B. FUNCTIONAL FORMULATION OF Q.F.T.

The functional formulation of QFT was initiated largely by Schwinger, but since it turns out to be equivalent to the path integral formulation of Feynman, we can treat them together. There are also interesting alternative formulations by the Russian schools of Bogoliubov & E.S. Fradkin, which we will also cover.

To properly understand the functional formulation, it is useful to see the key relationship it has with both probability theory & statistical mechanics, and with ordinary quantum mechanics, in its path integral formulation – both of these will be covered.

It is also important to see how it applies to specific theories. In this pedagogical introduction, we will only look at simple models, of either quantum mechanics or toy field theories. Once these are understood, one can go on to look at either relativistic gauge theories (with gauge fields coupled to bosons or fermions) or at non-relativistic field theories of N-particle systems.

B.1: THE GENERATING FUNCTIONAL $Z[J]$

: This is a key object in quantum field theory – as we shall see it reduces in ordinary Q.M. to a limiting form of the 1-particle propagator. However, before beginning, it is useful to look at another closely related object, viz., the generating functional in ordinary probability theory.

Recall that for a simple random variable ϕ, we can assign a probability distribution $P(\phi)$, so that the expectation value of any variable $A(\phi)$ that depends on ϕ is given by

$$\langle A \rangle = \int d\phi\, P(\phi)\, A(\phi) \tag{1}$$

where we assume that

$$\int d\phi\, P(\phi) = 1 \tag{2}$$

Now consider the generating function $\bar{Z}(J)$ defined by

$$\bar{Z}(J) = \int d\phi\, P(\phi)\, e^{\phi J} \tag{3}$$

From this def^n it immediately follows that

$$g_n \equiv \langle \phi^n \rangle \equiv \int d\phi\, P(\phi)\, \phi^n = \left.\frac{\partial^n \bar{Z}(J)}{\partial J^n}\right|_{J=0} \tag{4}$$

and that we can expand $\bar{Z}(J)$ as

$$\left.\begin{aligned} \bar{Z}(J) &= \sum_{n=0}^{\infty} \frac{J^n}{n!} \int d\phi\, P(\phi)\, \phi^n \\ &\equiv \sum_{n=0}^{\infty} \frac{1}{n!}\, g_n\, J^n \end{aligned}\right\} \tag{5}$$

so that $\bar{Z}(J)$ acts as a "generator" for the infinite sequence of moments g_n of the probability distribution $P(\phi)$.

For future reference it is also useful to recall how one may also expand the logarithm of $\bar{Z}(J)$; we write

$$\left.\begin{aligned} Z(J) &= e^{\bar{W}(J)} \\ \bar{W}(J) &= \ln \bar{Z}(J) = \ln \int d\phi\, P(\phi)\, e^{\phi J} \end{aligned}\right\} \quad (6)$$

Now suppose we make a power series expansion of $\bar{W}(J)$, ie., write

$$\bar{W}(J) = \sum_{n=0}^{\infty} \frac{1}{n!} C_n J^n \quad (7)$$

where the C_n, known as "cumulants", are just

$$C_n = \left.\frac{\partial^n \bar{W}(J)}{\partial J^n}\right|_{J=0} \quad (8)$$

The relationship between the cumulants C_n and the moments g_n is easily found. We notice that

$$\left.\begin{aligned} \frac{\partial \bar{W}(J)}{\partial J} &= \frac{\partial \ln \bar{Z}(J)}{\partial J} = \frac{1}{\bar{Z}} \frac{\partial \bar{Z}}{\partial J} \\ \frac{\partial^2 \bar{W}(J)}{\partial J^2} &= \frac{\partial}{\partial J}\left(\frac{1}{\bar{Z}} \frac{\partial \bar{Z}}{\partial J}\right) = \left[\frac{1}{\bar{Z}} \frac{\partial^2 \bar{Z}}{\partial J^2} - \frac{1}{\bar{Z}^2} \frac{\partial \bar{Z}}{\partial J}\right] \end{aligned}\right\} \quad (9)$$

and so on; from this, using (6), we immediately* find that, eg.,

$$\left.\begin{aligned} C_1 &= g_1 & C_3 &= g_3 - 3g_1 g_2 + 2g_1^3 \\ C_2 &= g_2 - g_1^2 & C_4 &= g_4 - 3g_2^2 - 4g_1 g_3 + 12 g_2 g_1^2 - 6g_1^4 \end{aligned}\right\} \quad (10)$$

Finally, note that in simple probability theory we can make a "Legendre transform" to change the dependent variables; if we write

$$\bar{W}(J) = \bar{\Gamma}(\phi) + J\phi \quad (11)$$

then in terms of the new function $\bar{\Gamma}(\phi)$, we have $\quad \frac{\partial \bar{\Gamma}(\phi)}{\partial \phi} = -J \quad (12)$

Later on we will explore the analogies between probability theory, field theory, and thermodynamics / statistical mechanics.

B.1(a) GENERATING FUNCTIONAL & PROPAGATORS for $\phi^4(x)$ THEORY

Suppose we are now dealing with a quantum field $\phi(x)$, in D dimensions. Then the analogue of the generating function (3) for ordinary probability distributions is the generating functional $Z[J]$, given by

$$\left.\begin{aligned} Z[J] &= N \int \mathcal{D}\phi\, A(\phi; J=0)\, e^{i\int d^D_x\, J(x)\phi(x)} \\ &\equiv N \int \mathcal{D}\phi\, e^{i/\hbar\, S[\phi]}\, e^{i\int d^D_x\, J(x)\phi(x)} \end{aligned}\right\} \quad (13)$$

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* To find the $\{c_j\}$ in terms of the $\{g_j\}$, expand $\ln \bar{Z}[J] = \ln\left\{\exp\left[\sum_n \frac{1}{n!} g_n J^n\right]\right\}$, and compare coefficients of J^n

and we see that the analogue of the probability distribution $P(\phi)$ is the "partition function", $Z[J=0,\phi]$, ie., the normalization factor N in (13) is:

$$\left.\begin{aligned} N^{-1} &= Z[J=0] = \int \mathcal{D}\phi\, e^{i/\hbar S[\phi]} \equiv \int \mathcal{D}\phi\, A(\phi) \\ \text{where: } A(\phi) &= e^{i/\hbar S[\phi]} \equiv e^{i/\hbar \int d^D x\, \mathcal{L}(\phi(x))} \end{aligned}\right\} \tag{14}$$

ie., the analogue of the probability $P(\phi)$ for a random variable to take the value ϕ, is now the AMPLITUDE $\exp^{i/\hbar}S[\phi]$ for a field to take a given configuration $\phi(x)$. We shall see later what we must write to deal with probabilities in QFT.

Note that a convergence factor has been left out of eqtns. (13) and (14). Just as in ordinary Q.M., we need to specify which configurations are being integrated over, and with which boundary conditions. In ordinary QM., this is done by either rotating the end points of the time integration to $\pm i\infty$, or by adding a term to the exponent to force the Green function $\int \mathcal{D}q\, e^{i/\hbar S[q]}$ to pick out the ground state (see notes on path integrals for Q.M.). The same is done here - we write

$$\left.\begin{aligned} Z[J] &= N\int \mathcal{D}\phi\, A(\phi; J=0)\, e^{i/\hbar \int d^D x\, [J(x)\phi(x) + i\epsilon\phi^2(x)]} \\ &\equiv N\int \mathcal{D}\phi\, e^{i/\hbar \int d^D x\, [\mathcal{L}(\phi(x)) + J(x)\phi(x) + i\epsilon\phi^2(x)]} \end{aligned}\right\} \tag{15}$$

I do not go through the demonstration that this gives the desired result, since it is similar to (but more complicated than) the demonstration for ordinary QM. In any case, the key point here is that

$$Z[J] = \langle 0 | 0 \rangle_{J(x)} \tag{16}$$

ie., $Z[J]$ is the vacuum expectation value for the theory, under the influence of an external source $J(x)$.

