APPENDIX: REGULARIZATION METHODS

In relativistic QFT we need to regularize the infinities that occur, in order to give meaning to the diagrams calculated in perturbative expansions, as well as to the theory as a whole. Various methods have been developed over the years to do this. The first systematic technique, now called "Pauli-Villars" regularization* (or sometimes called "covariant regularization") dates back to 1949. It is still commonly used, but we now have better techniques available, and in what follows I will first give some details of Pauli-Villars regularization, and then go on to discuss 2 modern techniques, viz.,

- Heat-kernel or "zeta-function" regularization, which was first discussed in the context of quantum gravity,** but rapidly found use in conventional QFT, since it is very useful in non-perturbative work; and

- Dimensional regularization,*** which is a very powerful tool for calculating diagrams in arbitrary dimensions; the diagrams are in fact calculated in a space of continuous complex dimension, and then analytically continued to the dimension of interest.

Both of these techniques, as well as Pauli-Villars, are of primary interest in relativistic QFT, and are mainly designed to deal with UV divergences (although dimensional regularization is also useful in dealing with IR divergences, both in relativistic theory and in a different non-relativistic version, called the "ϵ expansion"). In condensed matter systems UV divergences are not a problem, since there is almost always a natural cut-off in the problem; but IR divergences very commonly arise, and need to be regulated. Most commonly this is done as part of a more general renormalization scheme, typically using RG techniques - these are discussed in Chapter 7.

In what follows I first briefly describe the Pauli-Villars technique, and then go on to give more detail on the heat kernel and dimensional regularization methods. No attempt is made to give mathematically rigourous arguments - the emphasis is on practical application.

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* See W. Pauli, F. Villars, Rev. Mod. Phys <u>21</u>, 434 (1949)

** Dimensional regularization was invented for relativistic QFT more or less simultaneously in the following papers: G. 't Hooft, M. Veltman, Nucl. Phys. <u>B44</u>, 189 (1972); C.G. Bollini, J.J. Giambiagi, Phys. Lett. <u>40B</u>, 566 (1972); J.F. Ashmore, Nuovo Cim. Lett. <u>4</u>, 289 (1972); and G.M. Cicuta, E. Montaldi, Nuovo Cim. Lett. <u>4</u>, 329 (1972). For the ϵ-expansion, see, eg., K.G. Wilson, M.E. Fisher, Phys. Rev. Lett. <u>28</u>, 240 (1972).

*** For heat kernel methods (which have a long history), see JS Dowker, R. Critchley, Phys. Rev <u>D13</u>, 3324 (1976), SW Hawking, Comm. Math. Phys. <u>55</u>, 133 (1977)

I. PAULI-VILLARS REGULARIZATION

The basic idea of Pauli-Villars regularization is fairly simple, and is most easily explained by using an example. First, let us look at the general idea. In any relativistic QFT we are faced with UV divergent integrals, whose degree of divergence* depends on the dimensionality of the theory, and the form of the propagators. For example, we may be dealing with some integral over bosonic and fermionic propagators of form

$$\mathcal{D}_p = \prod_{\ell=1}^{m} \sum_{q_\ell} \left(\frac{i}{q_\ell^2 - \Omega^2 + i\epsilon} \right)^{m_\ell} \prod_{j=1}^{n} \sum_{k_j} \left(\frac{\gamma^\mu k_\mu + m}{k_j^2 + m^2} \right)^{n_j} \delta\Big(p - \sum_j n_j k_j - \sum_\ell m_\ell q_\ell\Big) \tag{1}$$

which describes some complicated diagram with external momentum p, and a set of internal lines with fermionic mass m and bosonic mass Ω. Some lines may be degenerate, ie. have the same internal momenta; in this case their multiplicities m_ℓ or n_j are greater than one. It is immediately obvious that for sufficiently large number of dimensions, such diagrams are UV divergent. Thus, in $d=4$, the simple integral

$$\mathcal{D}_0(x) = \sum_q e^{iqx} \frac{1}{q^2 - \Omega^2 - i\epsilon} \equiv \int_0^\Lambda \frac{d^dq}{(2\pi)^d} \frac{e^{iqx}}{q^2 - \Omega^2 + i\epsilon} \tag{2}$$

has a degree of divergence $\mathcal{D} = 2$, ie., it diverges like Λ^2 as $\Lambda \to \infty$; this is obvious if we let $x \to 0$.

