(A) SPIN AND SPIN REPRESENTATIONS

Spin is a fundamentally non-classical property of matter. It is like angular momentum in its algebraic properties, as well as its symmetries. In both cases one defines a vector operator ($\underline{L}$ or $\underline{S}$) which, when quantised, give a "ladder" of states. The crucial difference is that the ladder for spin is FINITE, and in the limit $\hbar \to 0$, spin disappears.

A.1 OPERATOR ALGEBRA for SPIN & ANGULAR MOMENTUM

We recall here the elementary results for angular momentum & spin algebra - these are covered in standard refs like Landau & Lifshitz [1], or Feynman [2]. The difference between spin & angular momentum is already clear in these fairly elementary results.

A.1.1 ANGULAR MOMENTUM

Angular momentum is a classical quantity - for a set of particles with canonical coordines $\{\underline{r}_j, \underline{p}_j\}$ in phase space we have

$$L = \sum_j \underline{r}_j \times \underline{p}_j \longrightarrow -i\hbar \sum_j \hat{\underline{r}}_j \times \underline{\nabla}_j \qquad (1)$$

where the second expression is the quantum-mechanical operator expression derived from the 1st classical expression.

From the usual result

$$[\underline{r}_j, \underline{p}_j] = i\hbar \qquad (2)$$

which is equivalent to the result

$$\underline{p}_j = -i\hbar \underline{\nabla}_j \qquad (3)$$

we derive the following set of results:

$$\left.\begin{aligned} \hat{L}_\alpha &= \epsilon_{\alpha\beta\gamma} \hat{r}_\beta \hat{p}_\gamma \qquad (\text{eg., } \hat{L}_x = \hbar l_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y) \\ &= \hbar l_\alpha \end{aligned}\right\} \qquad (4)$$

$$[L_\alpha, r_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} r_\gamma \qquad (\text{eg.}, [L_x, y] = i\hbar z) \qquad (5)$$

$$[L_\alpha, p_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} p_\gamma \qquad (\text{eg } [L_x, p_y] = i\hbar p_z) \qquad (6)$$

$$[L_\alpha, L_\beta] = i\hbar^2 \epsilon_{\alpha\beta\gamma} L_\gamma \qquad (7)$$

where the subscripts α, β, γ.. label the directions x, y, z in orbital space/real space. The angular momentum is quantized in units of $\hbar$. We also have

$$[L^2, L_\alpha] = 0 \qquad (8)$$

$$[L_\alpha^2, L_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} [L_\alpha L_\gamma + L_\gamma L_\alpha] \qquad (\text{eg } [L_y^2, L_z] = i\hbar(L_x L_y + L_y L_x)) \qquad (9)$$

where the operator $$\underline{L}^2 = \sum_\alpha L_\alpha^2 \tag{10}$$

Thus the angular momentum components do not commute with each other, and this gives the algebra a quite different character from that of real space operators.

The "ladder operators" are
$$\left.\begin{aligned} \hat{L}_\pm &= \hbar \hat{l}_\pm = \hbar(\hat{l}_x \pm i\hat{l}_y) \\ &= \hat{L}_x \pm i\hat{L}_y \end{aligned}\right\} \tag{11}$$

and they satisfy
$$[\hat{l}_+, \hat{l}_-] = 2l_z \tag{12}$$
$$[\hat{l}_z, \hat{l}_\pm] = \pm l_\pm \tag{13}$$

and we may write equivalently that
$$\left.\begin{aligned} &\hat{l}_z = \tfrac{1}{2}[l_+, l_-] \\ &l_x = \tfrac{1}{2}(\hat{l}_+ + \hat{l}_-) \qquad \hat{l}_y = -i/_2(\hat{l}_+ - \hat{l}_-) \end{aligned}\right\} \tag{14}$$

and also note that
$$L^2 = L_z^2 + \tfrac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+) \tag{15}$$

and that
$$[L^2, L_\pm] = 0 \tag{16}.$$

Finally, note that we can also write, using (7), that
$$\hat{\underline{L}} \times \hat{\underline{L}} = i\hbar \hat{\underline{L}} \tag{17}$$

A.1.2 ANGULAR MOMENTUM & ROTATIONS

Starting again from the classical angular momentum we can discuss the angular momentum in terms of its canonical conjugate rotation variable. To properly derive this, we proceed, in classical mechanics, by noting that the angular momentum is the generator of rotations: if we rotate the system by an infinitesimal angle $\delta\underline{\phi}$, then the radial coordinate changes according to

$$\left.\begin{aligned} &\underline{r} \to \underline{r}'(\delta\underline{\phi}) = \underline{r} - (\underline{r} \times \delta\underline{\phi}) = \underline{r} + \delta\underline{r} \\ &\text{and under a finite rot}^{n}\ \underline{\phi}, \quad \underline{r} \to \underline{r}'(\underline{\phi}) = \underline{r} - (\underline{r} \times \underline{\phi}) \end{aligned}\right\} \tag{18}$$

which we write as
$$\underline{r}'(\underline{\phi}) = \hat{R}(\underline{\phi})\underline{r} \tag{19}$$

where $\hat{R}(\underline{\phi})$ is a rotation operator.

