Appendix B3: DIAGRAMS for Non-Relativistic Systems

In what follows I do not go into any detail about the derivation of the diagram rules here, which are different for different Lagrangian or Hamiltonian - this is done in the main notes, when these theories are introduced.

In what follows the idea is to calculate a few diagrams in each theory, so one can see how the calculations are done. In this Appendix we deal with non-relativistic many-body systems, which are in some ways much more complicated to deal with than relativistic systems, because the 3-momentum integrals are often very messy. However if we do just the frequency integrals, the results are very illuminating.

There are a large number of different condensed matter systems of interest. On the one hand one has "itinerant" systems, in which mobile objects like electrons or phonons move around - often we can assume some sort of translational invariance. Then one has localized degrees of freedom, including static impurities as well as local spin variables, which may be either on random sites or in a lattice array. And of course we have problems where different fields interact, e.g., in the coupled electron-phonon system, or electrons coupled to either static disorder or fluctuating spin variables. In this Appendix we begin with simple homogeneous systems. The calculations and rules are exhibited here for interacting fermions, and for coupled boson-fermion systems (like the electron-phonon system). I then go on to discuss coupled local spin systems. The available degrees of freedom - and hence the diagram rules and graphical calculations - depend very much on what kind of Hamiltonian one is working with here. We cover both the Hubbard model and a lattice of localized spins, with and without anisotropy; finally, we look at a set of 2-level systems.

One other thing covered here is the Landau-Cutkowsky rules, which allow a very simple evaluation of the imaginary part of diagrams, which is then easily extended to cover the whole complex plane. A more sophisticated treatment of this for relativistic systems is given in "Diagrammar", by 't Hooft and Veltman.

App. B.3.1: INTERACTING FERMIONS & PHONONS

In what follows we will look at both fermions interacting via a static interaction $V(\mathbf{q})$, with symmetrized form $\bar{V}(\mathbf{q})$, and at fermions interacting via the coupling to simple longitudinal phonons (so as to avoid a plethora of indices, we ignore both the electron spin and the phonon polarization indices - the spin exchange effects will be taken care of by vertex symmetrization.)

The basic diagram rules depend on whether we are doing finite T calculations in the Matsubara formalism, or zero-T calculations in the Feynman formalism. As we will see, by making the appropriate analytic continuation, we can reduce any finite-T result, derived for some correlator in the complex z-plane of energy, to a zero-T Feynman result. It is actually simpler to start with finite-T Matsubara calculations, for then we are free to analytically continue these however we want. Moreover, with practice they are much easier to carry out, since we don't have to mess around with advanced and retarded parts of fermion propagators.

App. B.3.1 (a) FINITE-T MATSUBARA RULES

We assume the following diagram rules at finite-T (and to simplify things, we let $\hbar = 1$ when doing actual calculations.

I start off here with a set of fermions, interacting solely via the interaction V_q. Then we have the following (NB: these are a little different from those one gets by reading off things from the action - I have moved factors of i around for convenience.):

Rules for Fermions: In the finite-T formalism, we assign a set of Matsubara frequencies to the fermions, given by

$$\epsilon_n \;=\; (2n+1)\pi \, \frac{1}{\beta\hbar} \;=\; (2n+1)\pi \; kT/\hbar \qquad (1)$$

so that

$$\begin{aligned} \delta(t) \;&=\; \frac{1}{\beta\hbar} \sum_{\text{n=even}} e^{-i\epsilon_n \tau} \qquad (|\tau| < \beta\hbar) \\ \frac{1}{\beta\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \; e^{i\epsilon_n\tau} \;&=\; \delta(\epsilon_n) \end{aligned} \qquad (2)$$

The bare 1-particle fermion Green function $G_0(\mathbf{p}, i\epsilon_n)$ is then given, for a translationally invariant system, by

$$G_0^{\sigma\sigma'}(\mathbf{p}, i\epsilon_n) \;=\; \frac{\delta^{\sigma\sigma'}}{i\epsilon_n - (\epsilon^0_{\mathbf{p}} - \mu)/\hbar} \qquad (3)$$

so that

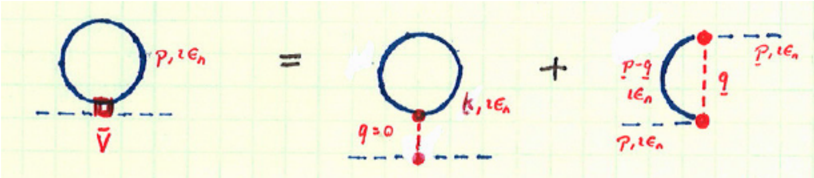
$$G_0^{\sigma\sigma'}(\mathbf{p}, \tau) \;=\; e^{-(\epsilon^0_{\mathbf{p}}-\mu)/\hbar}\left[f_{\mathbf{p}}\theta(-\tau) + (1-f_{\mathbf{p}})\theta(\tau)\right] \; \delta^{\sigma\sigma'} \qquad (4)$$

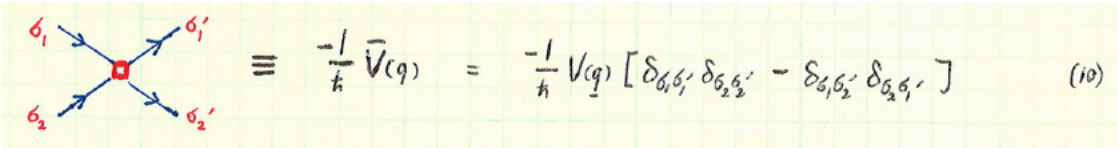
and also

$$G_0^{\sigma\sigma'}(\mathbf{r}, \mathbf{r}'; i\epsilon_n) \;=\; \sum_j \frac{\psi_j(\mathbf{r})\psi_j^*(\mathbf{r}')}{i\epsilon_n - (\epsilon_j - \mu)/\hbar} \, \delta^{\sigma\sigma'} \qquad (5)$$

with the obvious extension to imaginary time τ. Here σ, σ' are spin indices. In the last formula we assume some general set of eigenstates $\psi_j(\mathbf{r})$ of the Hamiltonian H_0, with eigenvalues ϵ^0_j; in the translationally invariant case we just get

$$G_0^{\sigma\sigma'}(\mathbf{r}, \mathbf{r}'; \tau) \;=\; \frac{1}{\beta\hbar}\sum_{\mathbf{p}}\sum_n \; e^{i[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}') - \epsilon_n\tau]} \frac{\delta^{\sigma\sigma'}}{i\epsilon_n - (\epsilon^0_{\mathbf{p}} - \mu)/\hbar} \qquad (6)$$





where as usual we have

$$\sum_{\mathbf{k}} \equiv \int \frac{\mathrm{d}^D k}{(2\pi\hbar)^D} \qquad (D \text{ dimensions}) \tag{7}$$

When it comes to fermion loops and integration over free internal momenta and frequency, we have the following rules

(i) If a diagram has L fermion loops, then we have a factor $(-1)^L$ for these multiplying the whole.