The Pauli-Villars prescription replaces the propagators in (1) and (2) with others, which allow the integrals to be UV convergent. What replacement we use depends on the degree of divergence, and so depends on the form of the propagators, and on the kind of integrals we are faced with (these being determined by the form of the Lagrangian and by the dimensionality). Thus the common replacements for the massive boson propagators are:

$$\frac{1}{q^2 - \Omega^2 + i\epsilon} \longrightarrow \begin{cases} \left(\dfrac{1}{q^2 - \Omega^2 + i\epsilon} - \dfrac{1}{q^2 - M^2 + i\epsilon} \right) = \dfrac{1}{q^2 - \Omega^2 + i\epsilon} \left(\dfrac{\Omega^2 - M^2}{q^2 - M^2} \right) & \text{(3a)} \\[2ex] \dfrac{1}{q^2 - \Omega^2 + i\epsilon} \left(\dfrac{M^2}{q^2 - M^2} \right)^2 & \\[2ex] \quad \equiv \dfrac{1}{q^2 - \Omega^2 + i\epsilon} - \left(\dfrac{Mq}{q^2 - M^2} \right)^2 + \left(\dfrac{q^2}{q^2 - M^2} \right)^2 & \text{(3b)} \end{cases}$$

* For a discussion of degrees of divergence, see Ch. 7.

Let's first note the effect of these replacements on the mathematical structure. If we assume that the "mass" $M \gg \Omega$, then for low moments, such that $q \ll M$, the effect of these extra factors can be ignored - they are unity. However for $q \gg M$, the extra factors suppress the divergence; they multiply the original propagator by either M^2/q^2, or M^4/q^4, depending on which form is chosen. Which form is chosen depends on how strongly we need things to converge at high q. One can do similar thing with fermion propagators.

Physically we can think of these extra terms as coming from extra "passive fields". This is obvious in the case of the regulator in the form (3a); we have, along with our massive scalar boson of mass Ω, another one with mass M. In the case of (3b) it is not so clear; but notice that we can rewrite (3b) in the form (dropping the $i\epsilon$ factors to make things less cluttered):

$$\left.\begin{aligned}\frac{1}{q^2-\Omega^2} \longrightarrow & \lim_{\substack{M_1=M_2=M \\ M\gg\Omega}} \left[\frac{1}{q^2-\Omega^2} - \frac{1}{M_1^2-M_2^2}\left(\frac{\Omega^2-M_2^2}{q^2-M_1^2} - \frac{\Omega^2-M_1^2}{q^2-M_2^2}\right)\right] \\ = & \lim_{M\gg\Omega}\left[\frac{1}{q^2-\Omega^2}\left(\frac{M^2-\Omega^2}{q^2-M^2}\right)^2\right] = \frac{1}{q^2-\Omega^2}\left(\frac{M^2}{q^2-M^2}\right)^2\end{aligned}\right\} \quad (4)$$

where the first form makes it look as though we have now added two extra boson fields, of masses M_1 and M_2 (both very large compared to Ω).

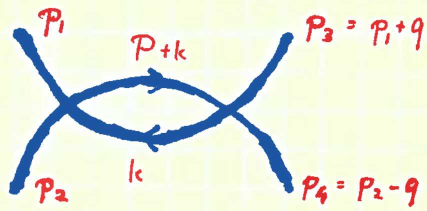
Thus the basic idea is to add one or more "heavy fields" to the initial field, and use these to regulate the integrals over the original field. These extra fields are not very physical - we notice that at least one of them must have a residue of the opposite sign to the original field, which would imply physically that they have switched kinetic and potential energy terms in the original Lagrangian (now with the added field(s)). There are other problems with Pauli-Villars regularization as well - notably that they lead to violation of gauge invariance in gauge theories, as we discuss in the main text. We note however that the Pauli-Villars technique is far superior to the simple expedient of imposing some arbitrary cut-off Λ in the theory - this violates both gauge invariance and Lorentz invariance.