How then does a wave-function transform? Under $\delta\underline{\phi}$ we have

$$\left.\begin{aligned} \psi(\underline{r}) \to \psi(\underline{r}') &= \psi(\underline{r}) + \delta\underline{r}\cdot\nabla\psi(\underline{r}) = \psi(\underline{r}) - (\underline{r} \times \delta\underline{\phi})\cdot\nabla\psi(\underline{r}) \\ &\equiv U(\delta\underline{\phi})\psi(\underline{r}) \end{aligned}\right\} \tag{20}$$

where the infinitesimal rot^n operator is

$$\left.\begin{aligned}\hat{U}(\delta\underline{\phi}) &= 1-(\underline{r}\times\delta\underline{\phi}).\underline{\nabla} \\ &= 1+\delta\underline{\phi}.(\underline{r}\times\underline{\nabla}) = 1+\frac{i}{\hbar}\underline{L}.\delta\underline{\phi}\end{aligned}\right\} \quad (21)$$

which we can also write in the form

$$-i\hbar\frac{\partial\psi}{\partial\underline{\phi}} = \underline{L}\psi \quad (22)$$

showing that the angular momentum $\underline{L}$ is conjugate to the angle $\underline{\phi}$. We can if we wish write out this result in terms of the polar & azimuthal angles:

$$\hat{L}_z = -i\hbar\,\partial_\phi \quad (23)$$

$$\hat{L}_\pm = i\hbar e^{\pm i\phi}[\cot\theta\,\partial_\phi \pm \partial_\theta] \quad (24)$$

and, using (15),

$$L^2 = -\hbar^2\left[\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\right] \quad (25)$$

From (23) we can immediately show that the numbers arising from equations like (6), or (12)-(14), are integers. Consider the eigenvalue eqtn (22) written along the $\hat{z}$-axis:

$$\left.\begin{aligned}-i\hbar\frac{\partial\psi}{\partial\phi} &= \hat{L}_z\psi \\ &= \hbar\hat{l}_z\psi\end{aligned}\right\} \quad (26)$$

The solution to the eqtn $-i(\partial_\phi\psi) = l_z\psi$ must have form $\psi = e^{il_z\phi}u(r,\theta)$, so that we immediately come to the conclusion that

$$\left.\begin{aligned}l_z &= m \qquad (m = 0, \pm1, \pm2, \ldots) \\ L_z &= \hbar m = M\end{aligned}\right\} \quad (27)$$

because ψ has to be single-valued.

It is fairly easy to demonstrate that the following results also derive from what we have so far. First, the eigenvalue of L^2 is uniquely given by

$$\left.\begin{aligned}\hat{L}^2\psi &= L(L+1)\psi = \hbar^2 l(l+1)\psi \\ \text{where } l &= 0, \pm1, \pm2, \ldots\end{aligned}\right\} \quad (28)$$

and for a given l, the values of m range over

$$-l \leq m \leq l \quad (29)$$

so that for a given l, there are $(2l+1)$ possible values of m. Next, one can show that a complete set of states for the system is given by a set of eigenfunctions $|L,M\rangle$, such that

$$\left.\begin{aligned} \hat{L}^2|L,M\rangle &= L(L+1)|L,M\rangle \\ \hat{L}_z|L,M\rangle &= M|L,M\rangle \end{aligned}\right\} \qquad (30)$$

These states are simply vectors in a $(2\ell+1)$-dimensional Hilbert space. More familiar to most is the inner product of these states with a set of states $|\theta,\phi\rangle$, which are localised at points on the unit sphere, according to

$$\langle\theta\phi|\theta'\phi'\rangle = \delta(\phi-\phi')\delta(\theta-\theta') \qquad (31)$$

and where

$$\langle\theta\phi|L,M\rangle = Y_{\ell m}(\theta,\phi) \qquad (32)$$

where $Y_{\ell m}(\theta,\phi)$ is the normalised spherical harmonic:

$$\begin{aligned} Y_{\ell m}(\theta,\phi) &= \Theta_{\ell m}(\theta)\,\Phi_m(\phi) \\ &= (-1)^m i^\ell\left[\frac{2\ell+1}{2}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_{\ell m}(\cos\theta)\frac{1}{\sqrt{2\pi}}e^{im\phi} \end{aligned} \qquad (33)$$

where we normalise $\Phi_m(\phi)$ as

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi} \qquad (34)$$

and the $P_{\ell m}$ are the associated Legendre polynomials:

$$\left.\begin{aligned} P_{\ell m}(\mu) &= (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_\ell(\mu) \\ P_\ell(\mu) &= \left(-\frac{1}{2}\right)^{\ell}\frac{1}{\ell!}\frac{d^\ell}{d\mu^\ell}(1-\mu^2)^\ell \end{aligned}\right\} \qquad (35)$$

in terms of the Legendre polynomials P_ℓ. The $Y_{\ell m}(\theta,\phi)$ satisfy the differential eqtn

$$\hat{L}^2\psi(\theta,\phi) = L(L+1)\psi(\theta,\phi) \qquad (36)$$

from (30), which when written out using (25) becomes

$$\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \ell(\ell+1)\right]\psi(\theta,\phi) = 0 \qquad (37)$$

For a general understanding of angular momentum we do not need to remember the detailed form of these functions. Note the "orthogonality" relation

$$\left.\begin{aligned} \langle L'M'|LM\rangle &\equiv \langle L'M'|\theta\phi\rangle\langle\theta\phi|LM\rangle \\ &\equiv \int d\theta\sin\theta\int d\phi\; Y^*_{\ell'm'}(\theta,\phi)\,Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'} \end{aligned}\right\} \qquad (38)$$

where we use the usual Dirac bra/ket notation & the "summation convention", according to which repeated indices or variables are summed/integrated over. Note also that an area element $d\underline{S}$ on the Bloch sphere is given by

$$dS = d\mu \, d\phi = \sin\theta \, d\theta \, d\phi \qquad (39)$$

One can go on from here to study the algebra of pairs of angular momenta, and so on.

Note finally that the classical limit for angular momentum, starting from a finite L^2, is given by

$$\left.\begin{array}{l} \underline{L} = \text{const} \\ \hbar \to 0 \qquad |\underline{\ell}| = \ell \to \infty \end{array}\right\} \qquad (40)$$

A.1.3 <u>SPIN ALGEBRA</u>

A lot of what is said above for angular momentum is also true for spin, except that the spin is entirely quantum-mechanical:

$$|\underline{S}| \xrightarrow[\hbar \to 0]{} 0 \qquad (41)$$

Spin is not associated with an extended set of rotating particles - instead, it is considered to be a property of "elementary" particles, and these have definite values of spin.

