NOTES ON FUNCTIONALS

The theory of functionals represents a fairly obvious generalization the idea of an ordinary function. Recall that we can have a function $f(x)$, or a function $f(x_1..x_n)$, of one or more variables $\{x_j\}$, such that

$$y = f(x_1..x_n) \tag{1}$$

maps the set of variables $\{x_j\}$ to a single number (real or complex) y. In other words, it is a many to one mapping.

A functional maps a set of functions $\{\phi_j(x)\}$ to a single number, ie., we have the relationship

$$y = F[\phi_1(x)..\phi_n(x)] \tag{2}$$

with a mapping now from a space of functions to the field of real or complex numbers. Now, since we can typically represent a function $\phi(x)$ as an infinite set of numbers [either simply as the set $\phi_1 = \phi(x_1)$, $\phi_2 = \phi(x_2)$, ... (where the $\{x_j\}$ take all possible values of x), or by a sum over orthonormal functions, as

$$\phi(x) = \sum_n \phi_n \chi_n(x) \tag{3}$$

where $\int dx\, \chi_n(x)\chi_m(x) = \delta_{nm}$, and so on, so that we define $\phi(x)$ by the infinite set $\{\phi_n\}$], we see that in a well-defined sense, a functional is like a generalization of (1) to infinite sets of variables.

In what follows we shall see various examples of functionals, but it is useful to give a few examples for orientation. Some common ones are

(i) Energy or free energy functionals in physics; for example

$$\mathcal{H}[\phi] = \int d\mathbf{r}^D \left(-\frac{\hbar^2}{2m}\phi(r)\nabla^2\phi(r)\right) + \int d\mathbf{r}^D \int d\mathbf{r}'^D\, \phi^2(r)V(r-r')\phi^2(r') \tag{4}$$

(ii) The action functional in field theory or ordinary classical mechanics:

$$\left.\begin{aligned} S[q,\dot{q}] &= \int dt\, L(q,\dot{q};t) \qquad &(\text{classical mechanics})\\ S[\phi] &= \int d^Dx\, \mathcal{L}(\phi(x), \partial^\mu x) \qquad &(\text{field theory})\end{aligned}\right\} \tag{5}$$

(iii) The propagator in quantum mechanics; one has

$$F[\psi] = \psi(\mathbf{r},t) = \int d\mathbf{r}'^D\, G(\mathbf{r}\mathbf{r}';tt')\psi(\mathbf{r}'t') \tag{6}$$

for the definition of $G(rr'tt')$, and we see that the wave-function $\psi(r,t)$ is a functional of its functional form $\psi(r't')$ at an earlier time; moreover we have

$$G[q] = G(\mathbf{r}t;\mathbf{r}'t') = \int_{q(t')=r'}^{q(t)=r} \mathcal{D}q(\tau)\, e^{i/\hbar \int dt\, L(q,\dot{q};\tau)} \tag{7}$$

which defines $G(rt,r't')$ as a FUNCTIONAL INTEGRAL over the set of all paths $q(\tau)$ between the end-points of the path integral at (ξ',t') and (ξ,t). The functional integral will be defined more generally below.

(iv) In probability theory we deal with probability functionals - these are the generalization to "processes" of ordinary probability functions. Thus, we are used to dealing with the probability $P(x)$ of the outcome of some random sampling. But suppose we want the probability that some process $\phi(x)$, a function of a variable x, occurs. Then we deal with a functional $P[\phi]$, assigning a probability to each possible process $\phi(x)$. And in the same way that the expectation value of a variable $A(x)$ that depends on the random variable x is given by $\langle A\rangle = \int dx\, P(x) A(x)$, we have

$$\langle A\rangle = \int \mathcal{D}\phi(x)\, P[\phi]\, A[\phi] \tag{8}$$

for the expectation value of some variable $A[\phi(x)]$ that depends on the form of the process $\phi(x)$. The most common form for the probability functional $P[\phi]$ is that for a "Gaussian random process", in which case

