J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Some properties of coherent spin states

J. M. RADCLIFFE†

School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, Sussex, England

MS. received 6*th November* 1970

Abstract. Spin states analogous to the coherent states of the linear harmonic oscillator are defined and their properties discussed. They are used to discuss some simple problems (a single spin in a field, a spin wave, two spin $\frac{1}{2}$ particles with Heisenberg coupling) and it is shown that their use may often give increased physical insight.

1. Introduction

The point of this paper is to show that there exist spin states analogous to the 'coherent' states of the harmonic oscillator. The latter have been studied extensively in recent years (see for example Carruthers and Nieto 1968) and appear to be useful in discussing the statistical mechanics and superfluid properties of boson fluids (Langer 1968); they also give a convenient description of the radiation from lasers. It is still an open question as to whether the spin states defined here will prove useful. They may, at the very least, give some physical insight into problems involving spins and their correlations.

2. Coherent states of the harmonic oscillator

Before defining the spin states it will be useful to look briefly at the problem of the one-dimensional harmonic oscillator.

In this case the coherent states are functions of a variable α which runs over the entire complex plane, and are given explicitly by

$$\begin{aligned}|\alpha\rangle &= \pi^{-1/2}\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{1/2}}|n\rangle \\ &= \pi^{-1/2}\exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n(\alpha^+)^n}{n!}|0\rangle \\ &= \pi^{-1/2}\exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)|0\rangle \qquad (2.1)\end{aligned}$$

where $|n\rangle$ is the nth energy eigenstate of the oscillator and a^+ the usual creation operator. These states form a complete set, in the sense that

$$\int d^2\alpha|\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbf{1} \qquad (2.2)$$

where the right hand side is the unit matrix. However, they are neither normalized nor orthogonal. In fact, from the definition (2.1), the overlap of two states $|\alpha\rangle$, $|\beta\rangle$ is given by

$$\begin{aligned}\langle\beta|\alpha\rangle &= \pi^{-1}\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2)\langle 0|\exp(\beta^*a)\exp(\alpha a^+)|0\rangle \\ &= \pi^{-1}\exp(\alpha\beta^*-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2). \qquad (2.3)\end{aligned}$$

Of course, these states $|\alpha\rangle$ do not span Hilbert space in the two-dimensional α plane.

† See *Note added in proof*, p. 323.

To see this explicitly, write $\alpha = \rho e^{i\phi}$. Then

$$|\alpha\rangle = |\rho;\phi\rangle = \pi^{-1/2}\exp(-\tfrac{1}{2}\rho^2)\sum_{n=0}^{\infty}\frac{\rho^n e^{in\phi}}{(n!)^{1/2}}|n\rangle. \quad (2.4)$$

The functions

$$f_n(\rho;\phi) \equiv \pi^{-1/2}\exp(-\tfrac{1}{2}\rho^2)\frac{\rho^n e^{in\phi}}{(n!)^{1/2}} \quad (2.5)$$

are just a subset of the eigenfunctions of the two-dimensional harmonic oscillator. In fact, a possible classification of the states of this system is by the energy $\epsilon = (m+1)$ and the angular momentum $L_z = l$ (l, m integers); for any given value of m, l can take the values $-m$, $-m+2 \dots m-2$, m. The functions $f_n(\phi)$ are clearly the subset of states corresponding to $l = m = n$.

It is clear that, quite generally, one can construct functions $|\boldsymbol{\xi}\rangle$ by making any (enumerably infinite) selection from any set of functions $\phi_n(\boldsymbol{\xi})$ which are orthonormal and complete in $\boldsymbol{\xi}$ space. Let the chosen subset be denoted by $\{n\}$, where the association of functions in this subset with a particular one-dimensional oscillator state is quite arbitrary. Now define

$$|\boldsymbol{\xi}\rangle = \sum_{\{n\}}\phi_n(\boldsymbol{\xi})|n\rangle. \quad (2.6)$$

These sets are complete in the oscillator Hilbert space:

$$\begin{aligned}\int d\boldsymbol{\xi}|\boldsymbol{\xi}\rangle\langle\boldsymbol{\xi}| &= \sum_{nn'}|n\rangle\langle n'|\int d\boldsymbol{\xi}\,\phi_{n'}^*(\boldsymbol{\xi})\,\phi_n(\boldsymbol{\xi})\\ &= \sum_n |n\rangle\langle n| = \mathbf{1}.\end{aligned} \quad (2.7)$$

However, the states $|\boldsymbol{\xi}\rangle$ are not orthogonal, and cannot be, since

$$\langle\boldsymbol{\xi}'|\boldsymbol{\xi}\rangle = \sum_{\{n\}}\phi_n^*(\boldsymbol{\xi}')\,\phi_n(\boldsymbol{\xi}). \quad (2.8)$$

Only if the subset $\{n\}$ runs over a complete set in $\boldsymbol{\xi}$ space will the right hand side be equal to $\delta(\boldsymbol{\xi}-\boldsymbol{\xi}')$. If the space $\boldsymbol{\xi}$ has two or more dimensions the subset is certainly not complete.