We may therefore write

$$[\hat{S}_\alpha, \hat{S}_\beta] = i\hbar^2 \epsilon_{\alpha\beta\gamma} \hat{S}_\gamma \qquad (42)$$

$$\hat{\underline{S}} = \hbar \underline{s} \qquad (43)$$

$$[\hat{S}^2, \hat{S}_\alpha] = 0 \qquad [\hat{S}^2_\alpha, \hat{S}_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} (\hat{S}_\alpha \hat{S}_\gamma + \hat{S}_\gamma \hat{S}_\alpha) \qquad (44)$$

Moreover, all the relations in (10)-(17) also hold for $\underline{S}$ and $\underline{s}$, with the obvious substitutions $\underline{L} \to \underline{S}$ and $\underline{\ell} \to \underline{s}$.

However, an important part of the theory for angular momenta is not directly applicable to spin states, just because we cannot simply write eqtns. like (22) or (26); spin is not considered to exist in real space.

In fact, one finds experimentally that spin is quantized in units of ONE HALF; we have

$$m_s = 0, \pm \tfrac{1}{2}, \pm 1, \pm \tfrac{3}{2}, \ldots \qquad (45)$$

$$S_z = \hbar s_z = \hbar m_s \equiv M_s \qquad (46)$$

Analagous results to (28)-(30) also follow, but we cannot treat the spin states in terms of the real space angular states as before.

Thus we have a set of states $|\chi(\underline{S})\rangle \equiv |S, M_s\rangle$, where

$$\left.\begin{aligned} \hat{S}^2 |S, M_s\rangle &= S(S+1)\,|S, M_s\rangle \\ \hat{S}_z |S, M_s\rangle &= M_s\,|S, M_s\rangle \end{aligned}\right\} \tag{47}$$

$$\text{with} \quad \left.\begin{aligned} S &= 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \\ -S &\leq M_s \leq S \end{aligned}\right\} \tag{48}$$

and we also have orthogonality of these states:

$$\langle S' M_s' | S\, M_s\rangle = \delta_{SS'}\,\delta_{M_s M_s'} \tag{49}$$

However, it should be noted that if we do perform a rotation in space, the spin state (usually called a "SPINOR") will transform. Since experimentally one finds that spin & angular momentum just add, to give a total angular momentum

$$\underline{J} = \underline{L} + \underline{S} \tag{50}$$

it follows that we still have
$$-i\partial_\phi \chi(\underline{S}) = m_s \chi(\underline{S}) \tag{51}$$

in analogy with (22), but since m_s can take half-integer values, the spinor wave-function is not necessarily single-valued as before.

Of particular interest is the spin-½ system, with $m_s = \frac{1}{2}$, for which we have

$$\left.\begin{aligned} \hat{s}_\alpha &= \tfrac{1}{2}\hat{\sigma}_\alpha \\ [\hat{\sigma}_\alpha, \sigma_\beta] &= i\epsilon_{\alpha\beta\gamma}\sigma_\gamma \\ \underline{s}^2 |\chi\rangle &= \tfrac{3}{4}\,|\chi\rangle \end{aligned}\right\} \tag{52}$$

where the $\{\hat{\sigma}_\alpha\}$ are the famous "Pauli spin operators". Note the following useful result, valid for any 2 vectors in "spin space", for spin-½ systems:

$$(\hat{\underline{S}}\cdot\underline{V}_1)(s\cdot\underline{V}_2) = \tfrac{1}{4}(\underline{V}_1\cdot\underline{V}_2) + \frac{i}{2}\underline{S}\cdot(\underline{V}_1\times\underline{V}_2) \tag{53}$$

A.1.4. MATRIX REPRESENTATION & MATRIX ELEMENTS

Finally, we can evaluate the matrix elements of these various operators between the different states. The operators $\underline{L}^2$, L_z, $\underline{S}^2$, and S_z are diagonal in the representation given above, so we already have these results in eqtns. (30), (47), & (48). However we also need the matrix elements of other operators. These are easily derived and found to be as follows [next page]:

$$\left.\begin{aligned} \langle M+1|L_x|M\rangle &= \langle M|L_x|M+1\rangle = \tfrac{1}{2}\left[(\ell+m+1)(\ell-m)\right]^{1/2} \\ \langle M+1|L_y|M\rangle &= -\langle M|L_y|M+1\rangle = \tfrac{-i}{2}\left[(\ell+m+1)(\ell-m)\right]^{1/2} \end{aligned}\right\} \quad (54)$$

and

$$\langle M+1|L_+|M\rangle = \langle M|L_-|M+1\rangle = \left[(\ell+m+1)(\ell-m)\right]^{1/2} \quad (55)$$

with analogous results for spin. In the special case of the spin-½ particle, we have $\hat{S}_\alpha = \frac{1}{2}\hat{\sigma}_\alpha$ from (52), with

$$\left.\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}\right\} \quad (56)$$

as the representation of the Pauli matrices in the basis where $\hat{S}_z$ is diagonal.

A.2. COHERENT STATES for SPIN & ANGULAR MOMENTA

Although we introduced an abstract set of states $|\theta,\phi\rangle$ that were supposed to pick out a particular direction, these are not terribly useful because for any finite ℓ or S, we cannot form them using a set of spin wave-functions (this is obvious, since we have at most $(2\ell+1)$ or $(2s+1)$ orthogonal states, and there is no way we can make a δ-function on the Bloch sphere with these).

For this reason one defines a set of states which are peaked around a given direction, and which in fact give the best way of representing a "wave-packet" localised in some particular direction.