(ii) A single closed loop, as shown in the diagram below, is associated with the expression (with a (-1) for the fermion loop):

$$-G_0(\mathbf{p}, i\epsilon_n)e^{i\epsilon_n\delta}$$

which ensures that

$$-\frac{1}{\beta\hbar}\sum_{n=-\infty}^{\infty} G_0(\mathbf{p}, i\epsilon_n)e^{i\epsilon_n\delta} \;=\; -f_{\mathbf{p}} \tag{8}$$

and also ensures that the result of doing the integration for the graph shown at left, which resolves into the direct Hartree term and the exchange Fock term as shown, is given correctly by

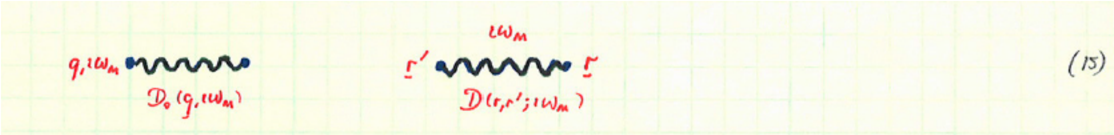
$$\begin{aligned} \frac{1}{\beta\hbar}\sum_n \bar{V}(\mathbf{q})G_0(\mathbf{p}, i\epsilon_n)e^{i\epsilon_n\delta} \;&=\; \frac{1}{\beta\hbar}\sum_n [V_0 G_0(\mathbf{k}, i\epsilon_n) - V_{\mathbf{q}} G_0(\mathbf{p}-\mathbf{q}, i\epsilon_n)] \\ &=\; (V_0 f_{\mathbf{k}} - V_{\mathbf{q}} f_{\mathbf{p}-\mathbf{q}}) \end{aligned} \tag{9}$$

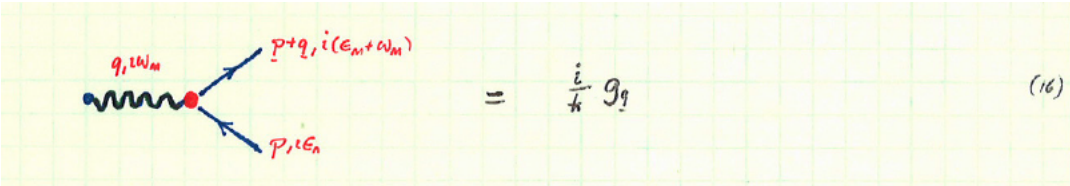
where $V_0 \equiv V(\mathbf{q}=0)$, and where the minus sign in front of $V_{\mathbf{q}}$ comes from the definition of the symmetrized graph, to which we assign the rule shown in (10).

This summarizes the rules for the fermionic graphs.

Rules for Phonons: These rules parallel those for fermions, with a few key differences. There is no factor phonon loops, and indeed we will not deal with them anyway. The Matsubara frequencies are

$$\omega_m \;=\; 2m\pi/\hbar\beta \;\equiv\; 2m\pi kT/\hbar \qquad \text{(bosons)} \tag{11}$$





and the phonon propagator is given by

$$D_0(\mathbf{q}, i\omega_m) \;=\; \frac{1}{2}\hbar\omega_{\mathbf{q}}\left[\frac{1}{i\omega_m - \omega_{\mathbf{q}}} - \frac{1}{i\omega_m + \omega_{\mathbf{q}}}\right] \;=\; -\hbar\frac{\omega_q^2}{\omega_m^2 + \omega_{\mathbf{q}}^2} \tag{12}$$

so that

$$D_0(\mathbf{q}, \tau) \;=\; e^{-(\omega_{\mathbf{q}}-\mu)\tau}\left[n_{\mathbf{q}}\theta(-\tau) + (1+n_{\mathbf{q}})\theta(\tau)\right] \tag{13}$$

and

$$D_0(\mathbf{r}, \mathbf{r}'; \tau) \;=\; -\sum_j \hbar\omega_{\mathbf{q}}^2 \frac{\phi_j(\mathbf{r})\phi_j^*(\mathbf{r}')}{\omega_m^2 + \omega_q^2} \tag{14}$$

and we represent these diagrammatically by the graphs shown in (15).

Finally, we introduce the interaction vertex between fermions and phonons, given by the graph in (16).

Note that the reason for the minus sign in our convention (10) for the fermion - fermion interaction is that we can think of it as coming from the exchange of a photon, with an effective coupling $-V(\mathbf{q}) \propto (i\alpha_q)^2$, where α_q is an electron-photon coupling.

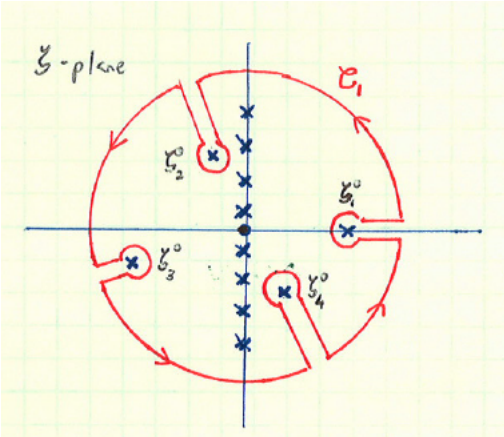
Frequency Sums: The sum over bosons and fermions at these specific frequencies is of course linked to the distribution functions, viz.,

$$f_{\mathbf{p}} \;\equiv\; f(\epsilon_{\mathbf{p}}) \;=\; \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\mu)} + 1} \qquad \text{(fermions)} \tag{17}$$

$$n_{\mathbf{q}} \;\equiv\; n(\omega_{\mathbf{q}}) \;=\; \frac{1}{e^{\beta(\omega_{\mathbf{q}}-\mu)} - 1} \qquad \text{(bosons)} \tag{18}$$

which have simple poles at the frequencies (1) and (11). Before we calculate any specific diagrams, we need to look in more detail at what happens when we do any kind of sum over frequencies, as we did in eq. (9). Suppose we have to evaluate the sum

$$I_{\mathcal{F}} \;=\; \frac{1}{\beta\hbar}\sum_n \mathcal{F}(i\epsilon_n) \tag{19}$$



and where $\mathcal{F}(z)$ is a meromorphic function, with simple poles at energies $\xi = \xi_j^0$. The key here is that we multiply the function $\mathcal{F}(\xi)$ by another function $f(\xi)$ which has poles at $\xi = i\epsilon_n$, which is of course the Fermi function. We then have

$$\frac{1}{\beta\hbar}\sum_n \mathcal{F}(i\epsilon_n) \;=\; \frac{1}{2\pi\hbar}\int \mathrm{d}\xi \; f(\xi)\mathcal{F}(\xi) \tag{20}$$

We wish to calculate this as a contour integral. One possibility is to use the contour shown in the top diagram. Now the key here is that the integral along the outer reaches of $\mathcal{C}_1$ is zero - this follows because both $f(\xi)$ and $\mathcal{F}(\xi)$ are assumed meromorphic. It then follows that the only poles that are picked up by the contour integral

$$I_{\mathcal{F}} \;=\; \frac{1}{\hbar}\oint^{\mathcal{C}_1} \frac{\mathrm{d}\xi}{2\pi i}\, f(\xi)\mathcal{F}(\xi) \tag{21}$$

are the ones along the imaginary axis that enclosed by $\mathcal{C}_1$, i.e., the poles of the Fermi function.