From now on all reference to the convergence factor in (15) is suppressed.

Continuing now in analogy with ordinary QM., and with the simple theory of probability, let us define the time-ordered correlators

$$\boxed{G_n(x_1 \ldots x_n) = \langle 0 | \hat{T}\{\phi(x_1)\ldots\phi(x_n)\} | 0 \rangle \equiv \frac{\int \mathcal{D}\phi\, e^{i/\hbar S[\phi]}\, \phi(x_1)\ldots\phi(x_n)}{\int \mathcal{D}\phi\, e^{i/\hbar S[\phi]}}} \tag{17}$$

where the denominator is added to normalize the result (we cannot in general assume the analogue of eqtn (2)); the corresponding power series expansion is then (cf. eqtn (5)):

$$\boxed{Z[J] = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int dx_1 \ldots \int dx_n\, G_n(x_1 \ldots x_n)\, J(x_1)\, J(x_2) \ldots J(x_n)} \tag{18}$$

and the G_n are related to $Z[J]$ by
$$\boxed{G_n(x_1 \ldots x_n) = (-i\hbar)^n \left.\frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\right|_{J=0}} \tag{19}$$

(cf. eqtn. (4)).

Noting that $Z(J)$ in ordinary probability theory is just the expectation value of the function $e^{\phi J}$, we see that in scalar field theory, we have

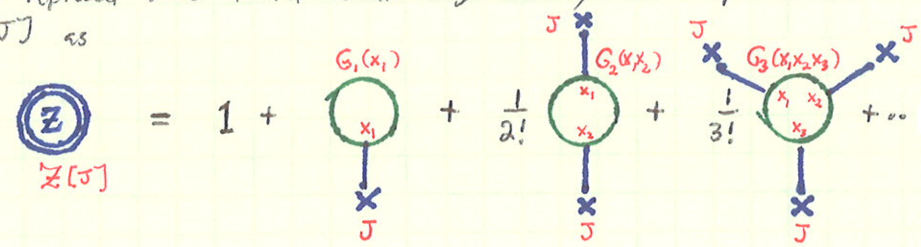
$$\boxed{Z[J] = \langle 0 | \hat{T}\{ e^{\frac{i}{\hbar}\int dx\, J(x)\phi(x)} \} | 0 \rangle} \qquad (20)$$

which when expanded out gives

$$Z[J] = \langle 0|0\rangle + \frac{i}{\hbar}\int d^Dx \langle 0|\phi(x)|0\rangle J(x) + \frac{1}{2!}\int d^Dx_1 \int d^Dx_2 \langle 0| T\{\phi(x_1)\phi(x_2)\}|0\rangle J(x_1) J(x_2) + \cdots \qquad (21)$$

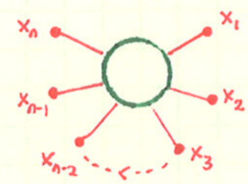
in agreement with (15) and (16).

It is useful to represent these formal results diagrammatically. We represent the vacuum functional $Z[J]$ as

$Z[J] =$ 

(22)

which just the series sum in (18) and (21); here we assume

The Green function are represented diagrammatically as. $G_n(x_1 .. x_n) =$ 

(23)

These diagrams are drawn according to well-defined rules, which can be read off from the relevant equations. Thus we have the correspondences:

$$f(x) \longrightarrow\!\!\times\; J \quad \equiv \quad \frac{i}{\hbar}\int d^Dx\, J(x)\, f(x) \qquad (24)$$

ie, we multiply whatever this "external leg" is attached to be the factor $\frac{i}{\hbar}J(x)$, and then integrate the combination over x. This is clear in (18) and (21). Note that whereas $G_n(x_1..x_n)$ has n external legs, $Z[J]$ has none – they are all integrated over.

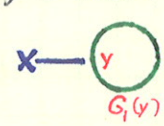
We get another useful relation, of recursive form, between $G_1(y)$ and the generating functional $Z[J]$ by comparing the 2 sides of (20) in series expanded form:

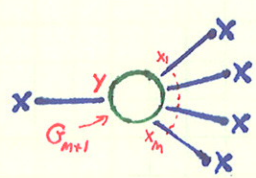
$$G_1(y) = -i\hbar \frac{\delta Z[J]}{\delta J(y)} = -i\hbar \frac{\delta}{\delta J(y)} \sum_{n=0}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\int dx_1 .. dx_n\, G_n(x_1..x_n) J_1(x_1) \cdots J(x_n) \qquad (25)$$

and reorganizing the labelling in the series, we get

$$G_1(y) = \sum_{m=0}^{\infty} \left(\frac{i}{\hbar}\right)^m \frac{1}{m!}\int dx_1 .. dx_m\, G_{m+1}(x_1..x_m; y)\, J(x_1) \cdots J(x_m) \qquad (26)$$

which is represented as (adding an external leg to each side of (26)):

$G_1(y) =$ 

$= \sum_{m=0}^{\infty} \left(\frac{i}{\hbar}\right)^m \frac{1}{m!}$ 

(27)

an eqtn. to which we will return. Note the simple result it corresponds to, in ordinary probability theory: from (5) we have

$$g_1 = \langle\phi\rangle = \int d\phi\, P(\phi)\, \phi e^{\phi J} = \sum_{n=0}^{\infty} \frac{J^n}{n!} \phi^{n+1} P(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} g_{n+1} J^n \qquad (28)$$

So far everything said has been true for any Lagrangian density $\mathcal{L}[\phi(x)]$, ie., we have not used a specific interaction like a $\phi^4(x)$ term. This will come later.

B.1(6) CONNECTED GREEN FUNCTIONS for $\phi^4(x)$ THEORY

: We have seen how the generating functional / partition function / vacuum expectation value $Z[J]$ corresponds to the generating function $Z(J)$ for simple probability distributions. Let's now continue this analogy, and write

$$\boxed{Z[J] = e^{\frac{i}{\hbar} W[J]}} \qquad (29)$$

(compare eqtn (6)). We now proceed in complete analogy with eqtns. (6) - (10), by defining what we call "connected" Green fns. (the name will become clear), as

$$\boxed{\mathcal{G}_n^{(c)}(x_1 \ldots x_n) = (-i\hbar)^n \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\Bigg|_{J=0}} \qquad (30)$$

and

$$\boxed{W[J] = -i\hbar \sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int dx_1 \ldots dx_n\, \mathcal{G}_n^{(c)}(x_1 \ldots x_n)\, J(x_1) \ldots J(x_n)} \qquad (31)$$

Comparing (19) and (30), we see that

$$G_n(x_1 \ldots x_n) = Z[J]\, \mathcal{G}_n^{(c)}(x_1 \ldots x_n) \qquad (32)$$

We see that the connected Green fns. here are the analogue of the "cumulants" that we defined for ordinary probability distributions — the 2 key differences being (a) we now use $W[J] = -i\hbar \ln Z[J]$ instead of $\bar{W}[J] = \ln Z[J]$, since we deal with quantum mechanics (see "NOTE" below); and (b) we now deal with functionals instead of functions.