One way to define these is to start off with a state which is quite well localised along the $\hat{z}$-direction, and then rotate it around the sphere. Thus we start with the state

$$|S,S\rangle \equiv |S, M_s = S\rangle \quad (57)$$

and we rotate it to the direction we want by applying a rotation operator:

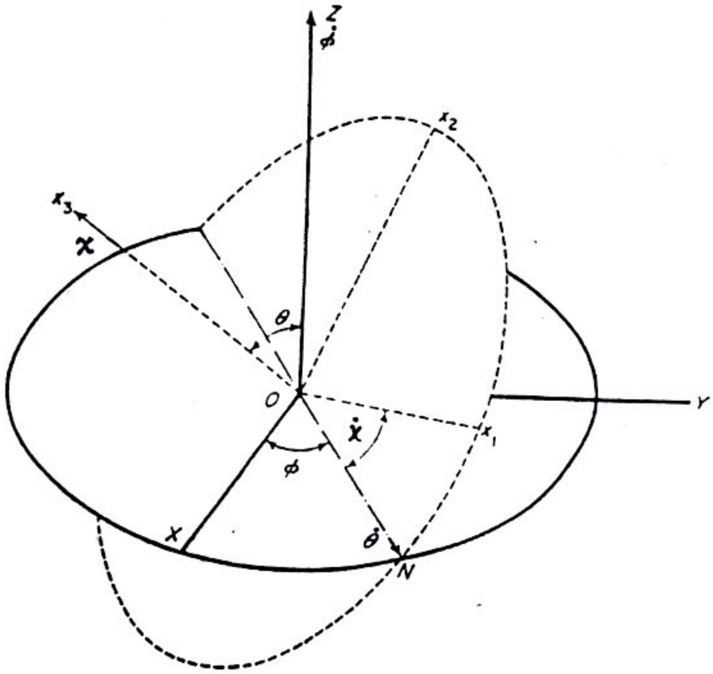
$$|\underline{\Omega}_s\rangle = R(\hat{\underline{\Omega}})\,|S,S\rangle \quad (58)$$

where $\hat{\underline{\Omega}} = (\theta,\phi)$ is the direction of the state on the Bloch sphere.

The standard way of parametrizing rotations, in either spin or orbital space, is to use the Euler angles – see Figure. The rotation operator can then be written in the form

$$R(\underline{\hat{\Omega}}) = e^{iS_z\phi}\, e^{iS_y\theta}\, e^{iS_z\chi} \quad (59)$$

in terms of the 3 Euler angles ϕ, θ, and χ. Note that the S_z operator is applied twice in this rotation operator.



THE EULER ANGLES

There are various ways of deriving explicit expressions for the coherent states. Before beginning it is useful to recall some salient features of coherent states that are usually derived for the harmonic oscillator. Suppose we have an oscillator with ground state $|0\rangle$ and excited state $|n\rangle$. The entire set $\{|n\rangle\}$ with $n = 0, 1, \ldots \infty$ forms a complete set of states for the SHO.

The the coherent states $|z\rangle$ for the SHO are defined as

$$|z\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}|z|^2} e^{za^+} |0\rangle \quad (59)$$

and if we call that the SHO state $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n |0\rangle$, we also have

$$|z\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/2}} |n\rangle \quad (60)$$

Note that $z = x + iy = z_0 e^{i\phi}$ is just a complex number – we have now moved the action to the z-plane, where in fact the coherent state functions look just like 2-d SHO wave-functions:

$$\left.\begin{aligned} |z\rangle &= \sum_{n=0}^{\infty} \psi_n(z)\, |n\rangle \\ \psi_n(z) &= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}z_0^2} \frac{z_0^n}{\sqrt{n!}} e^{in\phi} \end{aligned}\right\} \quad (61)$$

It is easily verified that the z-states form a complete set of states – in fact they are overcomplete, since we not only have

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \int d^2z\, |z\rangle\langle z| = 1 \quad (62)$$

but we also have a non-zero overlap between states of different z:

$$\langle z_1 | z_2 \rangle = \frac{1}{\pi} e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} e^{z_1^* z_2} \qquad (63)$$

Note that the completeness follows simply because the $\psi_n(z)$ are orthogonal, and so would happen no matter what set of $\{\psi_n(z)\}$ was chosen to make the $\{|z\rangle\}$ functions. The OVER-completeness follows because the dimension of z is larger than that of the original 1-d oscillator.

There are various reasons for choosing the $\psi_n(z)$ in the form given, which I won't go into here. Let us now turn to spin coherent states, which we develop first by direct analogy with oscillator coherent states. We define the <u>normalised</u> states

$$|Z\rangle = \frac{1}{(1+|Z|^2)^S} e^{Z\hat{S}_-} |S,S\rangle \qquad (63)$$

where $|S,S\rangle$ now acts as the vacuum state. If we write this as a sum over the states with different M we get

$$|Z\rangle = \frac{1}{(1+|Z|^2)^S} \sum_{M=-S}^{S} \left(\frac{(2S)!}{(S-m)!\,(S+m)!} \right)^{\frac{1}{2}} Z^{S-m} |S,M\rangle \qquad (64)$$

To get a proper complete set relationship we must include a weighting function in the integration over Z; we have

$$\frac{2S+1}{\pi} \int \frac{d^2 Z}{(1+|Z|^2)^2} |Z\rangle\langle Z| = \sum_{M=-S}^{S} |S,M\rangle\langle S,M| = 1 \qquad (65)$$

At first glance it is not clear what this complex variable Z is supposed to represent, but it becomes clearer once we make the standard Riemann construction

$$Z = \tan\frac{\theta}{2}\, e^{i\phi} \qquad (66)$$

which maps the Z-plane onto the unit sphere. We then find that on this Bloch sphere we can define a unit vector $\underline{n}$ with coordinates θ, ϕ as defined by (66), so that now

$$|\underline{n}\rangle = \left(\cos\frac{\theta}{2}\right)^{2S} \exp\left[\tan\frac{\theta}{2}\, e^{i\phi}\, \hat{S}_-\right] |S,S\rangle \qquad (67)$$

which, when written as a sum over M-states, gives

$$|\underline{n}\rangle = \sum_{m=-S}^{S} \left(\frac{2S!}{(S-m)!\,(S+m)!} \right)^{\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{S+m} \left(\sin\frac{\theta}{2}\right)^{S-m} e^{im\phi} |S,M\rangle \qquad (68)$$

with a completeness relation

$$\left.\begin{aligned} (2S+1)\int d\underline{n}\, |\underline{n}\rangle\langle\underline{n}| &\equiv (2S+1)\int \frac{d\phi\, d\theta}{4\pi} \sin\theta\, |\underline{n}\rangle\langle\underline{n}| \\ &= 1 \end{aligned}\right\} \qquad (69)$$