$$P[\phi] \longrightarrow |\det K(xx')|^{1/2}\, e^{-\frac{1}{2}\int dx \int dx'\, \phi(x) K(x-x') \phi(x')} \tag{9}$$

where the determinant, discussed below, is defined by

$$\int \mathcal{D}\phi(x)\, e^{-\frac{1}{2}\int dx \int dx'\, \phi(x) K(x-x') \phi(x')} = |\det K(x,x')|^{-1/2} \tag{10}$$

so that we have a normalized $P[\phi]$: $\quad \int \mathcal{D}\phi(x)\, P[\phi] = 1 \qquad (11)$

What we wish to do here is outline, rather informally, the basic features of the theory of functionals as it applies to simple problems in quantum mechanics, in statistical mechanics, & in field theory. The presentation will not be mathematically sophisticated, nor does it pretend to generality. In particular, the functions $\phi(x)$, $q(t)$, etc., being considered, will be assumed smooth; this is justified physically by the presence of inertial terms.

A. FUNCTIONAL INTEGRATION

: This will be considered as a simple generalization of ordinary integration over a finite set of variables, in the limit as the number of variables goes to infinity. To see what is involved, lets consider the example of a simple Gaussian integral. In one dimension this is just

$$I_0 = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Kx^2 + Jx} = \frac{1}{\sqrt{K}}\, e^{-\frac{1}{2}J^2/K} \tag{12}$$

and we want to generalize this to N dimensions, and then let $N \to \infty$. Let's first redefine the integration variable to get rid of the $\sqrt{2\pi}$ factor, ie., let $x \to \tilde{x} = x/\sqrt{2\pi}$, and now consider the integral

$$I = \int d^n\tilde{x}\, e^{-\frac{1}{2}\sum_{ij}\tilde{x}_i K_{ij}\tilde{x}_j + \sum_j J_j \tilde{x}_j} \equiv \int d\tilde{\underline{x}}\, e^{-\frac{1}{2}\tilde{\underline{x}}\underline{K}\tilde{\underline{x}} + \underline{J}\tilde{\underline{x}}} \tag{13}$$

for $\tilde{\underline{x}} = (\tilde{x}_1, \dots \tilde{x}_N)$. For $\underline{J} = (J_1 .. J_N) \equiv 0$, this is easily done, for we simply diagonalize $\underline{K} = K_{ij}$, to find

$$I_0 = \int d^n\tilde{x}\, e^{-\frac{1}{2}\tilde{x}_i K_{ij}\tilde{x}_j} = |\det K_{ij}|^{-1/2} \tag{14}$$

If we now add back the term $\underline{J}\tilde{\underline{x}}$ in the exponent, we see we can make the same manoeuvre by "completing the square", ie, writing (now suppressing the tilde over $\tilde{\underline{x}}$):

$$\left.\begin{aligned} S(x) &= \tfrac{1}{2}\mathbf{x}\mathbf{K}\mathbf{x} - \mathbf{J}\mathbf{x} = S(\mathbf{x}_0) + \tfrac{1}{2}(\underline{x}-\underline{x}^0)\underline{K}(\underline{x}-\underline{x}^0) \\ \text{where} \quad \underline{x}_0 &= \underline{J}\underline{K}^{-1} \qquad (\text{ie } x_i^0 = K_{ij}^{-1} J_j) \end{aligned}\right\} \tag{15}$$

and then redoing the Gaussian integration with $\underline{x}^0$ as the new "origin". Thus we get the result

$$I = \int d\underline{x}\, e^{-S(\mathbf{x})} = \int d\underline{x}\, e^{-\frac{1}{2}\mathbf{x}\mathbf{K}\mathbf{x} + \mathbf{J}\mathbf{x}} = \frac{1}{|\mathbf{K}|^{1/2}} e^{-\frac{1}{2}\mathbf{J}\mathbf{K}^{-1}\mathbf{J}} \tag{16}$$