The states $|\boldsymbol{\xi}\rangle$ are not normalized, but it may happen that the set $\{n\}$ can be chosen so that

$$\langle\boldsymbol{\xi}|\boldsymbol{\xi}\rangle = \sum_n|\phi_n(\boldsymbol{\xi})|^2 \quad (2.9)$$

is a constant. For example, in the case of the coherent states of the harmonic oscillator, the normalized wavefunctions in the set $\{n\}$ are

$$f_n(\rho;\phi) = \pi^{-1/2}\exp(-\tfrac{1}{2}\rho^2)\frac{\rho^n e^{in\phi}}{(n!)^{1/2}} \quad (2.5)$$

and so

$$\langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty}|f_n(\rho;\phi)|^2 = \pi^{-1}\exp(-\rho^2)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{n!} = \frac{1}{\pi}. \quad (2.10)$$

In practice it does not appear to matter whether or not the states do normalize to a constant.†

† It is of course equally possible to choose the states $|\boldsymbol{\xi}\rangle$ to be normalized and put in a weighting factor in the left hand side of the completeness relation (2.7). This is in fact the alternative we shall choose in the next section.

To conclude this section, we emphasize that the point of introducing states such as $|\boldsymbol{\xi}\rangle$ is that, being complete, they can be used perfectly well in the evaluation of such quantities as the partition function of the harmonic oscillator problem:

$$Z = \sum_n \langle n|\mathrm{e}^{-\beta\mathscr{H}}|n\rangle = \int \mathrm{d}\boldsymbol{\xi}\,\langle\boldsymbol{\xi}|\mathrm{e}^{-\beta\mathscr{H}}|\boldsymbol{\xi}\rangle.$$

Such states $|\boldsymbol{\xi}\rangle$ may well be better starting functions in 'perturbation' expansions of Z than the original oscillator states $|n\rangle$. For more details of applications and tricks in evaluating Z, see Carruthers and Nieto (1968) and Langer (1968).

3. Analogous spin states

We consider a single particle of spin S. Define the ground state $|0\rangle$ as the state such that $\hat{S}_z|0\rangle = S|0\rangle$, where $\hat{S}_z$ is the operator of the z component of spin. Then the operator $\hat{S}_- \equiv \hat{S}_x - \mathrm{i}\hat{S}_y$ creates spin deviations. In fact we have

$$(\hat{S}_-)^p|0\rangle = \left(\frac{p!\,2S!}{(2S-p)!}\right)^{1/2}|p\rangle \qquad 0 \leqslant p \leqslant 2S \tag{3.1}$$

where $|p\rangle$ is the eigenstate of $\hat{S}_z$ such that

$$\hat{S}_z|p\rangle = (S-p)|p\rangle. \tag{3.2}$$

Consider the state

$$|\mu\rangle \equiv N^{-1/2}\exp(\mu\hat{S}_-)|0\rangle = N^{-1/2}\sum_{p=0}^{2S}\left(\frac{2S!}{p!(2S-p)!}\right)^{1/2}\mu^p|p\rangle \tag{3.3}$$

where μ runs over the complex plane and N is a normalization factor. We have

$$\langle\mu|\mu\rangle = N^{-1}\sum_{p=0}^{2S}\frac{(2S)!}{p!(2S-p)!}|\mu|^{2p} = N^{-1}(1+|\mu|^2)^{2S} \tag{3.4}$$

and hence the normalized state is

$$|\mu\rangle = (1+|\mu|^2)^{-S}\exp(\mu\hat{S}_-)|0\rangle. \tag{3.5}$$

The overlap integral between two states $|\lambda\rangle$, $|\mu\rangle$ is

$$\langle\lambda|\mu\rangle = \frac{(1+\lambda^*\mu)^{2S}}{(1+|\lambda|^2)^S(1+|\mu|^2)^S} \tag{3.6}$$

and so

$$|\langle\lambda|\mu\rangle|^2 = \left(1-\frac{|\lambda-\mu|^2}{(1+|\lambda|^2)(1+|\mu|^2)}\right)^{2S}. \tag{3.7}$$

The states $|\mu\rangle$ defined by (3.5) do form a complete set, although it is necessary to include a weight function $m(|\mu|^2) \geqslant 0$ in the integral. We require

$$\int \mathrm{d}^2\mu\,|\mu\rangle m(|\mu|^2)\,\langle\mu| = \sum_{p=0}^{2S}|p\rangle\langle p| = \mathbf{1}. \tag{3.8}$$