Actually the closer analogy between $Z[J]$ and probability theory is with the "characteristic function" $Z(J)$ (as opposed to $\bar{Z}(J)$ in eqtns (1)-(9)):

$$Z(J) = \int d\phi\, P(\phi)\, e^{iJ\phi} = \langle e^{i\phi J}\rangle \qquad (33)$$

which can also be thought of as a simple Fourier transform of $P(\phi)$. To differentiate between all of these we put a bar over quantities like $\bar{W}$ or $\bar{Z}$ to indicate

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NOTE: It is common in the literature to write $Z[J] = e^{\bar{W}[J]}$, in very close analogy with eqtn (6). We then have the definitions & results as follows:

$$\bar{W} = \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\, \bar{\mathcal{G}}_n^{(c)}(x_1 \ldots x_n)\, J(x_1) \ldots J(x_n) \qquad \bar{\mathcal{G}}_n^{(c)}(x_1 \ldots x_n) = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}$$

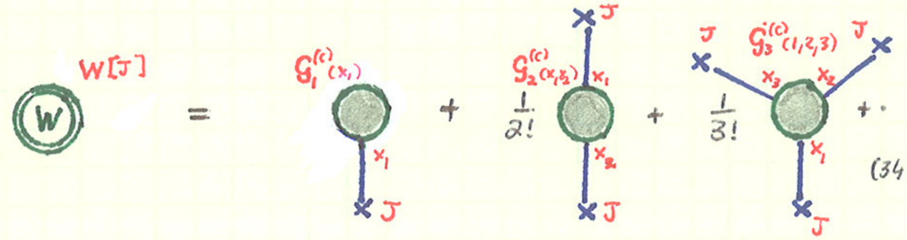
and $\dfrac{1}{Z[J]} \dfrac{\delta Z[J]}{\delta J} = \dfrac{\delta \bar{W}}{\delta J}$ and finally

$$G_1(y) = -i\hbar \frac{\delta Z[J]}{\delta J(y)} = Z[J]\, \mathcal{G}_1^{(c)}(y) = -i\hbar\, Z[J]\, \bar{\mathcal{G}}_1^{(c)}(y)$$

functions that are the results of exponentiation (like $\tilde{Z}(J) = e^{\tilde{W}(J)}$) as opposed to "Fourier transformation (like $Z(J) = e^{i/\hbar W(J)}$).

Now let's consider the relation between the connected Green functions and their graphical representation. Notice first of all from (31) that we can make the same sort of graphical expansion for $W[J]$ as we did for $Z[J]$, viz.,

$W[J] = -i\hbar \ln Z[J] =$



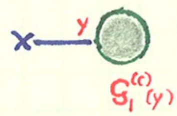
(34)

where the connected Green functions are shown as solid circles; and as before we can write that (cf. (26)):

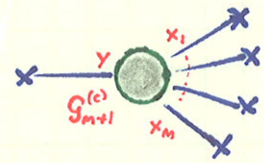
$$\mathcal{G}_1^{(c)}(y) = -i\hbar \frac{\delta W[J]}{\delta J(y)} = \sum_{m=1}^{\infty} \left(\frac{i}{\hbar}\right)^m \frac{1}{m!} \int dx_1 \cdots \int dx_m\, \mathcal{G}_m^{(c)}(x_1 \ldots x_m)\, J_1(x_1) \ldots J(x_m) \qquad (35)$$

or, in diagrams, that

$\mathcal{G}_1^{(c)}(y) =$

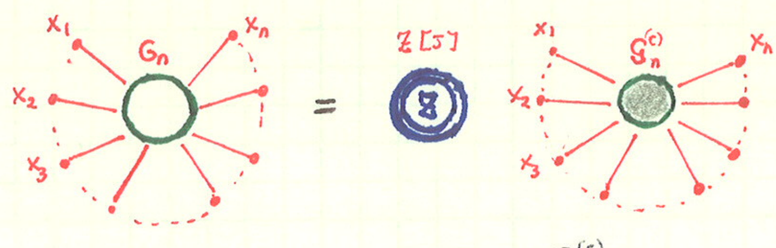


$= \sum_{m=1}^{\infty} \left(\frac{i}{\hbar}\right)^m \frac{1}{m!}$



(36)

So far so good. But now let's consider how we represent eqtn (32) graphically; we just have



$$G_n(x_1 \ldots x_n) = Z[J] \times \mathcal{G}_n^{(c)}(x_1 \ldots x_n) \qquad (37)$$

Now consider how to interpret this graphically. The left-hand side is the set of all graphs with n external legs (connected to the points $x_1, \ldots x_n$). On the right-hand side we imagine taking all n-leg graphs, and separating out all those that are "connected" graphs, ie., all those that cannot be seperated into 2 parts. Then the set of all n-leg graphs must be made from this connected set, multiplied by those parts disconnected from them. But this set of all disconnected parts has no external legs, and is therefore just $Z[J]$. Thus we get eqtn (37).

If this argument seems too glib to you, a more rigorous algebraic argument is given at the end of this section, in which the exponential in eqtn (29) is translated directly into a sum of products over connected graphs. An even more transparent discussion appears in the section on perturbative expansions and diagrammar.

B.1(c) PROPER VERTEX FUNCTIONS for $\phi^4(x)$ THEORY

So far we have expanded 2 functionals in infinite series, viz., the Vacuum Energy generating functional $Z[J]$ (the partition function), and the generating functional $W[J]$ for connected diagrams (analogous to the free energy in statistical mechanics), related by

$$W[J] = -i\hbar \ln Z[J] \tag{38}$$

with expansions

$$\left.\begin{aligned} Z[J] &= \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^{n} \int d^4x_j \, G_n(x_1..x_n) \, J(x_j) \\ W[J] &= \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \prod_{j=1}^{n} \int d^4x_j \, \mathcal{G}_n^{(c)}(x_1..x_n) \, J(x_j) \end{aligned}\right\} \tag{39}$$

Now we introduce a 3rd expansion, but this time the generating functional $\Gamma[\phi]$ is not a functional of the external currents, but of the field $\phi(x)$ itself. We write

$$\boxed{\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots \int d^4x_n \, \Gamma_n(x_1..x_n) \, \phi(x_1) \ldots \phi(x_n)}$$

so that

$$\boxed{\Gamma_n(x_1..x_n) = \left. \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1) \ldots \delta\phi(x_n)} \right|_{\phi=0}} \tag{40}$$

Now of course the way to change variables between the $J(x)$ and the $\phi(x)$ is by a Legendre transformation, of form given here by

$$\boxed{\Gamma[\phi] = W[J] - \int d^4x \, J(x) \phi(x)} \tag{41}$$

Thus we now have the pair of relations

$$\boxed{\left. \frac{\delta W[J]}{\delta J(x)} \right|_{\phi} = \phi(x) \qquad \left. \frac{\delta \Gamma[\phi]}{\delta\phi(x)} \right|_{J} = -J(x)} \tag{42}$$

Later on we will look in some detail at vertex functions, which turn out to be central to the application of QFT to the dynamics of real physical systems - used in everything from scattering theory to particle physics to transport theory in condensed matter systems.