Now this warmup allows us to jump to the limit $N \to \infty$. The determinant $|K| \equiv \det K_{ij}$ is well-defined when N is finite. What we will suppose is that in the limit as $N \to \infty$, we can still sensibly define quantities like $S(x)$ and $|K|$. Whether this is possible is a subtle mathematical question (it is certainly not possible in general!), but in physical applications, one typically always refers back to a case where N is finite - in any such application, there will typically be IR and UV cut-offs that make it so. Thus we now define the functional generalization of (16), to be

$$\left.\begin{aligned} I &= \int \mathcal{D}x(t)\, e^{-S[x]} \\ &\equiv \int \mathcal{D}x(t)\, e^{-\frac{1}{2}\int dt \int dt'\, x(t) K(t,t') x(t') + \int dt\, J(t) x(t)} \end{aligned}\right\} \tag{17}$$

where the continuously varying parameter t is now introduced as a proxy for the index j; the functional integral is then

$$\int \mathcal{D}x(t) \equiv \lim_{N\to\infty} \int d\tilde{\underline{x}} \equiv \lim_{N\to\infty} \frac{1}{(2\pi)^{N/2}} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dx(t_j) \tag{18}$$

where the times t_j are spaced by infinitesimal intervals, ie, we let $dt = t_{j+1} - t_j = T/N$, where T is the time interval involved (we now call t the "time").

Then the answer we get for the functional integral is

$$I = \int \mathcal{D}x(t)\, e^{-S[x]} = \frac{1}{|\det K(t,t')|^{1/2}} \exp\left\{-\tfrac{1}{2}\int dt \int dt'\, J(t) K^{-1}(t,t') J(t')\right\} \tag{19}$$

where the inverse function is defined as the obvious generalization of the inverse of a finite matrix, ie.,

$$\int dt'\, K(t_1,t')\, K^{-1}(t',t_2) \;=\; \delta(t_1-t_2) \tag{20}$$

Again, we note that objects like $\det K$, or the integral itself, will be infinite when $N\to\infty$. However in physical applications we will see that these infinite quantities divide out, by normalization. This was already obvious in eqtns (9)-(11) above, where we had to divide out the determinant; but this determinant simply normalized the probability distribution.

We also notice that there is no need to work with a specific basis when defining these functional integrals. Just as we can make a similarity transformation in an expression like (13) or (16), ie., rotate to a new orthonormal basis $\underline{y}$, where $\underline{y} = (y_1,\ldots y_N)$, we can rotate in functional space (ie., in the space of basis-functions) to rewrite a functional integral. Thus, suppose we Fourier transform $x(t)$, such that

$$x(t) \;=\; \int_{-\infty}^{\infty} \frac{dw}{2\pi}\, e^{-iwt}\, x(w) \tag{21}$$

Then we can Fourier transform the functional integral in the same way – we will then have

$$I \;=\; \int \mathcal{D}x(w)\, e^{-S[x]} \tag{22}$$

with a corresponding change in the measure in (18).

B. FUNCTIONAL DIFFERENTIATION

: We now want to define the inverse operation to functional integration. Just as functional integration is supposed to a generalization to a function space of ordinary integration, over an infinite set of functions, we can see functional deferentiation as an infinite-dimensional generalization of ordinary partial differentiation for a finite set of variables. Recall that, where differentiation is well-defined, we can write for a function $f(x)$ of a single variable x, that

$$f(x_0+\delta x) \;=\; f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n}\Big|_{x=x_0} (dx)^n \tag{23}$$

and for a function $f(x_1,\ldots x_N)$ of N variables, we have

$$f(\underline{x}^0+\delta\underline{x}) \equiv f(x_j^0+\delta x_j) \;=\; f(x^0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n f(x)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_N}}\Big|_{\underline{x}=\underline{x}^0} dx_{\alpha_1}\cdots dx_{\alpha_N} \tag{24}$$

Clearly these Taylor expansions don't work around singularities, but we are only going to discuss here cases where the functional differentials are well defined.