Another possible contour is shown in the lower diagram. Note that this time we circle the poles in the clockwise direction, and we exclude the poles of the Fermi function, but now include the poles of $\mathcal{F}(\xi)$. Summing over the 2 contours $\mathcal{C}_2^A$ and $\mathcal{C}_2^B$, we get

$$I_{\mathcal{F}} \;=\; -\frac{1}{\hbar}\oint^{\mathcal{C}_2^A} \frac{\mathrm{d}\xi}{2\pi i} \;+\; \oint^{\mathcal{C}_2^B} \frac{\mathrm{d}\xi}{2\pi i}\,[f(\xi)\mathcal{F}(\xi)] \tag{22}$$

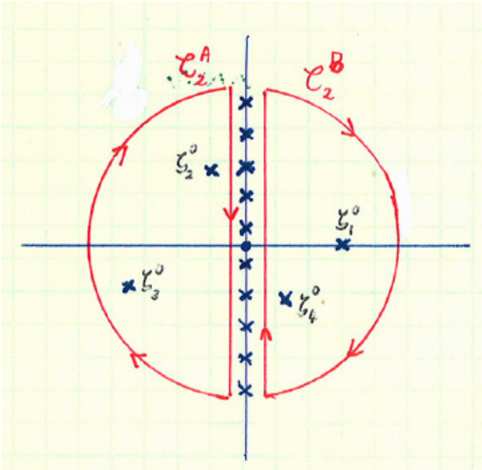
with the minus sign coming from the clockwise sense taken on the contour.

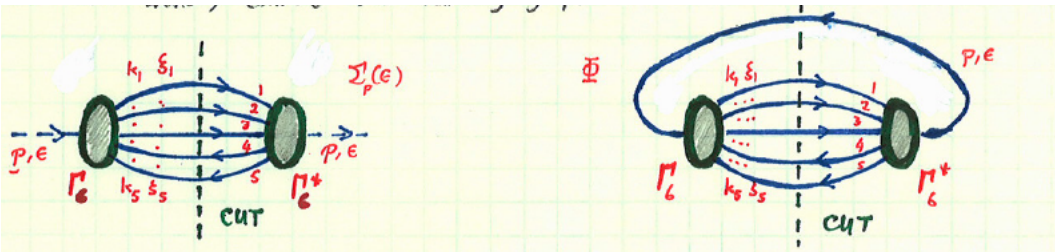
Either of these methods gives the same answer, viz., that

$$I_{\mathcal{F}} \;=\; \sum_j \mathbb{R}_{\mathcal{F}}(\xi_j^0) f(\xi_j^0) \;=\; \sum_j \frac{\mathbb{R}_{\mathcal{F}}(\xi_j^0)}{e^{\beta\xi_j^0}+1} \tag{23}$$

where $\mathbb{R}_{\mathcal{F}}(\xi_j^0)$ is the residue of the function $\mathcal{F}(\xi)$ at its poles; thus, in the case of a meromorphic function of form

$$\mathcal{F}(\xi) \;=\; \sum_{j=1}^{N} \frac{A_j}{\xi - \xi_j^0} \;+\; \phi(\xi) \tag{24}$$





we get the result

$$\mathbb{R}_{\mathcal{F}}(\xi_j^0) \;=\; \phi(\xi_j^0) \;+\; \sum_{k\neq j} \frac{A_k}{\xi_j^0 - \xi_k^0} \tag{25}$$

Thus, to evaluate any diagram, we can simply sum the residues of the diagram at the poles. The same argument goes through for bosons.

Landau-Cutkowsky Graphs: Given that graphs are meromorphic functions of their arguments, we can determine the entire graph from the pole structure, using Cauchy's theorem. To see how this works, consider the two graphs shown in the figure.

The first graph is a self-energy graph, and the 2nd is a graph for the thermodynamic potential. We can write them as (NB: $\xi_j > 0$ for particles, and $\xi_j < 0$ for holes):

$$\Sigma_{\mathbf{p}}(\epsilon + i\delta) \;=\; \prod_{j=1}^{5} \sum_{\mathbf{k}_j} \int \frac{d\xi_j}{2\pi} \, |\Gamma_{\mathbf{p},\{\mathbf{k}_j\}}(\epsilon,\xi_j)|^2 \, A_j(\mathbf{k}_j,\xi_j) \, \frac{1}{f(\epsilon)} \frac{f(\xi_j)}{(\epsilon - \sum_j \xi_j) + i\delta} \tag{26}$$

$$\Phi \;=\; -S_\Phi \prod_{j=1}^{5} \sum_{\mathbf{k}_j} \int \frac{d\xi_j}{2\pi} \sum_{\mathbf{p}} \int \frac{d\epsilon}{2\pi i} \, |\Gamma_{\mathbf{p},\{\mathbf{k}_j\}}(\epsilon,\xi_j)|^2 \, A_{\mathbf{p}}(\epsilon) A_j(\mathbf{k}_j,\xi_j) \, \frac{f(\epsilon) f_j(\xi_j)}{(\epsilon + \sum_j \xi_j)} \tag{27}$$

where $A_j(\mathbf{k}_j,\xi_j)$ is the 1-particle spectral function; for free particles,

$$A(\mathbf{k},\xi) \;\rightarrow\; -2\pi\delta(\epsilon_{\mathbf{k}}^0 - \xi) \tag{28}$$

and finally, S_Φ is a symmetry factor.

Now we observe that it is always easier to compute the imaginary parts of these graphs. Thus, e.g.,

$$\begin{aligned}
\mathrm{Im}\,\Sigma_{\mathbf{p}}(\epsilon+i\delta) &= \frac{1}{2i}\big(\Sigma_{\mathbf{p}}(\epsilon+i\delta)-\Sigma_{\mathbf{p}}(\epsilon-i\delta)\big) \\
&= -\pi\prod_{j=1}^{5}\sum_{\mathbf{k}_j}\int\frac{\mathrm{d}\xi_j}{2\pi}\,|\Gamma_{\mathbf{p},\{\mathbf{k}_j\}}(\epsilon,\xi_j)|^2\,A_j(\mathbf{k}_j,\xi_j)\,\frac{f(\xi_j)}{f(\epsilon)}\,\delta(\epsilon-\sum_j\xi_j) \\
\xrightarrow[\text{free internal lines}]{} & -\pi\prod_{j=1}^{5}\sum_{\mathbf{k}_j}\int\frac{\mathrm{d}\xi_j}{2\pi}\,|\Gamma_{\mathbf{p},\mathbf{k}_j}(\epsilon^0_{\mathbf{p}},\epsilon^0_{\mathbf{k}_j})|^2\,\frac{f(\epsilon^0_{\mathbf{k}_j})}{f(\epsilon^0_{\mathbf{p}})}\,\delta(\epsilon^0_{\mathbf{p}}-\sum_j\epsilon^0_{\mathbf{k}_j})
\end{aligned} \tag{30}$$

which is much simpler to evaluate than calculating the whole graph. In practical computations one then often calculates first the imaginary part of the graph, and then uses dispersion relations (ie., one calculates the Hilbert transform of the imaginary part) to determine the real part from this (for more details see the discussion of the analytic properties of propagators, in section A).

App. B.3.1 (b) COUPLED FERMIONS: EXAMPLES

To really understand graphs you have to calculate some of them. In what follows I will determine the frequency sums for a few simple graphs involving coupled fermions. The momentum sums are not done - they depend on the the dimensionality of the system, as well as its detailed underlying structure, and their evaluation can be quite lengthy (you can find details about momentum sums in the condensed matter literature).

We assume an interaction $V(\mathbf{r})$, and we assume a Fourier transform $V(\mathbf{q})$ for $V(\mathbf{r})$. In what follows I consider 4 examples - the first two involve a single fermion loop, the 3rd and 4th involve 2-loop diagrams. In what follows we set $\hbar = 1$.