All of this looks fairly complicated but will become clearer. From (68) we can calculate the overlap between 2 different states $|\underline{n}_1\rangle$ and $|\underline{n}_2\rangle$:

$$\left.\begin{aligned} \langle \underline{n}_1 | \underline{n}_2 \rangle &= \left[\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\, e^{-i(\phi_1-\phi_2)}\right]^{2S} \\ &= \left(\frac{1+\underline{n}_1\cdot\underline{n}_2}{2}\right)^{S} e^{iS\beta_{12}} \end{aligned}\right\} \quad (70)$$

where

$$\tan\frac{\beta_{12}}{2} = \tan\left(\frac{\phi_1-\phi_2}{2}\right)\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} \quad (71)$$

We see again that the coherent states are overcomplete. Now, suppose calculate the expectation value of the spin operators on these states. It is a useful exercise to show that

$$\left.\begin{aligned} \langle \underline{n} | \hat{S}_z | \underline{n} \rangle &= S\cos\theta \\ \langle \underline{n} | \hat{S}_\pm | \underline{n} \rangle &= S\sin\theta\, e^{\pm i\phi} \end{aligned}\right\} \quad (72)$$

from which we immediately have

$$\langle \underline{n} | \hat{\underline{S}} | \underline{n} \rangle = S\underline{n} \quad (73)$$

and one can show that more generally we can calculate expectation values of any spin operator $f(\underline{S})$ in the basis of coherent states:

$$\left.\begin{aligned} \langle f(\hat{S}) \rangle &= \sum_{MM'} \langle S,M | f(S) | S,M' \rangle \\ &= (2S+1)\int d\underline{n}\, \langle \underline{n} | f(\hat{S}) | \underline{n} \rangle \end{aligned}\right\} \quad (74)$$

From these results the meaning of these functions starts to become a little clearer. From (73) it will be no surprise to find that if we apply the rotation operator in (59) to the state $|\underline{n}\rangle = |\hat{\underline{z}}\rangle \equiv |S,S\rangle$, we get

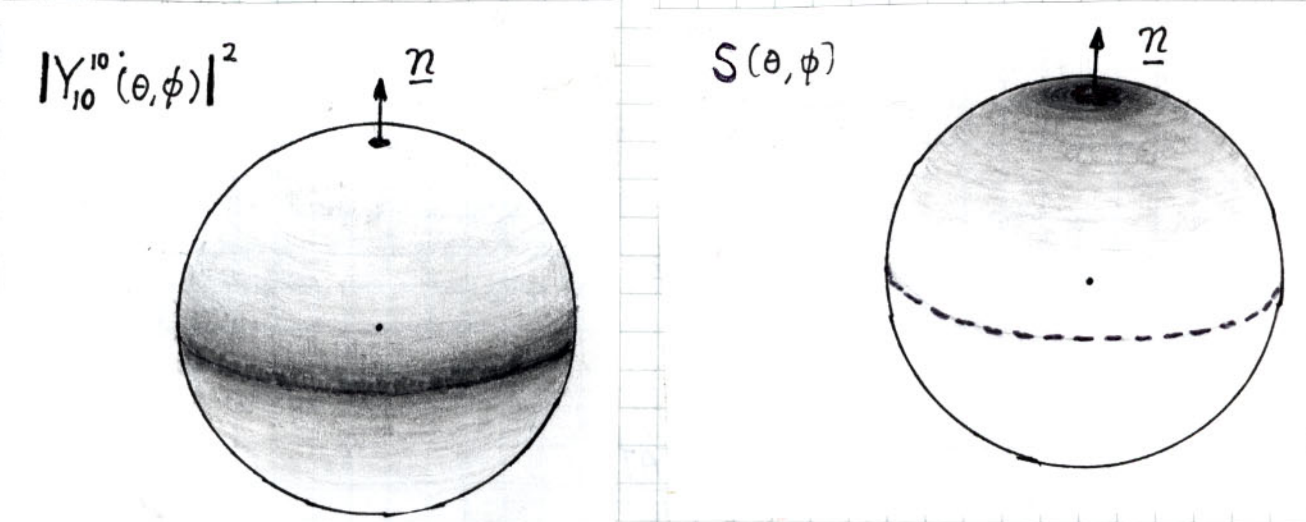
$$R(\hat{\underline{\Omega}})\,|\hat{\underline{z}}\rangle = |\underline{n}_\Omega\rangle \quad (75)$$

with $|n\rangle$ pointed along the vector $\hat{\underline{\Omega}}$. A direct calculation of this result is lengthy, but we will see shortly how it can be made a little simpler.