Now lets consider a variation of a function $\phi(x)$; we will call this variation $\delta\phi(x)$. The idea is that we start from some well-defined function

$\phi_0(x)$, and vary it infinitesimally, so that

$$\phi_0(x) \rightarrow \phi_0(x) + \delta\phi(x). \tag{25}$$

in the same way that, in (23) and (24) above, we let the <u>variable</u> $\underline{x}$ be varied, so that $\underline{x}_0 \rightarrow \underline{x}_0 + \delta\underline{x}$. We now want to know how some functional $F[\phi]$ changes under the change $\delta\phi(x)$. Clearly the appropriate generalization of (24) is just

$$\left.\begin{aligned} \delta F[\phi] &\equiv F[\phi_0 + \delta\phi] - F[\phi_0] \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left.\frac{\delta^n F[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}\right|_{\phi(x)=\phi_0(x)} \delta\phi(x_1)\cdots\delta\phi(x_n) \end{aligned}\right\} \tag{26}$$

where, if we wish to make sense of this expression by relating it back to (24), we should write $\phi(x)$ in terms of a set of orthonormal functions $\chi_n(x)$ (cf. eqtn. (3)), write the variation as

$$\delta\phi(x) = \sum_n \chi_n(x)\,\delta\phi_n \tag{27}$$

in terms of the coefficients $\delta\phi_n$, and then expand in powers of the $\delta\phi_n$, just we expanded in powers of the dx_{α_n} in (24). For the functional differentials to be well-defined, they must be independent of which basis set χ_n we use for the expansion in (27).

B.1 <u>BASIC RESULTS</u>:

Let's first establish some basic results for functional differentiation, and consider a few simple examples. We have already defined the n-th functional differential in eqtn (26) above, but it is a little abstract, since we have not fixed the form of $\delta\phi(x)$. To do this we make a particularly simple choice in (27), viz., we write

$$\left.\begin{aligned} \chi_n(x) &\rightarrow \chi_z(x) = \delta(x-z) \\ \delta\phi_n &\rightarrow \delta\phi_z = \epsilon \end{aligned}\;\right|\; \delta\phi(x) = \epsilon\,\delta(x-z) \;\Bigg\} \tag{28}$$

In functional language, this amounts to choosing as basis functions the infinite set of δ-functions which pick out different values of x; the function picks out $x=z$. Note that in QM, these are nothing but position eigenstates $|z\rangle$, so that

$$\langle x|z\rangle = \delta(x-z) \qquad \text{and} \qquad \int dx\, \langle x|z\rangle = 1 \tag{29}$$

Now let's define the derivative $\delta F/\delta\phi(x)$ using this form for $\delta\phi(x)$. We have, under a variation $\delta\phi(x)$, a change

$$\delta F[\phi] = F[\phi(x) + \delta\phi(x)] - F[\phi(x)] \tag{30}$$

and by definition
$$\delta F[\phi] = \int dx'\, \frac{\delta F[\phi]}{\delta\phi(x')}\,\delta\phi(x') \tag{31}$$

Then, with the assumption that $\delta\phi(x) = \epsilon\,\delta(x-x')$, as in (28), we get

$$\delta F[\phi] \longrightarrow \epsilon \int dx' \frac{\delta F[\phi]}{\delta\phi(x')}\,\delta(x'-x) = \epsilon \frac{\delta F[\phi]}{\delta\phi(x)} \tag{32}$$

so that we have
$$\frac{\delta F[\phi(x)]}{\delta\phi(x')} \equiv \frac{1}{\epsilon}\Big(F[\phi + \epsilon\,\delta(x-x')] - F[\phi]\Big) \tag{33}$$

This result is of practical use when it comes to calculating functional derivatives, or at least justifying results obtained by more heuristic means. We can use it derive a number of useful results; for example