EXAMPLE : ϕ^4 THEORY :

As always, we use the workhorse of scalar field theory to illustrate what is going on here. We will use the form (3a), writing it out fully as

$$D_0(q) \longrightarrow \left(\frac{1}{q^2-\Omega^2+i\epsilon} - \frac{1}{q^2-M^2+i\epsilon}\right) = \frac{\Omega^2-M^2}{(q^2-\Omega^2+i\epsilon)(q^2-M^2+i\epsilon)} \quad (5)$$

and, by way of example, we will set up the calculation of a simple 1-loop diagram for the 4-point vertex function $\Gamma_4(p_1, p_2; q)$; we will choose the diagram shown in the figure, which has the explicit expression



$P = p_1+p_2$

$$\Gamma(p_1, p_2; q) = \frac{(-ig_0)^2}{2} \sum_k G_0(p_1+p_2+k)\, G_0(k) \qquad (6)$$

which is a function only of the "centre of mass" 4-momentum $P = p_1+p_2$; writing it out explicitly, we have, in 4d, that

$$\Gamma(P) = \frac{(-ig_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(P+k)^2 - \Omega^2 + i\epsilon} \frac{i}{k^2 - \Omega^2 + i\epsilon} \qquad (7)$$

which is formally divergent; if we impose a UV cut-off Λ, we see that the integral $\sim O(\Lambda^2)$.

The first thing we do, in the Pauli-Villars technique, is to isolate out this divergence; in the standard way, we write $\Gamma(P)$ as an expansion in powers of P^2 around the origin, ie., we write

$$\Gamma(P) = \sum_{n=0}^{\infty} a_n \Gamma_n(P) \qquad (8)$$

where

$$\left.\begin{aligned} a_0 &= 1 \\ a_n &= 0 \qquad (n \text{ odd}) \\ a_n &= \frac{\partial^{2n}}{\partial P^{2n}} \Gamma(P) \Big|_{P=0} \qquad (n \text{ even}) \end{aligned}\right\} \qquad (9)$$

so that

$$\Gamma(P) = \Gamma_0 + a_2 \Gamma_2(P) + \cdots \qquad (10)$$

It is then obvious that the divergence is associated with Γ_0; the next term is logarithmically divergent, but we will now fix this with a Pauli-Villars regularization. We use the regularization in (3a), so that (7) becomes

$$\Gamma(P) = \frac{(-ig_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k+P)^2 - \Omega^2 + i\epsilon} \frac{1}{k^2 - \Omega^2 + i\epsilon} \frac{1}{k^2 - M^2 + i\epsilon} \qquad (11)$$

so that

$$\Gamma_0 = \tfrac{1}{2} g_0^2 M^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Omega^2 + i\epsilon)^2} \frac{1}{k^2 - M^2 + i\epsilon} \qquad (12)$$

which shows the UV divergence clearly, and

$$\Gamma_2^M(P) = -\frac{g_0^2}{2} M^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2-\Omega^2+i\epsilon}\,\frac{1}{k^2-M^2+i\epsilon}\right]\left[\frac{1}{(k+P)^2-\Omega^2+i\epsilon} - \frac{1}{k^2-\Omega^2+i\epsilon}\right] \qquad (13)$$

At this point we have brought the diagram to the form where we may apply Feynman parametrization to the integrals, and carry them out. This is done in the Appendix on diagram calculations, part B; one then finds that

$$\left.\begin{aligned}\Gamma_0 &= i\frac{g_0^2}{2} M^2 \frac{i}{(2\pi)^4}\int_0^1 d\alpha \,\frac{\alpha}{\alpha(\Omega^2-M^2)+M^2} \\ &\xrightarrow[M \gg \Omega]{} i\frac{g_0^2}{32\pi^2}\ln\frac{M^2}{\Omega^2}\end{aligned}\right\} \qquad (14)$$

which, as we discuss in Ch. 7, is a key part of the renormalization of the theory; and the convergent result, independent of M, for $\Gamma_2(P)$:

$$\Gamma_2(P) = i\frac{g_0^2}{2}\frac{1}{(2\pi)^4}\int_0^1 d\alpha\,(1-\alpha)(1+2\alpha)\,\frac{P^2}{\Omega^2-\alpha(1-\alpha)P^2-i\epsilon} \qquad (15)$$

which is evaluated in the diagram Appendix.