(i) Single Polarization Bubble: This is simply the bare propagator for a particle-hole pair, and so it is central to the calculation of many properties. We have the diagram in the form shown in the diagram, for which the formal expression (with $\hbar = 1$) is

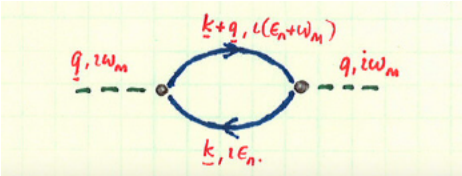
$$\pi_0(\mathbf{q},\omega_m) \;=\; \sum_{\mathbf{k}}\frac{1}{\beta}\sum_n\frac{1}{i\epsilon_n-\epsilon^0_{\mathbf{k}}}\,\frac{1}{i(\epsilon_n+\omega_m)-\epsilon^0_{\mathbf{k}+\mathbf{q}}} \tag{31}$$

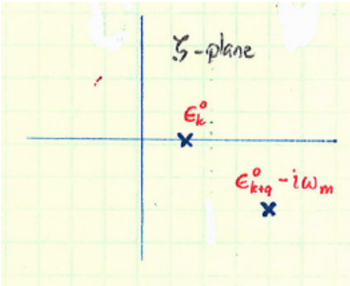
where I have set $\mu = 0$ (we will put it back at the end of the calculation).

The sum in (29) is converted to a contour integral, and we get

$$\pi_0(\mathbf{q},\omega_m) \;=\; \sum_{\mathbf{k}}\oint^{\mathcal{C}_1}\frac{\mathrm{d}\xi}{2\pi i}\,\frac{1}{\xi-\epsilon^0_{\mathbf{k}}}\,\frac{1}{\xi+i\omega_m-\epsilon^0_{\mathbf{k}+\mathbf{q}}}\,f(\xi) \tag{32}$$

and the contour $\mathcal{C}_1$ has to be taken around the 2 poles shown. The result is then given by





$$\pi_0(\mathbf{q}, i\omega_m) = \sum_{\mathbf{k}} \left[\frac{f_{\mathbf{k}}}{i\omega_m - (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})} + \frac{f(\epsilon^0_{\mathbf{k+q}} - i\omega_m)}{-i\omega_m + (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})} \right] \tag{33}$$

Now $f(\epsilon^0_{\mathbf{k+q}} \pm i\omega_m) = [e^{\beta(\epsilon^0_{\mathbf{k+q}} \pm i\omega_m)} + 1]^{-1} \equiv f(\epsilon^0_{\mathbf{k+q}})$, because of (11). Thus we get

$$\pi_0(\mathbf{q}, i\omega_m) = \sum_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k+q}}}{i\omega_m - (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})} \tag{34}$$

In books and papers you may see this written a little differently. There will be an extra factor 2, coming from a sum over spin indices; and it is also written as

$$\pi_0(\mathbf{q}, i\omega_m) = \sum_{\mathbf{k}} f_{\mathbf{k}}(1 - f_{\mathbf{k+q}}) \left[\frac{1}{i\omega_m - (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})} - \frac{1}{i\omega_m + (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})} \right] \tag{35}$$

where we have used invariance under inversion (ie., under $\mathbf{k} \to -\mathbf{k}$) of $f_{\mathbf{k}}$ and $\epsilon^0_{\mathbf{k}}$, and also swapped indices in the 2nd term. The latter form, in (33) makes it clear that we are dealing with the product of a particle and hole excitation. If we rewrite (35) as

$$\pi_0(\mathbf{q}, i\omega_m) = -\sum_{\mathbf{k}} f_{\mathbf{k}}(1 - f_{\mathbf{k+q}}) \frac{2(\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})}{\omega_m^2 + (\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})^2} \tag{36}$$

we see that it looks like the propagator of a boson, with energy $(\epsilon^0_{\mathbf{k+q}} - \epsilon^0_{\mathbf{k}})$, and frequency $i\omega_m$; but of course that is exactly what it is.

(ii) One-Loop Self-Energy graph: The simplest possible graph for the fermion self-energy, apart from the Hartree-Fock contribution, is the one involving a single polarization part.

In the diagram, the calculation is generalized somewhat so as the include an effective interaction $V(\mathbf{q}, i\omega_m)$, which also depends on frequency. The simplest example of such an interaction would be the point interaction

$$\begin{aligned} V(\mathbf{r}-\mathbf{r}', t-t') &\longrightarrow V_0\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \\ V(\mathbf{q}, i\omega_m) &\longrightarrow V_0 \end{aligned} \tag{37}$$

In any case, it is clear that we can write this contribution to the self-energy in 2 different ways; either

$$\begin{aligned} \Sigma(\mathbf{p}, i\epsilon_n) &= \frac{1}{\beta}\sum_m\sum_{\mathbf{q}} |V_{\mathbf{q}}(i\omega_m)|^2\, G_0\big(\mathbf{p}-\mathbf{q}, i(\epsilon_n-\omega_m)\big)\, \pi_0(\mathbf{q}, i\omega_m) \\ &= \frac{1}{\beta}\sum_m\sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}(i\omega_m)|^2 \frac{1}{i(\epsilon_n-\omega_m)-\epsilon^0_{\mathbf{p}-\mathbf{q}}} \frac{f_{\mathbf{k}}-f_{\mathbf{k}+\mathbf{q}}}{i\omega_m-(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})} \end{aligned} \tag{38}$$

or using (36), that

$$\Sigma(\mathbf{p}, i\epsilon_n) = \frac{1}{\beta}\sum_m\sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}(i\omega_m)|^2 \frac{f_{\mathbf{k}}(1-f_{\mathbf{k}+\mathbf{q}})}{i(\epsilon_n-\omega_m)-\epsilon^0_{\mathbf{p}-\mathbf{q}}} \frac{(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})}{\omega_m^2-(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})^2} \tag{39}$$

or, on the other hand, we can write, directly from the diagram, that

$$\begin{aligned} \Sigma(\mathbf{p}, i\epsilon_n) &= \frac{1}{\beta^2}\sum_{m,n}\sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}(i\omega_m)|^2 \\ &\qquad \times \frac{1}{i(\epsilon_n-\omega_m)-\epsilon^0_{\mathbf{p}-\mathbf{q}}} \frac{1}{i\epsilon_\ell-\epsilon^0_{\mathbf{k}}} \frac{1}{i(\epsilon_\ell+\omega_m)-\epsilon^0_{\mathbf{k}+\mathbf{q}}} \end{aligned} \tag{40}$$

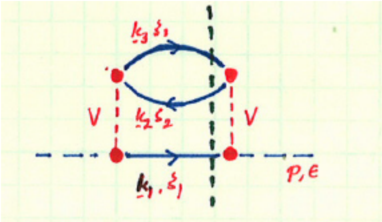
Starting from either (36) or (38), we do the sum over the bosonic frequency ω_m, and picking up the extra pole at $i\omega_m \to \xi = \epsilon^0_{\mathbf{p}-\mathbf{q}} + i\epsilon_n$, we get

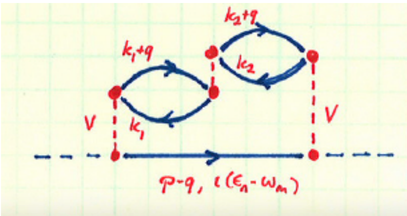
$$\Sigma_{\mathbf{p}}(i\epsilon_n) = \sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})|^2 \frac{(f_{\mathbf{k}}-f_{\mathbf{k}+\mathbf{q}})\left(\epsilon^0_{\mathbf{k}}-f_{\mathbf{p}-\mathbf{q}}-f(\epsilon^0_{\mathbf{k}+\mathbf{q}}+i\epsilon_n)\right)}{\epsilon^0_{\mathbf{p}-\mathbf{q}}-(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})-i\epsilon_n} \tag{41}$$

and since $f(\epsilon+i\epsilon_n) = -n(\epsilon)$, where $n(\epsilon)$ is the Bose function, this then becomes

$$\Sigma(\mathbf{p}, i\epsilon_n) = \sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})|^2 \frac{(f_{\mathbf{k}+\mathbf{q}}-f_{\mathbf{k}})\left(f_{\mathbf{p}-\mathbf{q}}+n(\epsilon^0_{\mathbf{k}}-\epsilon^0_{\mathbf{k}+\mathbf{q}})\right)}{i\epsilon_n-\left(\epsilon^0_{\mathbf{p}-\mathbf{q}}-(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})\right)} \tag{42}$$

Notice the form of this result - it tells us that the self-energy is like that of a system coupled to some boson, with one boson emitted and re-absorbed by the fermion; the fermion has intermediate state energy $\epsilon^0_{\mathbf{p}-\mathbf{q}}$; and the boson energy $(\epsilon^0_{\mathbf{k}+\mathbf{q}}-\epsilon^0_{\mathbf{k}})$. Thus an electron-phonon self-energy, to lowest order, will have the same structure.