Here we will simply establish the relationship between the connected Green functions $\mathcal{G}_n^{(c)}(x_1..x_n)$ and the proper vertices $\Gamma_n(x_1..x_n)$. Let's first do this for the simplest case, that of the 2nd-order correlators $\mathcal{G}_2^{(c)}(x_1,x_2)$ and $\Gamma_2(x_1 x_2)$. From their definitions, and from (42), we have

$$\left.\begin{aligned} \mathcal{G}_2^{(c)}(x_1 x_2) &= -\hbar^2 \left. \frac{\delta^2 W[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} = -\hbar^2 \left. \frac{\delta\phi(x_1)}{\delta J(x_2)} \right|_{J=0} \\ \Gamma_2(x_1 x_2) &= \left. \frac{\delta^2 \Gamma[\phi]}{\delta\phi(x_1) \delta\phi(x_2)} \right|_{\phi=0} = - \left. \frac{\delta J(x_1)}{\delta\phi(x_2)} \right|_{\phi=0} \end{aligned}\right\} \tag{43}$$

from which we immediately find that

$$\int d^4y\, \mathcal{G}_2^{(c)}(x_1,y)\,\Gamma_2(y,x_2) \;=\; \frac{\delta\phi(x_1)}{\delta\phi(x_2)} \;=\; \hbar^2\,\delta(x_1-x_2) \tag{44}$$

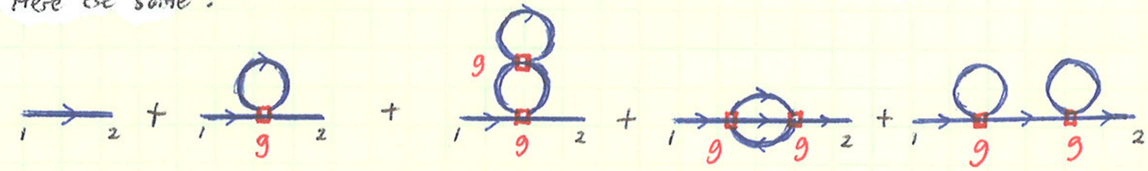
whose Fourier transform is

$$\mathcal{G}_2^{(c)}(p,-p) \;=\; \hbar^2/\Gamma_2(p,-p) \tag{45}$$

ie., the two functions are inverses of each other.

To understand this a little better it is helpful to look at them diagrammatically. Let's consider what kind of graphs make up $\mathcal{G}_2^{(c)}(x_1,x_2)$. As we saw above the $\mathcal{G}_n^{(c)}(x_1,..x_n)$ are "connected graphs", which cannot be seperated without cutting internal lines. Here are some:

$\mathcal{G}_2^{(c)}(x_1,x_2)$



+ etc. (46)

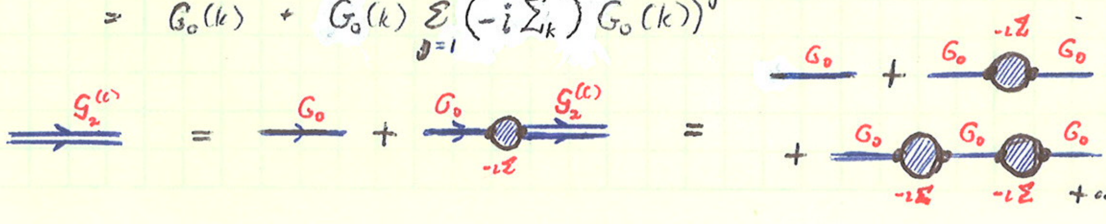
Now we notice that we can divide graphs like those above into 2 kinds. There are those graphs which, apart from their ingoing and outgoing lines, cannot be seperated into 2 parts by cutting any internal lines – or to put it differently, they can be viewed as "internal irreducible" parts, connected to 2 external legs. Note that all of these "irreducible vertex parts" are different from each other. Then we have graphs that can be seperated into 2 parts by cutting an internal line; these are called "reducible" graphs. We see that we can get all graphs for $\mathcal{G}_2^{(c)}(x_1,x_2)$ by simply stringing together the irreducible parts in all possible ways. Let us put this formally; we define the following 2 quantities

$$G_0(k) = \frac{i\hbar}{k^2-m^2} \qquad\qquad -i\Sigma(k) \equiv \text{sum of all irreducible vertex parts in } \mathcal{G}_2^{(c)}(k) \tag{47}$$

where $G_0(k)$ is of course just the free propagator, which we have already met in our discussion of 1-particle Green functions in QM. The diagram for $-i\Sigma(k)$ has no external legs – the dashed lines simply indicate where they would attach to $-i\Sigma(k)$. We work here in 4-momentum space, simply because momentum is conserved. Now we obtain the full $\mathcal{G}_2^{(c)}(k) \equiv \mathcal{G}_2^{(c)}(k,-k)$ as:

$$\begin{aligned}\mathcal{G}_2^{(c)}(k) &= G_0(k) + G_0(k)\left(-i\Sigma_k\right)\mathcal{G}_2^{(c)}(k)\\ &= G_0(k) + G_0(k)\sum_{j=1}^{\infty}\left(-i\Sigma_k\right)G_0(k))^j\end{aligned} \tag{48}$$

ie;



and so we have

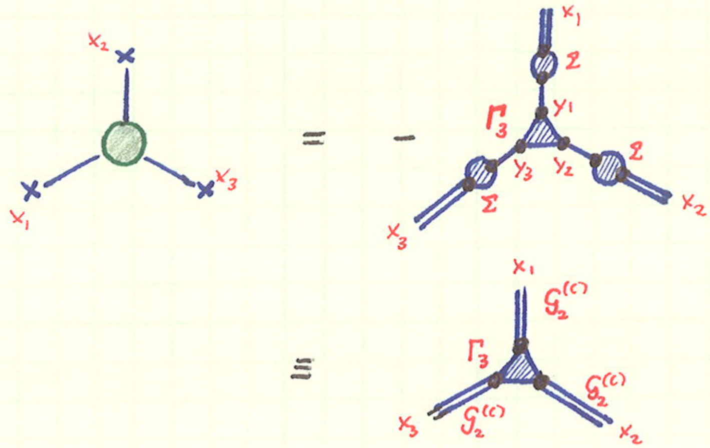
$$\left.\begin{aligned} \mathcal{G}_2^{(c)}(k) &= \frac{i\hbar}{k^2 - m^2 - \hbar\Sigma(k)} \\ \Gamma_2(k) &= -i\hbar\left(k^2 - m^2 - \hbar\Sigma(k)\right) \end{aligned}\right\} \quad (49)$$

and we see that the vertex part, apart from the term $(k^2 - m^2)$, can be identified with $-i\Sigma(k)$, which is of course just the self-energy.

One can continue this analysis for the higher-order functions $\mathcal{G}_n^{(c)}(x_1..x_n)$ and $\Gamma_n(x_1..x_n)$. Without going through the derivations, we find that

$$\left.\begin{aligned} \mathcal{G}_3^{(c)}(x_1 x_2 x_3) &= -\int d^4y_1 \int d^4y_2 \int d^4y_3 \; \Gamma_3(y_1 y_2 y_3)\, \mathcal{G}_2^{(c)}(x_1 y_1)\, \mathcal{G}_2^{(c)}(x_2 y_2)\, \mathcal{G}_2^{(c)}(x_3 y_3) \\ \Gamma_3^{(c)}(x_1 x_2 x_3) &= -\int d^4y_1 \int d^4y_2 \int d^4y_3 \; \mathcal{G}_3^{(c)}(y_1 y_2 y_3)\, \Gamma_2(y_1 x_1)\, \Gamma_2(y_2 x_2)\, \Gamma_2(y_3 x_3) \end{aligned}\right\} \quad (50)$$

or, in diagrams,



(50)

where we show the 1st eqn in (50).