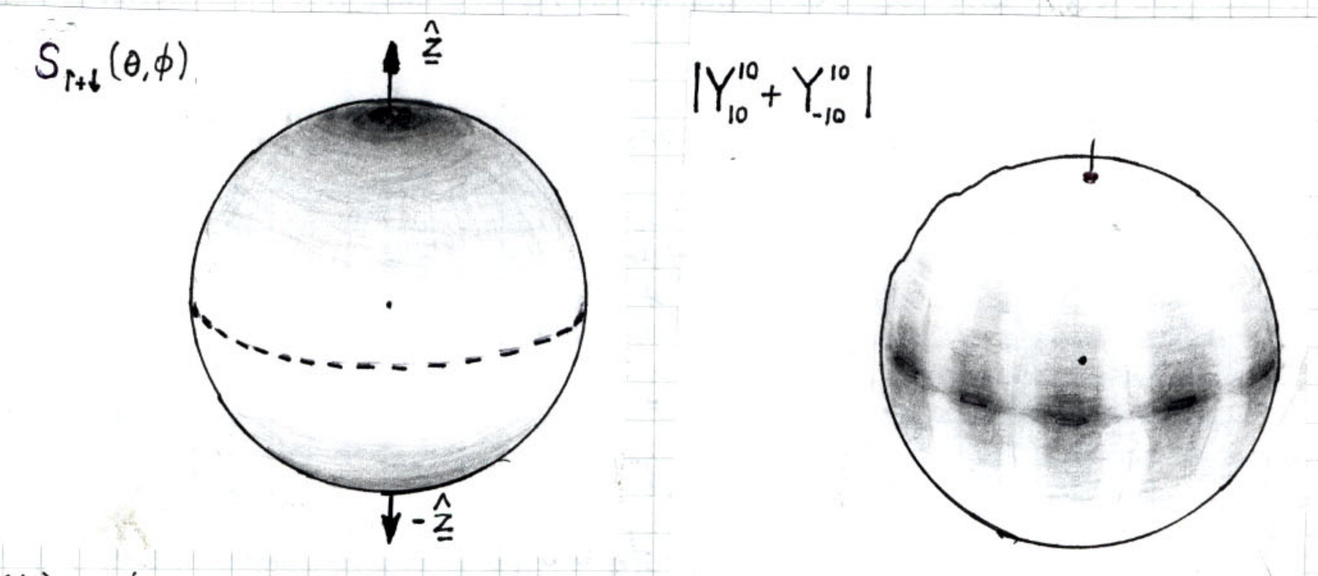
From equation (70) we see that we can think of the spin associated with the coherent states as being attached to a region on the Bloch sphere around the direction $\underline{n}$, and spread out over an area of roughly $4\pi/S$. Alternatively we can think of expanding the sphere to a radius S, in which case the state extends over an area $\sim 4\pi$.

This spreading reflects the finite-dimensional nature of our Hilbert space - we only have a finite number $(2S+1)$ states available to properly define a given position on the Bloch sphere; Thus the "smearing" of the spin coherent states $|n\rangle$ around the direction $\underline{n}$ on the Bloch sphere simply reflects the QM uncertainty in the definition of the spin.

I emphasize immediately that this does not mean that the actual coherent state wave-function is localised around the direction $\underline{n}$ on the Bloch sphere - only the spin is. To see this it is useful to draw a few pictures; we consider a spin $S = 10$ system.



(a) Consider first the vacuum state $|S,M\rangle = |10,10\rangle$; this has the magnitude shown at left, with the spin density shown at right. We have a "current" circulating around the equator.

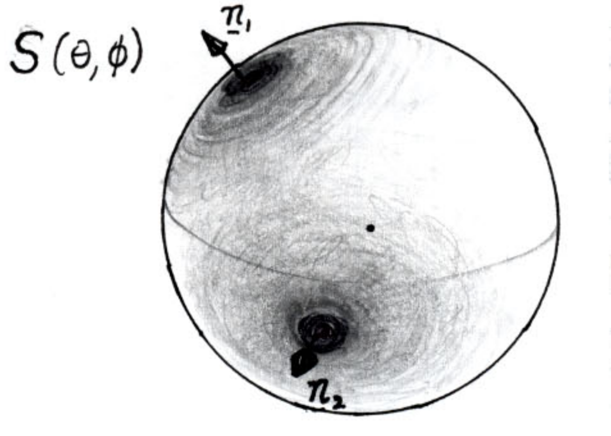


(b) Now suppose we form a superposition of states:

$$|\psi_{\uparrow+\downarrow}\rangle = \frac{1}{\sqrt{2}}\left(|10,10\rangle + |10,-10\rangle\right) \qquad (76)$$

From the point of view of spin, we are simply adding states which are polarised along $\hat{\underline{z}}$ and $-\hat{\underline{z}}$, hence the spin density shown at left. The actual coherent state wave-function is still concentrated around the equator, but now with a 'standing-wave' interference pattern between the 2 states.

We can of course superpose coherent states which are oriented in arbitrary directions; an example is shown at left. Although I have not drawn it, the coherent state wave-function for the sum of these is easily visualised – it is a sum of currents orbiting the 2 vectors $\underline{n}_1$ and $\underline{n}_2$, along the "equators" for these 2 vectors, and with interference between the two appearing where they overlap.



We shall see that the coherent state representation is particularly useful when we come to do semiclassical calculations for spin. In this context it is useful to look at the limit of large S, and look at bosonic representations for spin, which we now do.

A.3 BOSONIC REPRESENTATIONS for SPIN & ANGULAR MOMENTA

It is very tempting to try and construct either bosonic or fermionic reps. for spin degrees of freedom. As we shall see, certain problems arise when we try & do this, but nevertheless under the right circumstances these constructions are very useful, and when employed as part of a theory, lead to powerful methods of analysis.

We begin with 2 fairly well known representations, which we discuss for a single spin or angular momentum.