We can also get the above result using the Landau-Cutkowsky technique. This involves computing Im $\Sigma_{\mathbf{p}}(\epsilon + i\delta)$ using the "cut" across the graph, as follows:

Now, using the Landau-Cutkowsky rules, we immediately find that

$$\begin{aligned} \text{Im } \Sigma_{\mathbf{p}}(\epsilon + i\delta) &= \pi \int \frac{d\xi_1}{2\pi} \int \frac{d\xi_2}{2\pi} \int \frac{d\xi_3}{2\pi} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} (2\pi)^2 \, \delta(\epsilon^0_{k_j} - \xi_j) \\ &\quad \times \frac{1}{f(\epsilon)} f(\xi_1) f(-\xi_2) f(\xi_3) \, \delta(\epsilon - \xi_1 + \xi_2 - \xi_3) \qquad (43) \end{aligned}$$

and this is just

$$\text{Im } \Sigma_{\mathbf{p}}(\epsilon + i\delta) = \pi \sum_{\mathbf{k},\mathbf{q}} |V_{\mathbf{q}}|^2 \, \frac{1}{f(\epsilon)} f_{\mathbf{p}-\mathbf{q}} (1 - f_{\mathbf{k}}) f_{\mathbf{p}+\mathbf{q}} \, \delta(\epsilon^0_{\mathbf{p}} + \epsilon^0_{\mathbf{k}} - \epsilon^0_{\mathbf{k}+\mathbf{q}} - \epsilon^0_{\mathbf{p}-\mathbf{q}}) \qquad (44)$$

and it is actually fairly straightforward to see that (44) is the imaginary part of (42), if we continue $i\epsilon_n \to \epsilon + i\delta$ in (42).

(iii) A Two-loop Self-Energy Graph: Now let's compute a slightly more messy self-energy graph. This one actually brings in some new features - it is shown in the diagram, drawn as a Feynman graph with 2 polarization bubbles, with all the momenta and frequencies shown. Now the formal expression for this is:

$$\begin{aligned} \Sigma_{\mathbf{p}}(i\epsilon_n) &= \frac{1}{\beta^3} \sum_{m,n_1,n_2} \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} V^3_{\mathbf{q}} \, G_0\big(\mathbf{p} - \mathbf{q}, i(\epsilon_n - \omega_m)\big) G_0\big(\mathbf{k_1} + \mathbf{q}, i(E_{n_1} + \omega_m)\big) \\ &\quad \times G_0(\mathbf{k_1}, iE_{n_1}) G_0\big(\mathbf{k_2} + \mathbf{q}, i(E_{n_2} + \omega_m)\big) G_0(\mathbf{k_2}, iE_{n_2}) \\ &= \frac{1}{\beta} \sum_m \sum_{\mathbf{q}} V^3_{\mathbf{q}} \, G_0\big(\mathbf{p} - \mathbf{q}, i(\epsilon_n - \omega_m)\big) \, \pi^2_0(\mathbf{q}, i\omega_m) \qquad (45) \end{aligned}$$

where the 2nd form is obviously easier to evaluate. Writing these expressions out in full, we have

$$\Sigma_{\mathbf{p}}(i\epsilon_n) = \frac{1}{\beta^3}\sum_{m,n_1,n_2}\sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} V^3_{\mathbf{q}} \frac{1}{i(\epsilon_n-\omega_m)-\epsilon^0_{\mathbf{p+q}}}\frac{1}{i(E_{n_1}+\omega_m)-\epsilon^0_{\mathbf{k_1+q}}} \times \frac{1}{iE_{n_1}-\epsilon^0_{\mathbf{k_1}}}\frac{1}{i(E_{n_2}+\omega_m)-\epsilon^0_{\mathbf{k_2+q}}}\frac{1}{iE_{n_2}-\epsilon^0_{\mathbf{k_2}}} \tag{46}$$

or, collapsing the polarization bubbles, we have

$$\Sigma_{\mathbf{p}}(i\epsilon_n) = \frac{1}{\beta}\sum_m\sum_{\mathbf{q},\mathbf{k_1},\mathbf{k_2}} V^3_{\mathbf{q}} \frac{1}{i(\epsilon_n-\omega_m)-\epsilon^0_{\mathbf{p-q}}}\frac{f_{\mathbf{k_1}}-f_{\mathbf{k_1+q}}}{i\omega_m-(\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_1}})}\frac{f_{\mathbf{k_2}}-f_{\mathbf{k_2+q}}}{i\omega_m-(\epsilon^0_{\mathbf{k_2+q}}-\epsilon^0_{\mathbf{k_2}})} \tag{47}$$

Now at first glance, this calculation looks extremely easy, if we start from the form in (47). However, there is a slight problem - how are we to deal with the poles of $\pi_0^2(\mathbf{q}, i\omega_m)$, which appear to be double poles, i.e., of order 2 in a Laurent expansion?

This is where a resort to a Landau-Cutkowsky technique comes into its own. Let's first see how the problem manifests itself, and then show how the LC technique bypasses it.

Consider first the expression in (46). It takes a little time, but is otherwise quite straightforward, to do the 3 frequency sums, and get the following expression:

$$\Sigma_{\mathbf{p}}(i\epsilon_n) = \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} V^3_{\mathbf{q}} \frac{(f_{\mathbf{k_1}}-f_{\mathbf{k_1+q}})(f_{\mathbf{k_2}}-f_{\mathbf{k_2+q}})}{(\epsilon^0_{\mathbf{k_1}}-\epsilon^0_{\mathbf{k_1+q}}+\epsilon^0_{\mathbf{k_2}}-\epsilon^0_{\mathbf{k_2+q}})} \times \frac{\mathcal{F}(\mathbf{p},\mathbf{q},\mathbf{k_1},\mathbf{k_2};i\epsilon_n)}{\big(i\epsilon_n-\epsilon^0_{\mathbf{p-q}}-(\epsilon^0_{\mathbf{k_2+q}}-\epsilon^0_{\mathbf{k_2}})\big)\big(i\epsilon_n-\epsilon^0_{\mathbf{p-q}}-(\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_1}})\big)} \tag{48a}$$

where we have defined

$$\begin{aligned}\mathcal{F}(\mathbf{p},\mathbf{q},\mathbf{k_1},\mathbf{k_2};i\epsilon_n) = {}& \big(i\epsilon_n-\epsilon^0_{\mathbf{p-q}}-(\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_1}})\big)\, n(\epsilon_{\mathbf{k_2}}-\epsilon_{\mathbf{k_2+q}}) \\ & - \big(i\epsilon_n-\epsilon^0_{\mathbf{p-q}}-(\epsilon^0_{\mathbf{k_2+q}}-\epsilon^0_{\mathbf{k_2}})\big)\, n(\epsilon_{\mathbf{k_1}}-\epsilon_{\mathbf{k_1+q}}) \\ & + (\epsilon^0_{\mathbf{k_1}}+\epsilon^0_{\mathbf{k_2}}-\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_2+q}})\, f_{\mathbf{p-q}}\end{aligned} \tag{48b}$$

and of course (48) reduces to (47) after we do the sum over m in eq. (47).

