just the "k-limit" of $t_{pp'}(\underline{q},\omega)$ in (85), ie.,

$$a_{\underline{pp}'} = \lim_{|q|/\omega\to\infty} t_{pp'}(q,\omega) \tag{326}$$

and then defining the dimensionless quantity $A_{\underline{pp}'} = N^*(0)\, a_{\underline{pp}'}$, we have

$$A_{pp'} = F_{\underline{pp}'} - \sum_{\underline{k}} F_{\underline{pk}} A_{\underline{kp}'} \tag{327}$$

and we can rewrite (324) as

$$\frac{\partial \Sigma_p(\epsilon)}{\partial |\underline{p}|} = \sum_{\underline{k}} \frac{\underline{p}'}{m} A_{\underline{pp}'}$$

Clearly one can also derive expressions for the mixed expression in (321) in terms of the general function $t_{pp'}(\underline{q},\omega)$, by a generalization of these manoeuvres.

We see that the technique of applying infinitesimal transformations to vertices and/or correlation functions allows us to obtain lots of useful information about the different vertices, and to relate these to physically measurable quantities - this was particularly clear in the FLT example, where all the renormalization factors were incorporated, so that we have an extremely direct relationship. For relativistic field theories we will be able to see the same once we have done renormalization in Chapter 7.