Product Rule: For some product of functionals, we have

$$\frac{\delta}{\delta\phi(x)}\{F[\phi]\,G[\phi]\} = F[\phi]\frac{\delta G}{\delta\phi(x)} + G[\phi]\frac{\delta F}{\delta\phi(x)} \tag{34}$$

Chain Rule: Suppose we have a functional $G[F[\phi]]$ of a functional $F[\phi]$ of the function $\phi(x)$. Then the functional differential $\delta G/\delta\phi$ of the functional $G[F[\phi]]$ is

$$\frac{\delta G[F[\phi(x)]]}{\delta\phi(x)} = \int dx' \frac{\delta G}{\delta F[\phi(x')]}\,\frac{\delta F[\phi(x')]}{\delta\phi(x)} \tag{35}$$

If the functional $F[\phi]$ is just an ordinary function, $f(\phi(x)) = f(x)$, then (35) reduces to

$$\frac{\delta G[f]}{\delta\phi(x)} = \frac{\delta G[f]}{\delta f(\phi(x))}\,\frac{df}{d\phi(x)} \tag{36}$$

Functional Self-differentiation: The simplest functional $F[\phi]$ of $\phi(x)$ is the unit functional $F[\phi] = \phi(x)$. Then since

$$\delta\phi(x) = \int dx' \frac{\delta\phi(x)}{\delta\phi(x')}\,\delta\phi(x') \quad \text{and} \quad \delta\phi(x) = \int dx'\,\delta(x-x')\,\delta\phi(x') \tag{37}$$

we have
$$\frac{\delta\phi(x)}{\delta\phi(x')} = \delta(x-x') \tag{38}$$

Then, using the product rule, we have for any function $f(\phi(x))$ that is differentiable, and so can be written as $f(\phi(x)) = \sum_n f_n \phi^n(x)$, that the functional

$$F[\phi] = \int dx' f(\phi(x')) \tag{39}$$

has differential
$$\frac{\delta F[\phi]}{\delta\phi(x)} = \frac{df(\phi(x))}{d\phi(x)} \tag{40}$$

which follows from the simple result that

$$\frac{\delta \phi^n(x)}{\delta \phi(x')} = n\phi^{n-1}(x)\,\delta(x-x') \tag{41}$$

derived directly from (34) and (38).

From all these general results it is then fairly straightforward to derive some more specific results.

B.2 SIMPLE EXAMPLES :

In many applications we do not need to understand more than a few simple results. So in what follows I give a few of these, with remarks on how to get them. In another section we discuss a set of results that are crucial in quantum field theory.

(i) Derivatives of $\phi(x)$: We can imagine simple functionals involving derivatives of a function $\phi(x)$. The simplest example is the functional

$$F[\phi] = \int dx\,(\phi'(x))^n \equiv \int dx\left(\frac{d\phi(x)}{dx}\right)^n \tag{42}$$

It is simple to then derive the result (starting as usual from (33))

$$\left.\begin{aligned} \frac{\delta F[\phi]}{\delta\phi(x)} &= \int dx'\, n\,(\phi'(x'))^{n-1}\frac{d}{dx'}\delta(x'-x) \\ &\xrightarrow[\text{integration by parts}]{} -n\frac{d}{dx'}(\phi'(x'))^{n-1}\Big|_{x'=x} \end{aligned}\right\} \tag{43}$$

where in the last step we assume that $\phi'(x')$ and its derivatives can be ignored at the end points of the integral.

Now, by writing some arbitrary function $f(\phi'(x))$ as a power series in $\phi'(x)$, ie, write $f(\phi'(x)) = \sum_n f_n(\phi')^n$, we can immediately derive the result for a functional

$$F[\phi] = \int dx\, f(\phi'(x)) \tag{44}$$

that

$$\frac{\delta F[\phi]}{\delta\phi(x)} = -\frac{d}{dx'}\left(\frac{df(\phi')}{d\phi'(x')}\right)\Big|_{x'=x} \tag{45}$$

These results can be easily extended to encompass higher derivatives $\phi^{(n)}(x) \equiv d^n\phi(x)/dx^n$, or to multiple integrals over functionals of $\phi'(x)$, or to functionals of $\phi(\underline{r})$, where $\underline{r}$ exists in n-dimensional space; and so on.