From all of this we see that Pauli-Villars regularization works; it allows us to define convergent integrals. The meaning of all this, as physics, is discussed in Ch. 7.

2. HEAT KERNEL/ζ-FN REGULARIZATION

The problem of regularization in QFT is an example of the more general problem encountered in many areas of mathematics, of summing over a formally divergent series, or dealing with a formally divergent integral. Such problems were first addressed by Euler, and have, in the 280 yrs since then, been the subject of an enormous corpus of work.* Some of the well-known methods include Euler summation, Borel summation, Abel summation, ~~[illegible]~~ Cesaro-Riesz summation, and Padé summation. In all of

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* See the Mathematical Supplement on "series summation"; for a more thorough look at asymptotic & divergent series, and related topics, see., eg., "Divergent Series", by G.H. Hardy (Oxford, 1949); or "Asymptotic Expansions: their derivation & Interpretation", R.B. Dingle, (Academic, 1973).

these methods, a divergent sum is "regularized", or transformed into a convergent one, by multiplying the terms in the series by factors which fall off in some way as one goes to higher terms. Thus, eg., given a series

$$S_A = \sum_n a_n \tag{16}$$

which diverges, we can define a function $S_A(x) = \sum_n a_n x^n$ (17)

and the associated Borel function $B(x) = \sum_n \frac{a_n}{n!} x^n$ (18)

then we can define

$$S_B = \lim_{x \to 1-\epsilon} B(x) \tag{19}$$

which converges unless $|a_n|$ grows faster than a factorial.* Understanding of both divergent & asymptotic series is essential in theoretical physics.

Heat kernel regularization is associated with the mathematical technique of ζ-function regularization. If we take a series like S_A in (16) above, we define the function

$$\zeta_A(s) = \sum_n a_n^{-s} \tag{20}$$

which, if $a_n = n$, just becomes the Riemann ζ-function. Thus $\zeta_A(s)$, the generalized ζ-function, can be viewed as another means of regularizing the series S_A. As such it can be used to sum a large variety of divergent series (see the previous references to series summation); the original series S_A is produced by analytic continuation to $s = -1$:

$$S_A = \lim_{s \to 1} \zeta_A(s) = \zeta_A(-1) \tag{21}$$

if this is well-defined. Related forms of regularization are the techniques of Dirichlet and exponential regularization**, and the technique known as "heat kernel" regularization is related to the latter; we define

$$S_A^K(s) = \sum_n a_n e^{-s|\lambda_n|} \tag{22}$$

where the $\{\lambda_n\}$ are eigenvalues of a "heat kernel" operator $\hat{K}$, with eigenfunctions $\phi_n \equiv |n\rangle$, such that

$$\hat{K}|\phi_n\rangle = \lambda_n |\phi_n\rangle \tag{23}$$

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* To show that $B(x)$ converges to S_A, we define the inverse Borel transform $S_B(x)$ by $S_B(x) = \int_0^\infty dt\, e^{-t} B(tx)$; if S_A converges then $S_B(x) = \sum_n \frac{1}{n!} \int_0^\infty dt\, e^{-t} t^n a^n x^n = S_A(x)$.

** Exponential regularization writes $S_A(t) = \sum_n a_n e^{-tn}$

and is of use in QFT because the differential operators involved in QFT look like heat kernel operator when written in Euclidean form.

In what follows I make no attempt to develop the theory of heat kernel regularization in its most general form, but focus instead on its more restricted use in QFT, where it reduces to a form of ζ-function regularization. More general discussions can be found in the literature*

BASIC DEFINITIONS & IDENTITIES

: In QFT one is typically concerned with operators of form

$$\hat{K} = \sum_n |\phi_n\rangle \lambda_n \langle\phi_n| \equiv \sum_n |n\rangle \lambda_n \langle n| \tag{24}$$

and with their associated propagator or Green function $G_K(x,x';\tau)$, defined in imaginary time τ by

$$\left.\begin{aligned} G_K(x,x';\tau) &= \sum_n \langle x|n\rangle e^{-\lambda_n \tau} \langle n|x'\rangle \\ &\equiv \sum_n \phi_n(x)\, \phi_n^*(x')\, e^{-\lambda_n \tau} \end{aligned}\right\} \tag{25}$$

Not only do we see functions in Euclidean QFT and in statistical mechanics, but also in ordinary QM, in its imaginary time Euclidean form. In relativistic QFT we can also think of τ as a "5th time" or proper time.