One can continue to Γ_4, which can be written in terms of $\mathcal{G}_4^{(c)}$, $\mathcal{G}_3^{(c)}$, $\mathcal{G}_2^{(c)}$, Γ_3, and Γ_2; and so on; all the $\Gamma_n(x_1..x_n)$ have the same "tree" structure.

We have now written expression for $Z[J]$ in terms of 3 different sets of quantities, viz., the correlations $G_n(x_1..x_n)$, the connected correlations $\mathcal{G}_n^{(c)}(x_1..x_n)$, and the proper vertices $\Gamma_n(x_1..x_n)$. Now it turns out that each of these has an important physical meaning; but to better appreciate this we have to first go on with our formal development

B.1(d) FREE PROPAGATORS for ϕ-FIELD

: Before we ever look at the effect of interactions on the field dynamics (and this is of course the whole point of QFT), we need to establish the properties of the free field, without interactions. What this means, so far as we are concerned, is determining the form of the correlation functions. Thus we will begin again with the generating functional $Z_0[J]$

for the ϕ-field scalar theory, given in eqtns. (13) and (15) above. Let us write this out explicitly for the free system, viz.,

$$Z_0[J] = \frac{\int \mathcal{D}\phi \, e^{i/\hbar \int d^Dx \, [\mathcal{L}_0(\phi) + \phi(x) J(x) + i\delta\phi^2(x)]}}{\int \mathcal{D}\phi \, e^{i/\hbar \int d^Dx \, [\mathcal{L}_0(\phi) + i\delta\phi^2(x)]}} \tag{51}$$

where we include the convergence factor $i\delta\phi^2(x)$ in the exponent, so as not to forget it. Now actually we can evaluate (51) exactly because the functional integrals involve quadratic forms. To make this more explicit, let's rewrite the free field action, noting that (NB: $\partial^2 = \partial^\mu\partial_\mu$ is often written as $\Box$ or Δ):

$$\left.\begin{aligned} \int d^Dx \, (\partial^\mu\phi\partial_\mu\phi - m^2\phi^2) &= \int d^Dx \, [\partial_\mu(\phi\partial^\mu\phi) - \phi\partial^2\phi - m^2\phi^2] \\ &= \oint d^{D-1}\underline{x} \cdot (\phi\partial^\mu\phi) - \int d^Dx \, \phi(\partial^2 + m^2)\phi \end{aligned}\right\} \tag{52}$$

using the D-dimensional variant of Gauss's theorem. Thus we have the alternative form for $Z_0[J]$ as

$$Z_0[J] = \frac{\int \mathcal{D}\phi \, e^{-Q_J^0(\phi)}}{\int \mathcal{D}\phi \, e^{-Q_0(\phi)}} \tag{53}$$

where we drop the total derivative "surface integral" term in (52), simply assuming that $|\phi| \to 0$ fast enough at the boundaries of whatever system we deal with; and where we define

$$\left.\begin{aligned} Q_0(\phi) &= \frac{1}{2}\int d^Dx \, \phi(x)\left[\frac{i}{\hbar}(\partial^2 + m^2)\right]\phi(x) \quad \equiv (\phi, Q_0\phi) \\ Q_J^0(\phi) &= \int d^Dx \left\{\frac{1}{2}\phi(x)\left[\frac{i}{\hbar}(\partial^2+m^2)\right]\phi(x) - \frac{i}{\hbar}J(x)\phi(x)\right\} \quad \equiv (\phi, Q_J^0\phi) \end{aligned}\right\} \tag{54}$$

These are standard Gaussian functional integrals, and we immediately get

$$\left.\begin{aligned} \int \mathcal{D}\phi \, e^{-Q_0(\phi)} &= |\det Q_0|^{-1/2} \\ \int \mathcal{D}\phi \, e^{-Q_J^0(\phi)} &= |\det Q_0|^{-1/2} \, e^{\frac{1}{2}(j, Q_0^{-1} j)} \end{aligned}\right\} \tag{55}$$

where $Q_0 = i/\hbar \, (\partial^2 + m^2)$, and $j = i/\hbar \, J(x)$. The determinants cancel in (53), and we are left with

$$\boxed{Z_0[J] = e^{-\frac{i}{2\hbar}\int d^Dx \int d^Dx' \, J(x)\Delta_F(x-x')J(x')}} \tag{56}$$

where the "Feynman propagator" is: $\Delta_F(x) = -(\partial^2 + m^2 - i\delta)^{-1}$ (57)

ie., it is defined by:

$$(\partial^2 + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x) \tag{58}$$

where $\delta^D(x)$ is the D-dimensional δ-function. Thus we have reduced the generating functional $Z_0[J]$ to a simple function involving integrals over $J(x)$ and $\Delta_F(x-x')$.

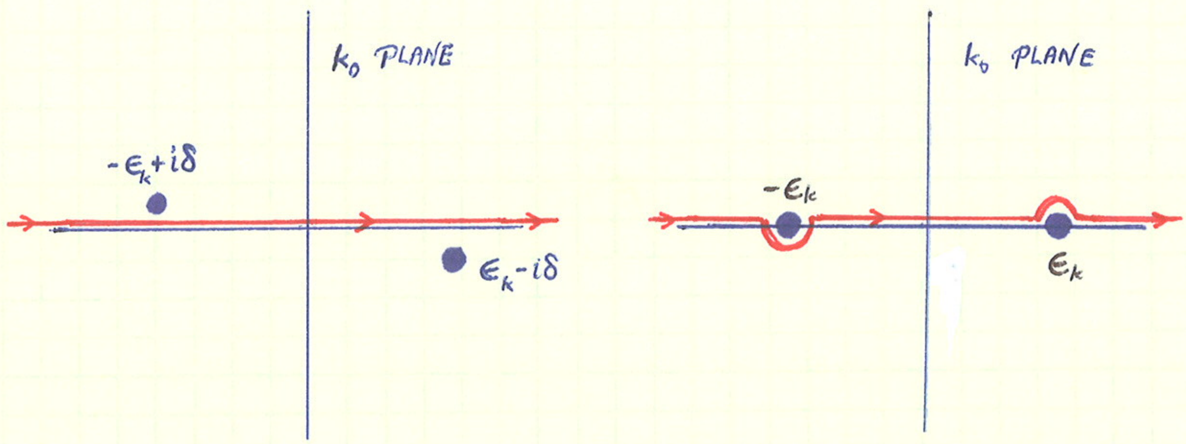
The Feynman propagator $\Delta_F(x-x')$ is the object that ties together currents $J(x)$ and $J(x')$, and it is therefore important to understand its properties. Since the free field system is translationally invariant, we Fourier transform it to get

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \Delta_F(k)\, e^{-ikx} \equiv \sum_k e^{-ikx} \Delta_F(k) \tag{59}$$

where

$$\boxed{\Delta_F(k) = \frac{1}{k^2 - m^2 + i\delta} \equiv \frac{1}{k_0^2 - (|\underline{k}|^2 + m^2) + i\delta}} \tag{60}$$

where in the last form we use the Lorentz invariance of the theory. From this last form we can see the pole structure of the theory in the frequency domain (ie., in k_0-space). Let's see what this means when we come to integrate over the k-space, as defined by (59). If we do the frequency integral first,



then we must follow the path shown above left - to see this, just rewrite (60) as

$$\Delta_F(k) = \frac{1}{2k_0}\left[\frac{1}{k_0 - (\epsilon_k - i\delta)} + \frac{1}{k_0 + (\epsilon_k + i\delta)} \right] \tag{61}$$

where

$$\epsilon_k^2 = (|\underline{k}|^2 + m^2) \tag{62}$$

The contour can also be taken as shown above right, if we desire to put the poles at $\pm(\epsilon_k - i\delta)$ onto the real axis (letting $\delta = 0$). Note that in both cases we are simply using artificial devices to deal with the real branch cut structure of the propagator – something we have already seen in ordinary QM.