(ii) Simple exponential functionals : In many applications, ranging from physics to economics, one has to deal with exponential functionals – the most important reason for this being in the generating functionals used in probability theory and quantum field theory. Thus we need to look at

$$F(\phi) = \exp\left\{\int dx\, J(x)\,\phi(x)\right\} \tag{46}$$

which we write as

$$F[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int dx\, J(x) \phi(x) \right)^n \qquad (47)$$

and then we easily find

$$\frac{\delta F[\phi]}{\delta \phi(x)} = J(x)\, e^{\int dx\, J(x)\phi(x)} = J(x) F[\phi] \qquad (48)$$

Notice that we can also write $F[\phi]$ as $\quad F[J] = e^{\int dx\, J(x)\phi(x)} \qquad (49)$

ie., $J(x)$ and $\phi(x)$ appear symmetrically, and we have

$$\frac{\delta F[J]}{\delta J(x)} = \phi(x)\, e^{\int dx\, J(x)\phi(x)} = \phi(x) F[J] \qquad (50)$$

We can also consider Gaussian functionals of these variables. These are central in both probability theory and in field theory and statistical mechanics. Thus, in quantum field theory one deals with

$$F[J] = e^{\frac{i}{2} \int dx_1 \int dx_2\, J(x_1) \Delta(x_1,x_2) J(x_2)} \qquad (51)$$

(here I have put $\hbar = 1$), and then one easily finds that

$$\left.\begin{aligned} \frac{\delta F[J]}{\delta J(x)} &= i \int dx'\, \Delta(x,x') J(x')\, e^{\frac{i}{2} \int dx_1 \int dx_2\, J(x_1) \Delta(x_1,x_2) J(x_2)} \\ &= i \int dx'\, \Delta(x,x') J(x')\, F[J] \end{aligned}\right\} \qquad (52)$$

and one can continue in this vein with more complicated exponential functionals.

(iii) Functionals of "Correlator" form : Often, in classical physics (eg., in the theory of electronic circuits, or in classical E&M theory, or in ordinary mechanical systems), and in quantum mechanics & quantum field theory, or in condensed matter physics, one deals with "response functions", in which one looks at some "correlator" $K(x,x')$ between events at 2 different spacetime positions. One then deals with functionals like

$$F[\phi(z)] = \int dz'\, K(z,z') \phi(z') \qquad (53)$$

and we already saw a simple example of this in eqtn (6). It is easy to then establish that

$$\frac{\delta F[\phi(z)]}{\delta \phi(x)} = K(z,x) \qquad (54)$$

and we easily generalize this to the functional $\quad F[\phi(z)] = \int dz'\, K(z,z') \phi^n(z') \qquad (55)$

to find

$$\left.\begin{aligned} \frac{\delta F[\phi(z)]}{\delta \phi(x)} &= \int dz'\, n \phi^{n-1}(z')\, K(z,z')\, \delta(x-z') \\ &= n K(z,x) \phi^{n-1}(x) \end{aligned}\right\} \qquad (56)$$

and from this we easily find that for a function $f(\phi(z))$ expandable as a polynomial in $\phi(z)$, the functional

$$F[\phi] = \int dz'\, K(z,z')\, f(\phi(z')) \tag{57}$$

has derivative

$$\frac{\delta F[\phi(z)]}{\delta\phi(x)} = K(z,x)\,\frac{df(\phi)}{d\phi(x)} \tag{58}$$

Finally, we can consider correlators of form

$$F[\phi] = \int dx_1 \int dx_2\, \phi(x_1)\, K(x_1,x_2)\, \phi(x_2) \tag{59}$$

and it is clear that

$$\frac{\delta F[\phi]}{\delta\phi(x)} = \int dx' \big(K(x,x') + K(x',x)\big)\,\phi(x') \tag{60}$$

and that

$$\frac{\delta^2 F[\phi]}{\delta\phi(x_1)\,\delta\phi(x_2)} = K(x_1,x_2) + K(x_2,x_1) \tag{61}$$

and one may continue in this vein with higher correlators involving integrals over n different ϕ-functions with different arguments $\{x_j\}$.