The propagator $G_K(x,x';\tau)$ obeys a differential eqtn of "heat diffusion" form, viz.,

$$(\partial_\tau + \hat{K})\, G_K(xx';\tau) = 0 \tag{26}$$

with boundary condition

$$G_K(x,x';\tau)\Big|_{\tau=0} = \delta(x-x') \tag{27}$$

This propagator is nothing but the matrix elements of an operator $\hat{G}_K(\tau)$ defined between eigenstates $|x\rangle$ and $|x'\rangle$, ie., defining

$$\hat{G}_K(\tau) = \sum_n |n\rangle e^{-\tau\hat{K}} \langle n| \tag{28}$$

we have

$$G_K(x,x';\tau) = \langle x|\, \hat{G}_K(\tau)\, |x'\rangle \tag{29}$$

We will also need the trace of $\hat{K}$, viz.,

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* For more on heat kernel & ζ-fn. regularization, see, eg., E. Elizalde, "Ten Physical Applications of spectral ζ functions" (Springer, 1995); "Cosmology, Quantum Vacuum, and ζ functions", ed. S.D. Odintsov, D. Sáez-Gómez, S. Xambó-Descamps (Springer, 2011); and E. Elizalde et al., "ζ Regularization techniques with Applications" (World Scientific, 1994).

$$\begin{aligned}\det|\hat{K}| &= e^{\mathrm{Tr}\ln\hat{K}} \\ &\equiv e^{\int d^dx\, \langle x|\ln\hat{K}|x\rangle} = \prod_n \lambda_n\end{aligned} \qquad (30)$$

as is easily seen by taking the logs of both sides.

We now define the generalized ζ-function for this problem as an expansion over the eigenvalues of K, viz.,

$$\begin{aligned}\zeta_K(xx'|s) &= \langle x|\hat{\zeta}_K(s)|x'\rangle \\ &= \sum_{n=0}^{\infty} \langle x|n\rangle \lambda_n^{-s} \langle n|x'\rangle\end{aligned} \qquad (31)$$

where the generalized ζ-operator $\hat{\zeta}_K(s)$ is

$$\begin{aligned}\hat{\zeta}_K(s) &= \sum_{n=0}^{\infty} |n\rangle \lambda_n^{-s} \langle n| \\ &\equiv \hat{K}^{-s}\end{aligned} \qquad (32)$$

Note that the generalized ζ-fn as defined by mathematicians is just the trace over the function we have defined here, ie.,

$$\begin{aligned}\zeta_K(s) &= \mathrm{Tr}\,\hat{\zeta}_K(s) \\ &= \int d^dx\, \zeta_K(x,x;s) = \sum_n \lambda_n^{-s}\end{aligned} \qquad (33)$$

We can also write $\zeta_K(s)$ and $\zeta_K(xx'|s)$ directly in terms of the heat function, using a Mellin transform on G_K; noting first that

$$\begin{aligned}\sum_{n=0}^{\infty} \int_0^{\infty} d\tau\, \tau^{s-1} e^{-\lambda_n \tau} &= \sum_{n=0}^{\infty} \Gamma(s)\, \lambda_n^{-s} \\ &= \Gamma(s)\, \zeta_K(s)\end{aligned} \qquad (34)$$

where $\Gamma(s)$ is the gamma function, we then have:

$$\begin{aligned}\zeta_K(xx'|s) &= \frac{1}{\Gamma(s)} \hat{M}_s\{G_K(xx';\tau)\} = \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau\, \tau^{s-1} G_K(x,x';\tau) \\ &\equiv \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau\, \tau^{s-1} \sum_n \phi_n(x)\phi_n(x')\, e^{-\lambda_n \tau}\end{aligned} \qquad (35)$$

for the Mellin transform of the heat kernel propagator, and

$$\left.\begin{aligned} \zeta_K(s) &= \frac{1}{\Gamma(s)} \hat{M}_s \{ \mathrm{Tr}\, \hat{G}_K(\tau) \} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1}\, \mathrm{Tr}\, e^{-\tau \hat{K}} \\ &\equiv \frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \int d^d x\, G_K(x,x';\tau) \end{aligned}\right\} \qquad (36)$$