Let's now look at what we get for the correlation functions in this free field theory. The simplest thing to do is simply expand the function in (56), to get

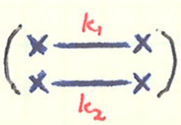
$$Z_0[J] = \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n \left(\int d^Dx \int d^Dx'\, J(x)\,\Delta_F(x-x')\,J(x')\right)^n \right\}. \qquad (63)$$

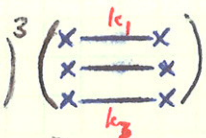
a series which is obviously better written in terms of the Fourier transforms of $J(x)$ and $\Delta_F(x)$; it is easy to show that we get

$$Z_0[J] = \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar}\right)^n \left(\sum_k J_k\,\Delta_F(k)\,J_{-k}\right)^n \right\} \qquad (64)$$

and we see that both of these expressions have obvious diagrammatic versions; we have

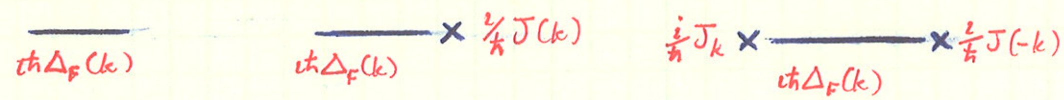
$Z_0[J] = \Big\{ 1 + \frac{1}{2}\Big($ 

$\Big) + \frac{1}{2!}\left(\frac{1}{2}\right)^2\Big($ 

$\Big) + \frac{1}{3!}\left(\frac{1}{2}\right)^3\Big($ 

$\Big) + \text{etc.} \Big\} \qquad (65)$

where the notation is pretty much as used in (24); we have



(66)

with integration over repeated indices (in this case the momenta).

To obtain the correlation functions we go back to the exponential form in (56), and just functionally differentiate. Let's do the function $G_2(x,x')$ in detail. From (19) we have

$$G_2^{(0)}(x,x') = -\hbar^2 \left.\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(x')}\right|_{J=0} \qquad (67)$$

Now, using the simple result

$$\frac{\delta}{\delta J(x)}\, e^{-\frac{i}{2\hbar}\int dx_1 \int dx_2\, J(x_1)\,\Delta_F(x_1-x_2)\,J(x_2)} = -\frac{i}{\hbar}\int dx_1\, \Delta_F(x-x_1)\,J(x_1)\, e^{-\frac{i}{2\hbar}\iint J\Delta_F J} \qquad (68)$$

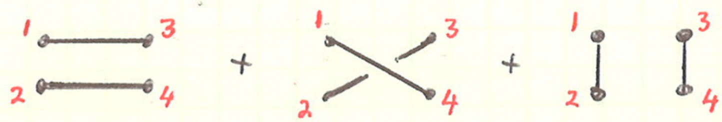
we easily find that

$$\boxed{G_2^{(0)}(x,x') = i\hbar\,\Delta_F(x-x')} \qquad (69)$$

By continuing this analysis we easily find that $G_3^0(x_1x_2x_3) = 0$, and that

$$G_4^{(0)}(x_1x_2x_3x_4) = \hbar^4\Big\{\Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3) - \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4)\Big\} \qquad (70)$$

whose diagrammatic interpretation appears on the next page:

$$G_4^{(0)}(x_1x_2x_3x_4) \;=\; $$  (71)

ad more generally we have

$$G^{(0)}_{2n+1}(x_1 \ldots x_{2n+1}) = 0 \tag{72}$$

$$\left.\begin{aligned} G^{(0)}_{2n}(x_1 \ldots x_{2n}) &= \sum_P G_2^{(0)}(x_{P_1} - x_{P_2}) \ldots\ldots G_2^{(0)}(x_{P_{2n-1}} - x_{P_{2n}}) \\ &= i^n \sum_P \Delta_F(x_{P_1} - x_{P_2}) \ldots\ldots \Delta_F(x_{P_{2n-1}} - x_{P_{2n}}) \end{aligned}\right\} \tag{73}$$

where $\sum_P$ means the sum over all permutations of the indices $P_1, P_2, \ldots P_{2n}$. This last result is actually an example of "Wick's theorem", which in the older canonical formulation of field theory, using field operators & their commutation relations, is actually quite clumsy to prove.

B.1(e) INTERACTIONS and $Z[J]$ in ϕ^4 THEORY

We have seen that the free field theory is simple to solve because we only have to deal with Gaussian integrals. This feature disappears as soon as we introduce interactions. However, we may use the same tricks to deal with these as we did for the simple 1-particle Green function for non-relativistic particle mechanics. What we will do in the following is derive 2 results for the full generating functional $Z[J]$ in ϕ^4 theory. The derivations will be short because they parallel very closely the derivations in ordinary QM.

Result 1 : Functional Derivative $\delta/\delta J(x)$: Let us recall the simple result for functional integrals, discussed in the Appendix and already used in eqtn (E.9) of the section on correlations in QM, that for a quadratic functional $Q_J[\phi]$ of the field $\phi(x)$ and a polynomial $P[\phi(x)]$ in $\phi(x)$, we have

$$\int \mathcal{D}\phi(x)\, P[\phi]\, e^{-Q[\phi]} = P(-\delta/\delta J) \int \mathcal{D}\phi(x)\, e^{-Q_J^0[\phi]} \tag{74}$$

where $Q_J^0[\phi]$ has the general form $\qquad Q_J[\phi] = \frac{1}{2}(\phi, Q_0 \phi) + J\phi + C \qquad (75)$

We now assume that the polynomial in question is just the exponential functional over the interaction term in the Lagrangian. Thus, suppose we have an action

$$S[\phi; J] = \int d^Dx \left\{ -\tfrac{1}{2}\phi(\partial^2 + m^2)\phi - V(\phi) + J(x)\phi(x) \right\} \tag{76}$$

where in the ϕ^4 theory, $\qquad V(\phi) = \dfrac{g}{4!}\phi^4(x) \qquad (77)$

Then our polynomial is just the exponential, ie.,

$$\mathcal{P}[\phi] = \exp\left\{\frac{-i}{\hbar}\int d^Dx\, V(\phi(x))\right\} \quad\longrightarrow\quad \exp\left\{-\frac{i}{\hbar}\, g/4! \int d^Dx\, \phi^4(x)\right\} \qquad (78)$$

so that, from (13), (14), and (74), we have the normalized $Z[J]$ in the form

$$\boxed{Z[J] = \frac{e^{-i/\hbar\int d^Dx\, V(-i\hbar\,\delta/\delta J(x))}\, Z_0[J]}{\left[e^{-i/\hbar\int d^Dx\, V(-i\hbar\,\delta/\delta J(x))}\, Z_0[J]\right]\Big|_{J=0}}} \qquad (79)$$

with $Z_0[J]$ given by (56); thus, for the ϕ^4 theory, we have

$$Z[J] = \left\{\frac{e^{-i/\hbar\int d^Dx\, g/4!\,(-i\hbar\,\delta/\delta J(x))^4}\; e^{-i/2\hbar\int d^Dx_1\int d^Dx_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)}}{\left[e^{-i/\hbar\int d^Dx\, g/4!\,(-i\hbar\,\delta/\delta J(x))^4}\; e^{-i/2\hbar\int d^Dx_1\int d^Dx_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)}\right]\Big|_{J=0}}\right\} \qquad (80)$$

which is a result that can be systematically expanded in powers of g, to give a perturbative theory for the effects of the ϕ^4 interaction. The same kind of expansion can then be given for the correlation functions, starting from the expansion for $Z[J]$. We will look at this in our section perturbation & diagrammatic expansions, applied to the ϕ^4 theory.