B.3 SHIFT OPERATIONS

: In field theory, and also elsewhere, it is important to be apply to apply different transformations to functional integrals that have come to be known as "shift operations" after they were used extensively by Schwinger in his work on QED in the early 1950's.

The simplest kind of "shift" or translation operation is that which generalizes the simple Taylor expansion beyond an infintesimal translation dx (cf eqtn (23). Thus for an ordinary function we have

$$e^{A_0\, d/dx} f(x) = f(x+A_0) \tag{62}$$

provided $f(x)$ is analytic for $|z| \leq A$, again by power series expansion, in powers of A_0 (as opposed to dx).

The functional generalization of this is just

$$\hat{Q}_1 F[\phi] \equiv e^{\int dx\, A(x)\, \delta/\delta\phi(x)} F[\phi] = F[\phi(x) + A(x)]. \tag{63}$$

There are various ways of demonstrating this – one of the fastest is to use the functional generalization of the usual definition of a delta-function, viz.,

$$\delta[f(x) - g(x)] = \int \mathcal{D}\phi(x)\, e^{i\int dx\, \phi(x)[f(x)-g(x)]} \tag{64}$$

and to write a general functional $F[\phi(x)]$ in the form of a functional power series, ie., write:

$$F[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int dx_j \, f_n(x_1 \ldots x_n)\, \phi(x_1) \cdots \phi(x_n) \tag{65}$$

which can also be written in the form

$$F[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \int dx_j \left[f_n(x_1 \ldots x_n) \left(-i \frac{\delta}{\delta J(x_1)}\right) \cdots \left(-i \frac{\delta}{\delta J(x_n)}\right)\right] e^{i \int dx\, \phi(x) J(x)} \Big|_{J=0} \tag{66}$$

and where we note that the coefficients $f_n(x_1 \ldots x_n)$ are just the functional derivatives of $F[\phi]$, ie., that

$$f_n(x_1 \ldots x_n) = (-i)^n \frac{\delta^n F[\phi]}{\delta\phi(x_1) \cdots \delta\phi(x_n)} \tag{67}$$

The formulae (66) and (67) are just the functional generalizations of what is written down in ordinary probability theory for the generating function and the moments of a probability distribution. Notice also that the "functional δ-function" in (64) is a special case of the "functional Fourier transform", viz.,

$$F[J] = \int \mathcal{D}\phi(x)\, e^{i \int dx\, \phi(x) J(x)}\, F[\phi] \tag{68}$$

which defines the characteristic functional in probability theory.

Now let's look at some more complicated shift operators. The next one up is the "quadratic shift operator", which we write as

$$\hat{Q}_2 = e^{i \hat{K}_\phi} = e^{\frac{i}{2} \int dx_1 \int dx_2 \frac{\delta}{\delta\phi(x_1)} K(x_1 x_2) \frac{\delta}{\delta\phi(x_2)}} \tag{69}$$

which is central in quantum field theory. Let's apply this to the simple exponential functional (bearing in mind we can write any functional using (68)); then we have

$$\left.\begin{aligned} \hat{Q}_2\, e^{i \int dx\, \phi(x) J(x)} &= e^{\frac{i}{2} \int dx_1 \int dx_2 \frac{\delta}{\delta\phi(x_1)} K(x_1 x_2) \frac{\delta}{\delta\phi(x_2)}}\; e^{i \int dx\, J(x) \phi(x)} \\ &= e^{-\frac{i}{2} \int dx_1 \int dx_2\, J(x_1) K(x_1 x_2) J(x_2)}\; e^{i \int dx\, J(x) \phi(x)} \end{aligned}\right\} \tag{70}$$

and by using the Fourier transform we can extend this to any functional in place of this exponential.