for the Mellin transform of its trace*

Finally, we see that we can define $\mathrm{Tr}\, \hat{K}$ directly in terms of the generalized ζ-function. Noting that

$$\zeta_K'(s) \equiv \frac{d}{ds} \zeta_K(s) = -\sum_{n=0}^\infty e^{-s \ln \lambda_n} \ln \lambda_n \qquad (37)$$

so that
$$\zeta_K'(0) = -\ln \prod_n \lambda_n \qquad (38)$$

we then have from (30) that
$$\det |\hat{K}| = e^{-\zeta_K'(0)} \qquad (39)$$

All the results above are written for some field, with eqtn. of motion given in terms of the operator $\hat{K}$, on a flat spacetime of dimension d. One can also do things on a more general manifold, which may describe a curved spacetime; this manifold may or may not have a boundary ∂M, upon which boundary conditions on $\hat{K}$ and its solutions need to be given. Much of what we will discuss here is not modified by these generalizations; where there are modifications I will say so.

PERTURBATION THEORY & PROPER TIME EXPANSIONS

We now move to practical questions of how to use the ζ-fn technique. As usual we want to ~~UV~~ regulate some sum or integral which is UV divergent. This may be of quite simple form, as for example occurs for the calculation of fluctuation corrections to a simple path integral like the 1-d SHO, or in the theory of the Casimir effect in a simple geometry; then we deal with a sum of form $S_\omega = \sum_n \omega_n$, and to regularize it we make the transformation

$$S = \sum_n \lambda_n \longrightarrow \sum_n \lambda_n |\lambda_n|^{-s} = \sum_n \lambda_n^{1-s} \equiv \zeta_K(s-1) \qquad (40)$$

* The Mellin transform of a function $f(\tau)$ is $\hat{M}_s\{f(\tau)\} = \int_0^\infty d\tau\, \tau^{s-1} f(\tau)$; writing $F_M(s) = M_s\{f(\tau)\}$, the inverse transform is then given by $f(\tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \tau^{-s} F_M(s)$, with the vertical line at $\mathrm{Re}\, s = c$ to the left of poles. The Gamma fn. is, as usual, $\Gamma(s+1) = \int_0^\infty d\tau\, \tau^s e^{-\tau}$.

which is just that described in eqtn (20). Alternatively we can use the heat kernel form of regularization defined in eqtn (22); this is used often both in conventional QFT (as in our discussion of anomalies in Ch.6), and in quantum gravity (where it was first developed). Thus, eg., in the discussion of chiral anomalies in Ch.6 we deal with the following regularization:

$$\frac{1}{2} C(x, \bar{A}_n) = -\sum_n \bar{\phi}_n(x) \gamma^5 \phi_n(x) \rightarrow -2 \sum_n \bar{\phi}_n(x) \gamma^5 \phi_n(x) e^{-\lambda_n^2/\Lambda^2} \quad (41)$$

which is clearly of the heat kernel form in (22). In both (40) and (41), the eigenvalues are associated with a differential operator $\hat{K}$.

From what has been said so far, it is clear that two essential stages in doing a heat kernel or ζ-function regularization are

(i) solve the heat eqtn for G_K, either in closed form (if $\hat{K}$ and its boundary conditions are simple) or in approximate form);

(ii) Use a Mellin transform on this solution for G_K, to get a solution for $\zeta_K(s)$; this then gives a regularized sum/integral.

In what immediately follows we deal with two crucial techniques for solving the heat eqtn; this cannot, except for trivial problems, be done in closed form.

(a) PERTURBATION EXPANSION :

Typically we can write the operator $\hat{K}$ which appears in the heat kernel eqtn (26) as a sum of a simple operator K_0, for which the eqtn can be solved, and a perturbation ΔK on this - which hopefully is small. Writing

$$\hat{K} = \hat{K}_0 + \Delta\hat{K} \quad (42)$$

we then have

$$\hat{K}^{-s} = \hat{K}_0^{-s} - s K_0^{-(1+s)} \Delta\hat{K} + O(\Delta\hat{K})^2 \quad (43)$$

so that

$$\left.\begin{aligned} \zeta_K(xx'|s) = \langle x|\hat{K}_0^{-s}|x'\rangle - s\langle x|\hat{K}_0^{-(1+s)}\Delta K|x'\rangle \\ + \text{ etc.} \end{aligned}\right\} \quad (44)$$

and so on.