Result 2 : Functional Derivative $\delta/\delta\phi(x)$: As we already saw in dealing with ordinary QM, we can also transform expressions like (79) and (80) into expressions involving functional derivatives with respect to $\phi(x)$, as opposed to $J(x)$. This relies on the identity, discussed in the Appendix on functionals, given by

$$f[-i\,\delta/\delta J(x)]\; g[J(x)] = g[-i\,\delta/\delta\phi(x)]\; f[\phi]\, e^{i\int dx\,\phi(x)J(x)}\Big|_{\phi=0} \qquad (81)$$

for 2 functions $\phi(x)$ and $J(x)$. This means that we can write

$$e^{-i/\hbar\int d^Dx\, V(-i\hbar\,\delta/\delta J(x))}\, Z_0[J] = Z_0[-i\hbar\,\delta/\delta\phi(x)]\, e^{-i/\hbar\int d^Dx\,[V(\phi)-\phi(x)J(x)]}\Big|_{\phi=0} \qquad (82)$$

from which

$$\boxed{Z[J] = \left\{\frac{Z_0[-i\hbar\,\delta/\delta\phi(x)]\, e^{-i/\hbar\int d^Dx\,[V(\phi)-\phi(x)J(x)]}\Big|_{\phi=0}}{Z_0[-i\hbar\,\delta/\delta\phi(x)]\, e^{-i/\hbar\int d^Dx\, V(\phi(x))}\Big|_{\phi=0}}\right\}} \qquad (83)$$

ie., we have converted the whole expression to one in which functional derivatives act directly on the fields $\phi(x)$ themselves, rather than on the currents. For the the ϕ^4 theory with the interaction in (77), eqtn (83) reduces to

$$Z[J] = \left\{ \frac{e^{-\frac{i}{2\hbar}\int d^Dx_1 \int d^Dx_2\, \Delta_F(x_1-x_2)\, \delta/\delta\phi(x_1)\, \delta/\delta\phi(x_2)}\; e^{-\frac{i}{\hbar}\int d^Dx \left[\frac{g}{4!}\phi^4(x) - J(x)\phi\right.}}{e\;\; e^{-\frac{i}{2\hbar}\int d^Dx_1 \int d^Dx_2\, \Delta_F(x_1-x_2)\, \delta/\delta\phi(x_1)\, \delta/\delta\phi(x_2)}\; e^{-\frac{i}{\hbar}\int d^Dx\, \frac{g}{4!}\phi^4(x)}\Big|_{\phi=0}} \right\} \tag{84}$$

with again a perturbative expansion in g in the offing, both for $Z[J]$ and for the correlation functions.

B.1 (f) PHYSICAL MEANING OF THESE FUNCTIONS

We will only learn the full meaning of the various functions introduced here so far as we go along, gaining experience with them & seeing how they are deployed. However we can learn a lot by comparing what we have with other similar mathematical structures, and look at their physical meaning. We can in particular look at the analogies with thermodynamics & statistical mechanics, on the one hand, and with the ordinary problem of a single QM particle in a "noise field", on the other.

In what follows we will (i) look at the analogies with thermodynamics & statistical methanics (& at the same time give a more complete discussion of connected graphs and $W[J]$), and then (ii) compare results here with those for the 1-particle Green function.

(i) CORRESPONDENCE with STATISTICAL MECHANICS / THERMODYNAMICS

At the very beginning of this section, we saw the simple way in which the structure of simple probability functions corresponds to the generating functional of QFT. We can make this correspondence more precise in one of two ways, viz;

(i) Reduce the QFT to a simple theory of functions by taking the classical limit, ie., by restricting ourselves purely to the classical solution $\bar{\Phi}(x)$ that minimizes the action.

(ii) By generalizing the simple probability theory given earlier, from a theory dealing with probabilities of outcomes $P(\phi)$ for some random variable ϕ, to probabilities for outcomes $P[\phi(x)]$ of some random process $\phi(x)$, itself a function of some variable x. The outcomes $P[\phi(x)]$ are functionals of $\phi(x)$, whereas $P(\phi)$ is a function of ϕ.

The correspondence with statistical mechanics or thermodynamics comes because these subjects deal precisely with probabilities for outcomes. Ordinary thermodynamics deals with simple functions like the free energy $F(T)$, a function of temperature T; statistical mechanics, on the other hand, deals with functionals $F[\phi]$ of some configuration $\phi(x) = \phi(\underline{r}, t)$ of the system. In a non-relativistic condensed matter system this configuration could be written in terms of the quantum field describing the system, with probabilities $P[\Phi]$ for different such configurations; or it might be written in terms of the probabilities $P_i[\phi_i(x)]$ for the system to be in different eigenstates of the Hamiltonian – it amounts to the same thing. In a relativistic QFT we will always deal with field configurations.

In what follows we will make things simple by focussing on the classical limit. Later, when we have gotten used to the tools of QFT, we will look in a little more detail at the correspondence between QFT and full-blooded statistical mechanics.

The classical limit of a QFT for a simple scalar ϕ-field is defined by minimizing the action, ie., for an action

$$S[\phi, J] = S[\phi] - \int d^D\!x\, J(x)\,\phi(x) \tag{85}$$

we define the classical solution $\bar{\phi}_J(x)$ by

$$\left.\frac{\delta S[\phi, J]}{\delta\phi(x)}\right|_{\phi=\bar{\phi}} = \left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\bar{\phi}} + J(x) = 0 \tag{86}$$

The generating <u>functional</u> $Z[J]$ then becomes a generating function, viz.,

$$Z[J] \rightarrow Z_{cl}(J) = e^{\frac{i}{\hbar}\left(S_{cl}(\bar{\phi}) + \int d^D\!x\, \bar{\phi}(x) J(x)\right)} \tag{87}$$

where we have lost the functional integration over paths $\phi(x)$ because there is now only one path $\bar{\phi}(x)$, and the action $S[\phi] \rightarrow S_{cl}(\bar{\phi})$, for that path.