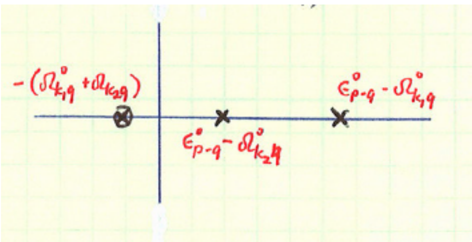
Now in both (47) and (48) there is a problem in doing the integrals. To see this, we look at the poles in these 2 expressions, which are shown on the next page. Consider first the divergent term in the final energy denominator in eq. (48). We introduce a short-hand notation, defining

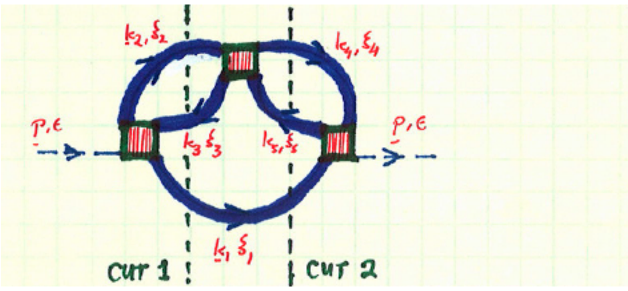
$$\Omega^0_{\mathbf{kq}} = \epsilon^0_{\mathbf{k+q}}-\epsilon^0_{\mathbf{k}} \tag{49}$$

we then have a denominator given by

$$\Omega^0_{\mathbf{k_1q}}+\Omega^0_{\mathbf{k_2q}} = \epsilon^0_{\mathbf{k_1+q}}+\epsilon^0_{\mathbf{k_2+q}}-\epsilon^0_{\mathbf{k_1}}-\epsilon^0_{\mathbf{k_2}},$$

and we notice that this goes to zero precisely when energy is conserved in the series of 2 polarization loops.





(51)

The 2nd pair of zeroes in the denominator (i.e., poles in the graph) enforce energy conservation in the 2 intermediate states in the graph, i.e., they enforce

$$i\epsilon_n = \begin{Bmatrix} \epsilon^0_{\mathbf{p-q}} + \Omega^0_{\mathbf{k_1q}} & \text{first bubble} \\ \epsilon^0_{\mathbf{p-q}} + \Omega^0_{\mathbf{k_2q}} & \text{second bubble} \end{Bmatrix} \tag{50}$$

However we notice now that when we come to do the 3-momentum integrals, viz., $\sum_{\mathbf{k_1}}\sum_{\mathbf{k_2}}\sum_{\mathbf{q}}$, we will have to integrate over 3 poles, all of which are on the real axis, and that indeed these poles will overlap when energy and momentum are conserved! This makes the contour integration very tricky to do.

Things are much easier using the Landau-Cutkowsky technique. To show its full generality, let's first evaluate the graphs assuming we have full irreducible vertices in place of $V(\mathbf{q})$, and that the internal $\mathcal{G}_2$ lines are fully dressed, so that we get the graph shown in the diagram. The internal lines thus have spectral functions $A(\xi_j, \mathbf{k}_j)$, and we now evaluate the LC graph by summing over the 2 possible cuts that exist for this graph. We get:

$$\begin{aligned} \text{Im}\, \Sigma_{\mathbf{p}}(\epsilon + i\delta) &= \pi \prod_{j=1}^{5} \int \frac{d\xi_j}{2\pi} \sum_{\mathbf{k}_j} \frac{A(\mathbf{k}_j, \xi_j)}{f(\epsilon)}\, I_4(\mathbf{p}, \epsilon, \mathbf{k}_3, \xi_3; \mathbf{k}_1, \xi_1, \mathbf{k}_2, \xi_2) \\ &\quad \times\, I_4(\mathbf{k}_2, \xi_2, \mathbf{k}_5, \xi_5; \mathbf{k}_3, \xi_3, \mathbf{k}_4, \xi_4)\, I_4(\mathbf{k}_1, \xi_1, \mathbf{k}_4, \xi_4; \mathbf{k}_5, \xi_5, \mathbf{p}, \epsilon) \\ \times \Bigg[&\frac{(f_2 - f_3)(f_1 f_4 f_{-5})}{\epsilon - \xi_1 - (\xi_2 - \xi_3) + i\delta} \delta(\epsilon - \xi_1 - \xi_4 + \xi_5) + \frac{(f_4 - f_5)(f_1 f_2 f_{-3})}{\epsilon - \xi_1 - (\xi_4 - \xi_5) + i\delta} \delta(\epsilon - \xi_1 - \xi_2 + \xi_3) \Bigg] \end{aligned} \tag{52}$$

Now this result is very complicated - I wanted you to see just once how bad it can get for a real graph, containing dressed internal lines and vertices. But suppose we now make the assumption of free particles in the internal lines, and let the irreducible vertices be bare vertices, i.e., we let

$$\begin{aligned} A(\mathbf{k}_j, \xi_j) &\longrightarrow A_0(\mathbf{k}_j, \xi_j) = -2\pi\delta(\xi_j - \epsilon^0_{\mathbf{k_j}}) \\ I_4(1,2;3,4) &\longrightarrow V(\mathbf{q}) \end{aligned} \tag{53}$$

Then the graph in (51), and the expression in (52), collapse to

$$\begin{aligned} \text{Im}\ \Sigma_{\mathbf{p}}(\epsilon + i\delta) \longrightarrow\ & \pi \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V^3_{\mathbf{q}} \frac{1}{f(\epsilon)} \\ & \times \left\{ \frac{(f_{\mathbf{k_1+q}} - f_{\mathbf{k_1}}) f_{\mathbf{p-q}} f_{\mathbf{k_2+q}} (1 - f_{\mathbf{k_2}})}{\epsilon - \epsilon^0_{\mathbf{p-q}} - (\epsilon^0_{\mathbf{k_1+q}} - \epsilon^0_{\mathbf{k_1}}) + i\delta}\, \delta(\epsilon - \epsilon^0_{\mathbf{p-q}} - \epsilon^0_{\mathbf{k_2+q}} + \epsilon^0_{\mathbf{k_2}}) \right. \\ & \left. + \frac{(f_{\mathbf{k_2+q}} - f_{\mathbf{k_2}}) f_{\mathbf{p-q}} f_{\mathbf{k_1+q}} (1 - f_{\mathbf{k_1}})}{\epsilon - \epsilon^0_{\mathbf{p-q}} - (\epsilon^0_{\mathbf{k_2+q}} - \epsilon^0_{\mathbf{k_2}}) + i\delta}\, \delta(\epsilon - \epsilon^0_{\mathbf{p-q}} - \epsilon^0_{\mathbf{k_1+q}} + \epsilon^0_{\mathbf{k_1}}) \right\} \end{aligned} \tag{54}$$

which can be reduced to the imaginary part of (48), with a few manipulations, once we have analytically continued (48) down to the real axis, i.e., let

$$i\epsilon_n \longrightarrow \epsilon + i\delta \tag{55}$$

in (48), and takes the imaginary part. Then, to get back (48), we use Cauchy's theorem.