A.3.1 HOLSTEIN-PRIMAKOFF REPRESENTATION :

Suppose we again assume that our vacuum state is the "maximally up" state $|S,S\rangle$. Then it seems intuitively natural model small fluctuations about this state as though they were like small fluctuations of an oscillator. This tactic was first used in the very early days of quantum mechanics, and a particularly useful formulation was given by Holstein & Primakoff in 1940. We write

$$\left.\begin{aligned} \hat{S}_z &= S - b^+b \\ \hat{S}^+ &= (2S)^{1/2}\left[1-\tfrac{1}{2S}b^+b\right]^{1/2} b \\ \hat{S}^- &= (2S)^{1/2}\, b^+\left[1-\tfrac{1}{2S}b^+b\right]^{1/2} \end{aligned}\right\} \qquad (77)$$

where the bosonic operators satisfy the usual commutation relationships:

$$[b,b^+] = 1 \qquad\qquad [b,b] = 0 \qquad (78)$$

These expressions should only be used if the occupation number $n = \langle b^+b\rangle$ is small, ie., $n \ll 2S$ (although in many applications it is actually found that accurate calculations can be done for quite large n). In this case it makes sense to expand the square roots in $\hat{S}_\pm$, to get

$$\left.\begin{aligned} \hat{S}_+ &= (2S)^{1/2}\left[1 - \frac{1}{4S}b^+b - \frac{1}{32S^2}b^+bb^+b + \cdots\right]b \\ \hat{S}_- &= (2S)^{1/2}\, b^+\left[1 - \frac{1}{4S}b^+b - \frac{1}{32S^2}b^+bb^+b + \cdots\right] \end{aligned}\right\} \qquad (79)$$

This "$1/S$ - expansion" forms the basis of spin wave expansions and the theory of interacting magnons.

In passing, we note that these bosonic oscillator states are also obtained directly from the coherent spin states of the last sub-section. Suppose we write

$$z_s = (2S)^{1/2} Z \qquad (80)$$

where Z is the complex variable which appears in the coherent spin state in (63). Now, if we take the lowest order term in the $1/S$ expansion (79) for S_- (ie., that $\hat{S}_- = (2S)^{1/2} b^+$) we get for the coherent state in (63) that

$$|Z\rangle \rightarrow |z_s\rangle = \frac{1}{(1+|z_s|^2/2S)^S}\, e^{z_s b^+}\, |0\rangle \qquad (81)$$

where we have now renamed the vacuum state: $|S,S\rangle \equiv |0\rangle$. In the large S limit

$$(1+|z_s|^2/2S)^S \xrightarrow[S\to\infty]{} e^{\frac{1}{2}|z_s|^2} \qquad (82)$$

so that $$|z_s\rangle \xrightarrow[S\to\infty]{} e^{-\frac{1}{2}|z_s|^2} e^{zb^+}|0\rangle \qquad (83)$$

so that we recover the SHO coherent state in (59) (apart from a normalization factor).

A.3.2 SCHWINGER BOSONS

We have already seen how it is possible to map between spin & oscillator states in one way, and in fact it is quite fun to play around with different possible ways of doing this. For example, you can verify quite easily that by writing some component of angular momentum $L_k = \epsilon_{ijk} r_i p_j$, and then writing this in terms of oscillator states, we end up with an expression like $L_k \propto i(b_j^+ b_i - b_i^+ b_j)$, in terms of 2 different bosons b_i and b_j.

The treatment of Schwinger is a refinement of this idea. We consider two sets of bosons, created by operators a^+ and b^+, and represent the spin operators as

$$\left.\begin{aligned} \hat{S}_x &= a^+b + b^+a \qquad \hat{S}_y = -i(a^+b - b^+a) \\ \hat{S}_z &= \tfrac{1}{2}(a^+a - b^+b) = (\hat{n}_a - \hat{n}_b)/2 \end{aligned}\right\} \qquad (84)$$

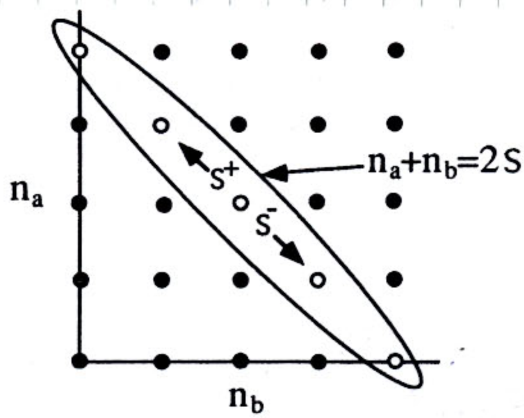
so that $$\left.\begin{aligned} \hat{S}_+ &= a^+b \\ \hat{S}_- &= b^+a \end{aligned}\right\} \qquad (85)$$

and we have the constraint $$S = \tfrac{1}{2}(a^+a + b^+b) = \tfrac{1}{2}(\hat{n}_a + \hat{n}_b) \qquad (86)$$

The vacuum state here is a little different, as we see if we construct the spin states that we are used to. The point is that we are not allowed to have no bosons in the system - some of the states must always be occupied. In fact we must have, from (86), a set 2S bosons at any time in the system, whereas the vacuum state has none. From (84) we have

$$|S, M_s\rangle = \frac{1}{[(S+m)!(S-m)!]^{1/2}} (a^+)^{S+m} (b^+)^{S-m} |0\rangle \qquad (87)$$

where $|0\rangle$ is the vacuum state. The rather peculiar construction of the Hilbert space is shown in the commonly-used diagram shown at left. Any ladder operator automatically increases one set of bosons, taking away from the other.



If we eliminate the "a" bosons using the constraint (86), we just get back the Holstein-Primakoff bosons, with $b_{Schwinger} \leftrightarrow b_{HP}$, and

$$\left. \begin{aligned} a &= (2S)^{1/2} \left[1 - \frac{1}{2S} b^+ b\right]^{1/2} \\ &\equiv a^+ \end{aligned} \right\} \qquad (88)$$

A.3.3 SPIN ROTATIONS IN THE BOSONIC REPRESENTATIONS:

There is a very unwieldy algebra associated with rotations in spin space, but things are not quite so bad when dealing with bosonic representations. From what you know about spin algebra the following results may seem fairly obvious, but it is useful to derive them.