An important case of such a perturbation arises when we vary the Kernel $\hat{K}$ by some infinitesimal $\delta\hat{K}$; we would then like to know what is the infinitesimal change $\delta\zeta_K(s)$ in the ζ-function.

Symbolically we have

$$([\hat{K} + \delta\hat{K}] + \partial_\tau)[\hat{G}_K + \delta G_K] = 0 \quad (45)$$

from which we have

$$(\hat{K} + \partial_\tau)\, \delta G_k(xx';\tau) \;+\; \delta\hat{K}\, G_k(xx';\tau) \;=\; 0 \qquad (46)$$

which to $\sim O(\delta K)$ has the solution

$$\delta G_k(xx'\tau) \;=\; -\int_0^\tau d\tau' \int d^d y\; G(x,y;\tau-\tau')\, \delta\hat{K}\, G(y,x';\tau') \qquad (47)$$

Now, from (36) this immediately gives us the result, via Mellin transformation of (47), that

$$\delta \zeta_k(s) \;=\; -\frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} \int_0^\tau dt' \int d^d x \int d^d y\; \mathrm{Tr}\left[G_k(x,y;\tau-\tau')\, \delta\hat{K}\, G_k(y,x;\tau') \right] \qquad (48)$$

which, using the eigenfunction expansion of $G_k(x,x';\tau)$ from (25), and then doing the integral over t', gives us*

$$\delta\zeta_k(s) \;=\; -s \int d^d y \sum_{n=0}^{\infty} \lambda_n^{-(1+s)}\, \phi_n^*(y)\, \delta\hat{K}\, \phi_n(y) \qquad (49)$$

which we would also have derived from (44) by letting $\Delta\hat{K} = \delta\hat{K}$ and letting $K_0 = \hat{K}$. Expression like this turn out to be very useful in both ordinary QFT and in quantum gravity (once appropriate modifications have been made to deal with spacetime curvature).

(b) PROPER TIME EXPANSION: Since we are interested in the UV behaviour of whatever theory we are dealing with, and singularities associated with this, it is the short time behaviour of $G_k(xx'|\tau)$ that interests us. Accordingly we carry out a "proper time" expansion of $G_k(x,x'|\tau)$ around $\tau = 0$, of form

$$G_k(x,x'|\tau) \;=\; G^0(xx'|\tau) \sum_{n=0}^{\infty} a_n(x,x')\, \tau^n \qquad (50)$$

where $G^0(x,x'|\tau)$ is the known solution for K_0 in (42). Clearly, to make this expansion we need to decide on what general form for $\hat{K}$ we are interested. In flat space this will be taken to be

$$\hat{K} \;=\; \hat{K}_0 + \hat{}(x) \qquad (51)$$

where $U(x)$ will be some x-dependent "potential", and K_0 will either take

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* Alternatively, from the $\zeta_k(s)$ in (36), we have $\delta\zeta_k(s) = -\frac{1}{\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1}\, \mathrm{Tr}\, \delta G_k$ from which we also get (48).

the simple form

$$K_0 = \partial^2 = \partial_\mu \partial^\mu \tag{52}$$

or the somewhat more complicated

$$K_0 = \mathcal{D}^2 = (\partial_\mu + iA_\mu)(\partial^\mu + iA_\mu) \tag{53}$$

where we deal with gauge theories.*

To see how things work, let's take the simple form in (52) for $\hat{K}_0$. Then from

$$(\partial_\tau + \hat{K}_0)\, G_k^0(xx';\tau) = 0 \tag{54}$$

from (26), with the usual boundary condition in (27), we find

$$G_k^0(xx';\tau) = \sum_k e^{ik(x'-x)} e^{\tau k^2} = \frac{1}{(4\pi\tau)^{d/2}} e^{-(x-x')^2/4\tau} \tag{55}$$

which is the standard heat diffusion propagator.