(iv) Two-loop contribution to Thermodynamic Potential $\Omega(T)$: Finally, let's compute a graph which at $T = 0$ contributes to the ground state energy, and at finite T contributes to the thermodynamic potential $\Omega(T)$. We choose a 2-loop graph for $\Omega(T)$, with fermion-fermion interactions, as shown.

The formal expression for this graph, assuming bare internal lines, is then just (NB: the factor 1/4 is a symmetry factor):

$$\begin{aligned} \Omega(T) &= \frac{1}{4\beta^3} \sum_{n_1,n_2,m} \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} |V_{\mathbf{q}}|^2 G_0\big(\mathbf{k}_1 + \mathbf{q}, i(\epsilon_{n_1} + \omega_m)\big) G_0(\mathbf{k}_1, i\epsilon_{n_1}) \\ &\qquad \times G_0\big(\mathbf{k}_2 + \mathbf{q}, i(\epsilon_{n_2} + \omega_m)\big) G_0(\mathbf{k}_2, i\epsilon_{n_2}) \qquad (56) \\ &= \frac{1}{4\beta} \sum_m \sum_{\mathbf{q}} |V_{\mathbf{q}}|^2\ \pi^2_0(\mathbf{q}, i\omega_m) \qquad (57) \end{aligned}$$

or, writing (56) explicitly,

$$\begin{aligned} \Omega(T) = & \frac{1}{4\beta^3} \sum_{n_1,n_2,m} \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} |V_{\mathbf{q}}|^2 \\ & \times \frac{1}{i(\epsilon_{n_1} + \omega_m) - \epsilon^0_{\mathbf{k_1+q}}} \frac{1}{i\epsilon_{n_1} - \epsilon_{\mathbf{k_1}}} \frac{1}{i(\epsilon_{n_2} + \omega_m) - \epsilon^0_{\mathbf{k_2+q}}} \frac{1}{i\epsilon_{n_2} - \epsilon_{\mathbf{k_2}}} \end{aligned} \tag{58}$$

and, writing (57) explicitly, we have

$$\Omega(T) = \frac{1}{4\beta}\sum_m \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} |V_\mathbf{q}|^2 \frac{1}{i\omega_m - (\epsilon^0_{\mathbf{k_1+q}} - \epsilon^0_{\mathbf{k_1}})} \frac{1}{i\omega_m - (\epsilon^0_{\mathbf{k_2+q}} - \epsilon^0_{\mathbf{k_2}})} \tag{59}$$

or, even, using (57),

$$\begin{aligned}\Omega(T) = \frac{1}{\beta}\sum_m \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} &|V_\mathbf{q}|^2 f_{\mathbf{k_1}}(1-f_{\mathbf{k_1+q}}) f_{\mathbf{k_2}}(1-f_{\mathbf{k_2+q}}) \\ &\times \frac{(\epsilon^0_{\mathbf{k_1+q}} - \epsilon^0_{\mathbf{k_1}})}{\omega_m^2 + (\epsilon^0_{\mathbf{k_1+q}} - \epsilon^0_{\mathbf{k_1}})^2} \frac{(\epsilon^0_{\mathbf{k_2+q}} - \epsilon^0_{\mathbf{k_2}})}{\omega_m^2 + (\epsilon^0_{\mathbf{k_2+q}} - \epsilon^0_{\mathbf{k_2}})^2}\end{aligned} \tag{60}$$

in which we see the role of the Fermi "blocking" functions fully displayed, and in which the true bosonic form of the particle-hole pair propagator is exposed.

To vary things a little, let's do this using the $T = 0$ formalism - this will show you how it is done, and also show us a neat piece of mathematics. Recall that we can write the $T = 0$ fermion propagator as

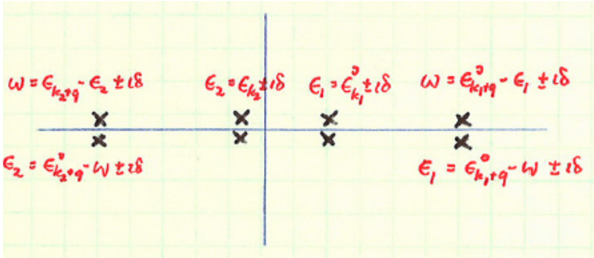
$$G_0(\mathbf{p}, \epsilon) = \frac{1 - f_\mathbf{p}}{\epsilon - \epsilon^0_\mathbf{p} + i\delta} + \frac{f_\mathbf{p}}{\epsilon - \epsilon^0_\mathbf{p} - i\delta} \tag{61}$$

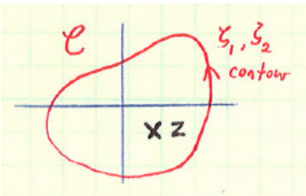
so we can write yet another expression for the diagram in (56), viz.,

$$\begin{aligned}\Omega(T) = \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{q}} &\iiint \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi} |V_\mathbf{q}|^2 \\ \times &\Bigg[\left(\frac{1-f_{\mathbf{k_1+q}}}{\epsilon_1 + \omega - \epsilon^0_{\mathbf{k_1+q}} + i\delta} + \frac{f_{\mathbf{k_1+q}}}{\epsilon_1 + \omega - \epsilon^0_{\mathbf{k_1+q}} - i\delta}\right)\left(\frac{1-f_{\mathbf{k_1}}}{\epsilon_1 - \epsilon^0_{\mathbf{k_1}} + i\delta} + \frac{f_{\mathbf{k_1}}}{\epsilon_1 - \epsilon^0_{\mathbf{k_1}} - i\delta}\right) \\ &\cdot \left(\frac{1-f_{\mathbf{k_2+q}}}{\epsilon_2 + \omega - \epsilon^0_{\mathbf{k_2+q}} + i\delta} + \frac{f_{\mathbf{k_2+q}}}{\epsilon_2 + \omega - \epsilon^0_{\mathbf{k_2+q}} - i\delta}\right)\left(\frac{1-f_{\mathbf{k_2}}}{\epsilon_2 - \epsilon^0_{\mathbf{k_2}} + i\delta} + \frac{f_{\mathbf{k_2}}}{\epsilon_2 - \epsilon^0_{\mathbf{k_2}} - i\delta}\right)\Bigg]\end{aligned} \tag{62}$$

where when $T \to 0$, $f_\mathbf{k} \to \theta(\mu - \epsilon^0_\mathbf{k})$. From (62) we see why it is often better to use the finite-T Matsubara technique; there is a total of 8 poles here, if we look at the 3 energy variables combined. The figure cheats a bit, because it shows all 8 poles, even though in any given frequency integration only some of these will come in.

However there is a key point that comes in when we look at this. This is that, yet again, we have the possibility of "overlapping poles", which causes ambiguity in how we treat the integration (and once we also take into account the momentum integration, these become overlapping branch cuts). Consider, e.g., integration in the ω-plane. We have 2 poles, at $\omega = \epsilon_{\mathbf{k_1+q}} - \epsilon_1 \pm i\delta$ and $\omega = \epsilon_{\mathbf{k_2+q}} - \epsilon_2 \pm i\delta$. The problem than arises when these poles coincide, ie., when $\epsilon_{\mathbf{k_1+q}} - \epsilon_1 = \epsilon_{\mathbf{k_2+q}} - \epsilon_2$; what does one then do?