Now there is a very precise correspondence between the functions $Z_{cl}(J)$, $W_{cl}(J)$, and $\Gamma_{cl}(\bar{\phi})$, on the one hand, and the thermodynamic functions $F(T)$ or $F(H)$ (free energy as functions of temperature T or applied field H), the partition functions $Z(T)$ or $Z(H)$, and the functions $U(S)$ (internal energy as a function of entropy S) and $G(M)$ (thermodynamic Gibbs free energy as a function of magnetization). These links can be summarized in the following table:

SEMICLASSICAL Q.F.T.	STATISTICAL MECHANICS/THERMODYAMICS
$Z_{cl}(J)$ Generating Function.	$Z[T]$ or $Z(H)$ Partition function
$\bar{\phi}$; J (field; external current)	S; T (entropy; temperature) $\underline{M}$, $\underline{H}$ (magnetization; external field).
$W(J) = -i\hbar \ln Z_{cl}(J)$ $= \Gamma_{cl}(\bar{\phi}) - \int d^D\!x\, \bar{\phi}(x) J(x)$	$F(T) = -kT \ln Z(T)$ $= E(S) - TS$ $F(H) = -kT \ln Z(\underline{H})$ $= G(M) - \int d^D\!r\, \underline{M}(\underline{r}) \cdot \underline{H}(\underline{r})$

(88)

Thus we see that in these cases, the generator $W_{cl}(J)$ of connected graphs corresponds to the free energy $F(T)$ or $F(H)$, both functions of "external" (and intensive) degrees of freedom, like J. On the other hand a Legendre transformation to extensive degrees of freedom like S and M corresponds to the transformation of the field theory to $\Gamma_{cl}(\phi)$, a function of the field itself, which is also an extensive variable.

This analogy between statistical mechanics & QFT is one we will pursue quite often, with increasing sophistication. But notice the key difference here, which is the correspondence

$$\boxed{i\hbar \quad \longleftrightarrow \quad kT} \tag{89}$$

between the two. This difference is in one way quite crucial - probabilities in the stat. mech. system correspond to amplitudes in the QM or QFT system. One can of course change this by rotating to imaginary time, to get a Euclidean QFT - we will also explore this later on.

We have already explored, in our discussion of 1-particle QM and correlation functions, the connection between $G(0,0|J)$, or $Z[J]$, for a particle in the presence of a "noise source" $J(t)$, and the path integral expression for this propagator. Without going into too many details (which we will do later), it is very revealing to look at the result for a harmonic oscillator subject to a noise force $J(t)$. As we saw, this is given by the AMPLITUDE ("vacuum correlator"):

$$Z[J(t)] = e^{-\frac{i}{2\hbar}\int dt \int dt' J(t) D_0(t-t') J(t')} \tag{90}$$

where $D_0(t)$ is the simple SHO correlator. Now consider the expression which gives the PROBABILITY for some random process $\xi(t)$ to follow some particular "path". This is described by a generating functional

$$Z[k(t)] = \int \mathcal{D}\xi(t)\, \mathcal{P}[\xi(t)]\, e^{i\int dt\, \xi(t) k(t)} \tag{91}$$

where the probability for a given $\xi(t)$ is given, for a Gaussian random walk, by

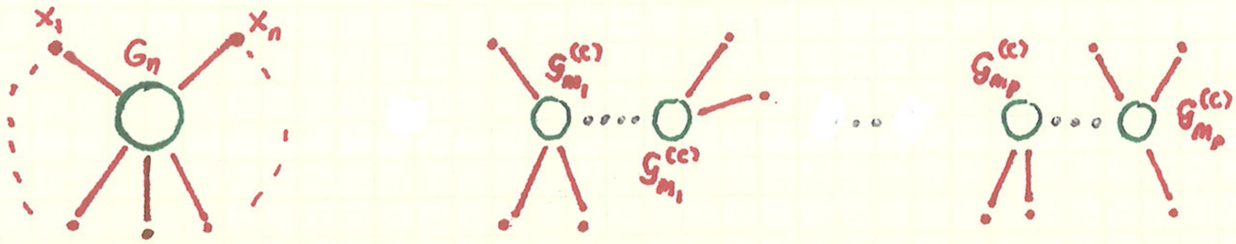
$$\mathcal{P}[\xi] = e^{-\frac{i}{2\sigma}\int dt \int dt' \xi(t) K_0^{-1}(t-t') \xi(t')} \tag{92}$$

where $K_0(t-t')$ is some correlator. We see already how we can start to make connection between random walks, constrained in one way or another, and QFT - again, we will explore this more later.

(ii) CONNECTED GRAPHS:

Finally, let's briefly sketch the connection between the functional $W[J]$ and the connected graphs - this also helps us understand the meaning of the function we have defined. Now the relationship between the set of all connected n-point graphs, which we have called $\mathcal{G}_n^{(c)}(x_1..x_n)$, and the set of all graphs with n external legs, which we have called $G_n(x_1..x_n)$, is intuitively rather obvious, and we have already seen it shown graphically (cf. eqn (37)). Clearly we can get all graphs for $G_n(x_1..x_n)$ by taking the set of all graphs for $\mathcal{G}^{(c)}_{m_1}(x_1..x_{m_1})$, $\mathcal{G}^{(c)}_{m_2}(x_1..x_{m_2}), \dots \mathcal{G}^{(c)}_{m_p}(x_1..x_{m_p})$, such that $\sum_{j=1}^p m_j = n$, and combining them in all possible ways, with all possible choices for $m_1 .. m_p$ summed over.

For those who do not trust the graphical argument given in (37), a direct calculation may be more convincing. So let's do this. Schematically we have



(93)

where we imagine graphs for $\mathcal{G}^{(c)}_{m_1}$ repeated q_1 times, and so on, and fix the condition that

$$\sum_{j=1}^{p} q_j m_j = q_1 m_1 + q_2 m_2 + \dots q_p m_p = n \tag{94}$$

Now we just have a combinatorial problem. At order n, the number of different graphs is

$$N_n = \frac{n!}{(m_1!)^{q_1} q_1! \dots (m_p!)^{q_p} q_p!} \tag{95}$$

and then summing over all n, we get for $Z[J]$ the expression

$$\left.\begin{aligned}
Z[J] &= \sum_n \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int d^Dx_1..d^Dx_n\, J(x_1)\cdots J(x_n) \sum_{q_1m_1+\dots q_pm_p=n}\!\!\sum\sum \mathcal{G}^{(c)}_{m_1}(x_1..x_{m_1})\cdots\mathcal{G}^{(c)}_{m_p}(x_1..x_{m_p}) \\
&= \sum_n \left(\frac{i}{\hbar}\right)^n \sum_{q_1m_1+q_pm_p=n}\!\!\sum\sum\ \prod_{j=1}^{p} \frac{1}{q_j!}\left(\frac{\int d^Dx_1..\int d^Dx_{m_j}\, \mathcal{G}^{(c)}_{m_j}(x_1..x_{m_j})\, J(x_1)..J(x_{m_j})}{m_j!}\right)^{q_j} \\
&= \sum_{q_j}\prod_j \left(\frac{i}{\hbar}\right)^{q_j}\frac{1}{q_j!}\left(\frac{\int d^Dx_1\dots\int d^Dx_{m_j}\, \mathcal{G}^{(c)}_{m_j}(x_1..x_{m_j})\, J(x_1)\cdots J(x_{m_j})}{m_j!}\right)^{q_j} \\
&= \exp\left[\sum_{n=1}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{1}{n!}\int d^Dx_1\dots\int d^Dx_n\, \mathcal{G}^{(c)}_n(x_1..x_n)\, J(x_1)..J(x_n)\right]
\end{aligned}\right\} \tag{96}$$

so that, comparing with (31), we get again that

$$Z[J] = e^{i/\hbar\, W[J]} \tag{97}$$

which makes the connection we wanted.