To find the coefficients $\{a_n(x,x')\}$ in (50), we make the assumption that we work in flat spacetime, and go over to a momentum-space representation, so that we represent $G_k(xx'\tau)$ in the form**

$$\left.\begin{aligned} G_k(xx'\tau) &= \sum_k e^{-ikx} e^{-\tau\hat{K}} e^{ikx'} \\ &= \sum_k e^{-ikx} e^{-\tau(\partial^2 + V(x))} e^{ikx'} \\ &= \sum_k e^{ik(x'-x)} e^{\tau k^2} e^{-\tau[\partial^2 + 2ik_\mu\partial^\mu + V(x)]} \end{aligned}\right\} \tag{56}$$

where the 1st part of (56) is just $G_k^0(xx'\tau)$. We now expand the exponential in the last part of (56) in powers of τ, to get

$$\left.\begin{aligned} G_k(xx'\tau) = \sum_k e^{ik(x'-x)} e^{\tau k^2} \Big\{ & 1 + (\partial^2 + V)\tau \\ & + \tfrac{1}{2}\left[(\partial^2+V)^2 - 4(k_\mu\partial^\mu)^2\right]\tau^2 \\ & + \tfrac{1}{3!}\big[(k_\mu\partial^\mu)^2(\partial^2+V) + (\partial^2+V)(k_\mu\partial^\mu)^2 \\ & \qquad + k_\mu\partial^\mu(\partial^2+V)k_\mu\partial^\mu\big]\tau^3 \\ & + O(\tau^4)\Big\} \end{aligned}\right\} \tag{57}$$

---

* In dealing with gauge theories we note that terms with $A_\mu(x)$ in them may also have to be absorbed into $V(x)$

** Here we use $\partial_\mu e^{ikx'} = e^{ikx'}(ik_\mu + \partial_\mu)$, and also
$\partial^2 e^{ikx'} = e^{ikx'}(ik_\mu + \partial_\mu)(ik^\mu + \partial^\mu)$

and if we calculate these out, we get the results *

$$
\left.\begin{aligned}
a_0(x,x') &= 1 \\
a_1(x,x') &= V(x) \\
a_2(x,x') &= \tfrac{1}{2}\Big(V(x) + \tfrac{1}{3}[\partial_\mu, [\partial^\mu, V(x)]] \\
&\qquad + \tfrac{1}{4}[\partial_\mu, \partial_\nu][\partial^\mu, \partial^\nu]\Big) \\
&\text{etc.}
\end{aligned}\right\} \qquad (58)
$$

These coefficients are called the Hamideyw, or deWitt-Seeley coefficients, and have a long history; we are only scratching the surface of a very large topic here.** In particular, we have not discussed

- how all this works in curved spacetime - both for QFT, and for quantum gravity
- The complex role of boundary conditions and different kinds of boundary

for which one may consult the more specialized references given in footnotes.

3. DIMENSIONAL REGULARIZATION

Invented in 1972, this is a very handy method for calculating diagrams or more general quantities in QFT. It is a big improvement over the Pauli-Villars technique (also used for diagrammatic calculations) in that the singular contributions in diagrams are easily isolated - they appear as poles in the complex d-plane, where d is the dimensionality.

In order to dimensional regularization, we need to define the quantities appearing in the integrals in any diagram for a space of arbitrary dimension. For this we need a number of identities.

First, we define the volume of a hypersphere in a space of integer dimensions d, as

$$V_d(r) = \prod_{j=1}^{d} \int dx_j \; \theta(r^2 - \sum_j x_j^2) \qquad (59)$$

---

* Here we use the easily derived results: $\sum_k k_\mu k_\nu e^{-k^2\tau} = \frac{1}{2}\delta_{\mu\nu}\frac{1}{(4\pi)^{d/2}}\tau^{-(1+d/2)}$

and $\sum_k k_\mu k_\nu k_\alpha k_\beta e^{-k^2\tau} = \frac{1}{4}(\delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})\frac{1}{(4\pi)^{d/2}}\tau^{-(2+d/2)}$

** Apart from the references already given, some of the key early mathematical references are S. Minakshisundaram, A. Pliejel, Can. J. Math. 1, 242 (1949); R.T. Seeley, Proc. Symp. Pure Math. 10, 288 (1967); D.B. Ray, IM Singer, Adv. Math 7, 145 (1971).