The quadratic shift operator plays a particular role in field theory, when it is applied to more than one functional (representing, eg., some n-point correlation function). Let us consider, for example, the case where the functional is a product of 2 functionals:

Let

$$F[\phi] = G_1[\phi]\, G_2[\phi] \tag{71}$$

Then we have

$$\hat{Q}_2\, G_1[\phi] G_2[\phi] = e^{\frac{i}{2}\int dx_1 \int dx_2 \frac{\delta}{\delta J(x_1)} K(x_1,x_2) \frac{\delta}{\delta J(x_2)}}\, G_1[\phi] G_2[\phi] \tag{72}$$

and if we work this out, we get

$$\hat{Q}_2\, G_1[\phi] G_2[\phi] = e^{i\hat{K}_{12}^{\phi}} \left\{ e^{i\hat{K}_{11}^{\phi_1}} G_1[\phi_1]\, e^{i\hat{K}_{22}^{\phi_2}} G_2[\phi] \right\}\Big|_{\phi_1=\phi_2=\phi} \tag{73}$$

where the operators are

$$K_{ij}^{\phi} = \frac{1}{2}\int dx \int dx' \frac{\delta}{\delta\phi_i(x)} K(x,x') \frac{\delta}{\delta\phi_j(x')} \tag{74}$$

and the result in (73) requires, in field theory, a clear diagrammatic interpretation; the operators K_{ij}^{ϕ} are "linking" operators, which join together pairs of ϕ-factors (ie., "ϕ-fields"); the exponentiation then allows all possible pairs, an arbitrary number of times. These are then linked between the 2 different functionals $G_1[\phi]$ and $G_2[\phi]$ by the operator K_{12}^{ϕ}.

This can be generalized to a product $F[\phi] = \hat{\prod}_{j=1} G_j[\phi]$ quite easily, and all possible linkings are then generated.

As a final example, let's look at the action of $\hat{Q}_2$ on a simple Gaussian functional. We can solve this starting from (70), or otherwise. We consider the form

$$\left.\begin{aligned} &\hat{Q}_2\, e^{\frac{i}{2}\int dx_1 \int dx_2\, \phi(x_1)\Delta(x_1,x_2)\phi(x_2) + i\int dx\, J(x)\phi(x)} \equiv \hat{Q}_2 F[\phi] \\ &\equiv e^{\frac{i}{2}\int dx \int dx' \frac{\delta}{\delta\phi(x)} K(xx') \frac{\delta}{\delta\phi(x')}} \left[e^{\frac{i}{2}\int dx_1 \int dx_2\, \phi(x_1)\Delta(x_1,x_2)\phi(x_2) + i\int dx\, J(x)\phi(x)} \right. \end{aligned}\right\} \tag{75}$$

If we work this out, we get a key result in field theory, viz.,

$$Q_2 F[\phi] = A \exp\left\{ \frac{i}{2}\int dx \int dx' \left[\phi(x) \tilde{G}^{\phi\phi}_{(xx')}\phi(x') + 2\phi(x) G^{\phi J}_{(xx')} J(x') + J(x) G^{JJ}_{(xx')} J(x') \right] \right\} \tag{76}$$

where the correlators are

$$\left.\begin{aligned} G^{\phi\phi}_{(xx')} &= \left(\frac{\Delta}{1+\Delta K}\right)_{xx'} \\ G^{\phi J}(xx') &= (1+\Delta K)^{-1}_{xx'} \\ G^{JJ}(xx') &= \left(\frac{K}{1+\Delta K}\right)_{xx'} \end{aligned}\right\} \tag{77}$$

and

$$A = \exp\left\{ -\tfrac{1}{2} \mathrm{Tr} \ln (1+\Delta K)_{xx'} \right\} \tag{78}$$

which is a result much used in field theory.