Consider again the rotation defined by the operator in (58). We wish to apply this either to a state like $|S, S\rangle$ (or perhaps even $|S, M_s\rangle$), or perhaps applied directly to one of the "excited" states. One way to do this is to start from the Schwinger boson representation itself, and ask how these operators transform. This must be a unitary transformation, and the standard result is that

$$R(\underline{\Omega}) \begin{pmatrix} a \\ b \end{pmatrix} R^{-1}(\underline{\Omega}) = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\phi/2} & \sin\frac{\theta}{2} e^{-i\phi/2} \\ -\sin\frac{\theta}{2} e^{i\phi/2} & \cos\frac{\theta}{2} e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad (89)$$

so that a pair of Schwinger bosons transforms as $a \to (u(\underline{\Omega}) a + v(\underline{\Omega}) b)$, and $b \to (-v^*(\underline{\Omega}) a + u^*(\underline{\Omega}) b)$, with

$$\left. \begin{aligned} u(\underline{\Omega}) &= \cos\frac{\theta}{2} e^{i\phi/2} \\ v(\underline{\Omega}) &= \sin\theta/2\, e^{-i\phi/2} \end{aligned} \right\} \qquad (90)$$

Actually this result is not quite the most general we can write down, as can be seen if we start from the form for $R(\underline{\Omega})$ in terms of Euler angles given in (59). Suppose, for example, we rotate from $\hat{z}$ to $\hat{n}$, so that we want to know what happens to operators like S_z. This transforms in the usual way

as
$$\left.\begin{aligned}\hat{S}_z &\to R(\Omega)\, S_z\, R^{-1}(\Omega) \\ &= \hat{\underline{S}}.\hat{\underline{n}} \equiv \left[S_x \sin\theta\cos\phi + S_y \sin\theta \sin\phi + \hat{S}_z \cos\theta \right]\end{aligned}\right\} \quad (91)$$

If we now compare this with what we get by direct application of (59), we see that we can actually have the more general result

$$\left.\begin{aligned} u(\underline{\Omega}) &= \cos\frac{\theta}{2}\, e^{i\phi/2}\, e^{i\chi/2} \\ v(\underline{\Omega}) &= \sin\theta/2\, e^{-i\phi/2}\, e^{i\chi/2}. \end{aligned}\right\} \quad (92)$$

This extra degree of freedom, the angular variable χ, is usually of no interest - it is a "gauge variable", corresponding to rotations about $\underline{S}$, and usually we will fix $\chi = 0$.

Using these results we can now also re-derive our previous result for the form of the coherent state vector $|\underline{n}\rangle$ (see eqtns (67) and (68)). To do this, we note that in Schwinger boson language, using (87), we have

$$\left.\begin{aligned} |\hat{\underline{n}}\rangle &= \frac{(\hat{a}^+)^{2S}}{(2S!)^{1/2}}\,|0\rangle = \frac{(ua^+ + vb^+)^{2S}}{(2S!)^{1/2}}\,|0\rangle \\ &= \sum_{m=-S}^{S} \left(\frac{2S!}{(S-m)!\,(S+m)!} \right)^{1/2} u^{S+m}\, v^{S-m}\, |S, M_s\rangle \end{aligned}\right\} \quad (93)$$

where $\hat{a} = RaR^{-1}$ is the transformed operator, and the u and v are as in (90). Note that the result (93) is precisely what was derived in (68).

A.3.4 MAJORANA SPINS on BLOCH SPHERE :

Finally I discuss quite briefly a rather cute representation of a general spin state for a spin-S system, due to Majorana in 1932; it seems to be very little known (and should not be confused with the Majorana spins commonly used in the theory of strongly correlated systems).

The general idea can be appreciated by taking again a single spin, and writing the general state:

$$|\psi\rangle = \sum_{m=-S}^{S} \alpha_m |S, M_s\rangle \quad (94)$$

where the $\{\alpha_m\}$ are all complex, and $\sum_m |\alpha_m|^2 = 1$. Now define the polynomial

$$P(x) = \sum_{m=-S}^{S} c_m x^{S-m} \equiv c_S + c_{S-1}x + c_{S-2}x^2 + \cdots c_{-S}x^{2S+1} \quad (95)$$

$$c_m = \left(C^{2S+1}_{S-m} \right)^{1/2} \alpha_M \quad (96)$$

which has the complex roots $\{z_j\}$:

$$P(x) = (x - z_1)(x - z_2) \cdots (x - z_{2S+1}) \quad (97)$$

In equations (95) and (96), the number $\mathcal{C}_m^{2S+1}$ is just a binomial coefficient, ie.,

$$\mathcal{C}_m^{n} = \frac{n!}{m!\,(n-m)!} \qquad (98)$$

Thus the polynomial $P(x)$ is a type of generating function for the state $|\psi\rangle$ and its coefficients.

Now we can take the $2S+1$ complex roots of $P(x)$ and map them back onto the Bloch sphere using the Riemann construction in (66). This then defines our original state as a set of $2S+1$ points on the Bloch sphere (rather than a single point, as was done for the coherent state $\hat{n}$).

The inner product between 2 states $|\psi_A\rangle$ and $|\psi_B\rangle$ is then given by

$$\left.\begin{aligned} \langle \psi_A | \psi_B \rangle &= \sum_M A_m (C_m^A)^* (C_m^B) \\ A_s &= (-1)^{S-m} \, 1/(\mathcal{C}_{S-m}^{2S+1})^2 \end{aligned}\right\} \qquad (99)$$

We may at some stage use this representation later on, in which case I will develop its properties further.

REFERENCES to (A)

L.D. Landau · E.M. Lifshitz : "Quantum Mechanics"

R.P. Feynman : "Feynman Lectures in Physics, vol. 3"

A. Auerbach : "Interacting Electrons & Quantum Magnetism"

R. Penrose : "Shadows of the Mind"

J.M. Radclife : J. Phys A4, 313 (1971)

The books by Landau & Lifshitz & by Feynman are well-known, extremely good, and need no comment. The book of Auerbach is rather specialised, & a little too focussed on certain formal techniques, but worth looking at. The book of Penrose is a highly speculative discussion of the problems associated with Q.M., a mixture of quite amazing insights along with some real nonsense when he goes outside his own field. It contains the only easily accessible I know of the Majorana spin representation. The paper of Radclife first introduced coherent spin states, & is well worth studying.