There are 2 ways to deal with this question. One way is to use the device of displacing the branch cuts from each other - this will be described below. The other way is much simpler, if one is prepared to take a theorem on trust. This is the famous "Poincare-Bertrand" theorem, which we can describe as follows:

Consider the contour integral:

$$I_{\mathcal{F}}^{\mathcal{C}} = \oint_{\mathcal{C}} d\xi_1 \oint_{\mathcal{C}} d\xi_2 \, \frac{\mathcal{F}(\xi_1, \xi_2)}{(\xi_1 - z)(\xi_2 - z)} \tag{63}$$

in which we integrate over both ξ_1 and ξ_2 along the same contour $\mathcal{C}$. At first this looks like a rather trivial integral. However suppose that (a) we put z either on, or at an infinitesimal distance away from, the contour $\mathcal{C}$; and (b) we then ask what happens when ξ_1 and ξ_2 happen to be equal to each other?

The answer is provided by the "Poincare-Bertrand" theorem, which says that the value of (63) is (here $\mathbb{P}$ denotes "principle value")":

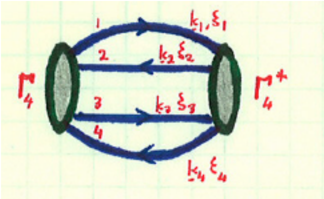
$$\begin{aligned} I_{\mathcal{F}}^{\mathcal{C}}(z) &= \oint^{\mathcal{C}} d\xi_1 \oint^{\mathcal{C}} d\xi_2 \, \hat{K}_z(\xi_1, \xi_2) \mathcal{F}(\xi_1, \xi_2) \\ &= \mathbb{P} \oint_{\mathcal{C}} d\xi_1 \oint_{\mathcal{C}} d\xi_2 \, \frac{\mathcal{F}(\xi_1, \xi_2)}{(\xi_1 - z)(\xi_2 - z)} - \pi^2 \mathcal{F}(z, z) \end{aligned} \tag{64}$$

i.e., we have

$$\hat{K}_z(\xi_1, \xi_2) = \mathbb{P} \frac{1}{(\xi_1 - z)(\xi_2 - z)} - \pi^2 \delta(\xi_1 - z) \delta(\xi_2 - z) \tag{65}$$

a formula which we might have guessed from the usual result that

$$\int_{\mathcal{C}} d\xi \, \frac{f(\xi)}{\xi - z \pm i\delta} = \int_{\mathcal{C}} d\xi \, \left[\mathbb{P} \frac{1}{\xi - z} \mp i\pi\delta(\xi - z) \right] f(\xi) \tag{66}$$



Now the application of the Poincare-Bertrand result to an integral like (62) will be obvious. By looking at the pole structure of (62) we see that we can drop terms where all poles in integrand of an integral (whether it be $\int d\epsilon_1$, $\int d\epsilon_2$, or $\int d\omega$) are on the same side of the real axis. Actually to go through all the terms in (62) is rather tedious, so I will just give the answer:

$$\begin{aligned}\Omega(T) &= -\sum_{\mathbf{k_1},\mathbf{k_2}}\sum_{\mathbf{q}} |V_{\mathbf{q}}|^2 f_{\mathbf{k_1}}(1-f_{\mathbf{k_1+q}}) f_{\mathbf{k_2}}(1-f_{\mathbf{k_2+q}}) \\ &\quad \times \left[\mathbb{P}\,\frac{1}{\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_1}}+\epsilon^0_{\mathbf{k_2+q}}-\epsilon^0_{\mathbf{k_2}}} - \pi^2\delta(\epsilon^0_{\mathbf{k_1}}+\epsilon^0_{\mathbf{k_2}}-\epsilon^0_{\mathbf{k_1+q}}-\epsilon^0_{\mathbf{k_2+q}})\right]\end{aligned} \tag{67}$$

Let us now observe that if we take the Landau-Cutkowsky results on trust, then we could have obtained this result almost immediately. Up to now I have used these rules to calculate the imaginary part of self-energy graphs, by using (30), which is the imaginary part of (26). But we can also calculate the total value of a closed graph like $\Omega(T)$, which of course must be real, by using (27) adapted to our specific diagram. Let's start from the general graph shown in the diagram, which is written in terms of some general 4-point vertex $\Gamma_4(1,2,3,4) = \Gamma_4(\mathbf{k}_1,\xi_1;\mathbf{k}_2,\xi_2;\mathbf{k}_3,\xi_3;\mathbf{k}_4,\xi_4)$, and fully renormalized lines, i.e., $\mathcal{G}_2(\mathbf{k}_j,\xi_j)$. At the end we shall reduce this calculation to that of the diagram in (56).

According to eq. (27), this graph is given by (here we let $(\mathbf{p},\epsilon)\to(\mathbf{k_1},\xi_1)$):

$$\Phi = -S_\Phi \prod_{j=1}^{4}\sum_{\mathbf{k}_j}\int \frac{d\xi_j}{2\pi}\,|\Gamma_4(\mathbf{k}_j,\xi_j)|^2\,A(\xi_j)\,\frac{f(\xi_j)}{\sum_{j=1}^4 \xi_j} \tag{68}$$

where we use the convention, as before, that for a hole line ξ_j is negative, and S_Φ is the symmetry factor (which for this graph is actually 1/4). We now simply substitute as follows:

$$\begin{aligned} A(\xi_j) &\longrightarrow -2\pi\delta(\epsilon^0_{\mathbf{k_j}}-\xi_j) \\ I_4(\mathbf{k}_j,\xi_j) &\longrightarrow V(\mathbf{q})\end{aligned} \tag{69}$$

and, taking account of the "Poincare-Bertrand" lemma, and the fact that we will have overlapping poles when we do the integrations over $\mathbf{k_1}$ and $\mathbf{k_3}$, we get

$$\Delta\Omega(T) = -\frac{1}{4}\prod_{j=1}^{4}\sum_{\mathbf{k}_j}|V_{\mathbf{q}}|^2\, f_1 f_{-2} f_3 f_{-4} \times \left[\frac{1}{\epsilon_1^0 - \epsilon_2^0 + \epsilon_3^0 - \epsilon_4^0} - \pi^2\delta(\epsilon_1^0 - \epsilon_2^0 + \epsilon_3^0 - \epsilon_4^0)\right] \quad (70)$$

where $f_1 = f(\epsilon_{\mathbf{k_1}})$ and $\epsilon_1 = \epsilon_{\mathbf{k_1}}$. If we now take account of momentum conservation, and allow the 4 possible combinations of momenta in (70) (these being: $1,2,3,4 = \{\mathbf{k_1}, \mathbf{k_1}+\mathbf{q}; \mathbf{k_2}, \mathbf{k_2}+\mathbf{q};\}$, or $\{\mathbf{k_1}+\mathbf{q}, \mathbf{k_1}; \mathbf{k_2}+\mathbf{q}, \mathbf{k_2}\}$, or $\{\mathbf{k_1}, \mathbf{k_1}+\mathbf{q}; \mathbf{k_2}+\mathbf{q}, \mathbf{k_2}\}$, or $\{\mathbf{k_1}+\mathbf{q}, \mathbf{k_1}; \mathbf{k_2}+\mathbf{q}, \mathbf{k_2}\}$, all just being relabelling of the graphs) then we get back in (67).

App. B.3.1 (c) COUPLED FERMIONS/SCALAR BOSONS: EXAMPLES

Things change a little bit when we include phonons in the theory. The basic diagram rules do not change from what we have already discussed, but the details do change.

In what follows we will look at a number of standard graph for this kind of theory. This include (i) the lowest self-energy graphs for the fermion and bosonic propagators, and (ii) the lowest non-trivial 3-point vertex describing the fermion-boson interaction. The theory to be discussed will be a set of spinless ......