By doing the angular integration and putting $|\mu| = \rho$, one finds

$$\begin{aligned}\int \mathrm{d}^2\mu\,|\mu\rangle m(|\mu|^2)\langle\mu| &= 2\pi\sum_{p=0}^{2S}|p\rangle\langle p|\frac{(2S)!}{p!(2S-p)!}\int_0^\infty \mathrm{d}\rho\,\frac{\rho^{2p}}{(1+\rho^2)}2Sm(\rho^2)\\ &= \sum_{p=0}^{2S}|p\rangle\langle p|\frac{(2S)!}{p!(2S-p)!}I(p,S)\end{aligned} \tag{3.9}$$

where

$$I(p;\,S) \equiv \pi \int_0^\infty \mathrm{d}\sigma \frac{\sigma^p}{(1+\sigma)^{2S}}\, m(\sigma). \tag{3.10}$$

Now one seeks a form for $m(\sigma)$ such that $I(p;\,S) = \{p!(2S-p)!/(2S)!\}$. A little thought shows that a suitable choice is

$$m(\sigma) = \frac{2S+1}{\pi}\,\frac{1}{1+\sigma^2}. \tag{3.11}$$

So, finally, the completeness relation is

$$\frac{2S+1}{\pi}\int \frac{\mathrm{d}^2\mu}{(1+|\mu|^2)^2}\,|\mu\rangle\langle\mu| = \sum_{p=0}^{2S} |p\rangle\langle p| = \mathbf{1}. \tag{3.12}$$

This result can be obtained more neatly by transforming back from a different parametrization of the states, namely $\mu = \tan(\theta/2)\mathrm{e}^{i\phi}$, which is used later on. It is convenient to work with μ while drawing analogies with the harmonic oscillator. The case of the oscillator is obtained in the limit $S \gg 1$. To see this, write

$$\hat{S}_- \to (2S)^{1/2} a^+ \tag{3.13}$$

(which is the high-spin limit of the Holstein–Primakoff transformation) and

$$\mu \to \alpha/(2S)^{1/2}. \tag{3.14}$$

The normalized states $|\alpha\rangle_{(S)}$ are then

$$|\alpha\rangle_{(S)} = \left(1+\frac{|\alpha|^2}{2S}\right)^{-S} \exp(\alpha a^+)|0\rangle. \tag{3.15}$$

But we have

$$\lim_{S\to\infty}\left(1+\frac{|\alpha|^2}{2S}\right)^{S} = \exp(\tfrac{1}{2}|\alpha|^2) \tag{3.16}$$

and so

$$\lim_{S\to\infty} |\alpha\rangle_{(S)} = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(a\alpha^+)|0\rangle \tag{3.17}$$

which apart from normalization is precisely a coherent state of the harmonic oscillator (cf. equation (2.1) above). It is easy to show that, for example, the spin state overlap integrals go to the correct limit.

We conclude this section with a remark on the completeness of the spin states. For consistency, we must have

$$\frac{2S+1}{\pi}\int \frac{\mathrm{d}^2\mu}{(1+|\mu|^2)} f(\mu)\langle\mu|\lambda\rangle = f(\lambda) \tag{3.18}$$

where in general the $f(\mu)$ is an overlap of the state $|\mu\rangle$ on some (arbitrary) spin state. This relation does hold so long as $f(\mu)$ is of the general form $P(\mu)/(1+|\mu|^2)^S$, where $P(\mu)$ is an arbitrary polynomial in μ with terms up to μ^{2S}. Now, in fact, only functions of precisely this form can occur in calculations if one stays within the Hilbert space appropriate to a particle of spin S, so in all such cases (3.18) is valid.

4. Some typical matrix elements

Define

$$\hat{p} \equiv S - \hat{S}_z \qquad \hat{S}_+ \equiv \hat{S}_x + i\hat{S}_y. \tag{4.1}$$

Then we have the following relations:

(i)

$$\langle\mu|\hat{p}|\mu\rangle = (1+|\mu|^2)^{-2S} \sum_{p=0}^{2S} \frac{(2S)!}{p!(2S-p)!} |\mu|^{2p} p$$

$$= \frac{2S|\mu|^2}{1+|\mu|^2}. \tag{4.2}$$

The second equality can be derived either by direct computation or by the observation that the sum can be written in the form

$$\rho \frac{\partial}{\partial \rho}(1+\rho)^{2S} \qquad \rho \equiv |\mu|^2.$$

(ii)

$$\langle\mu|\hat{S}_+|\mu\rangle = (1+|\mu|^2)^{-2S} \frac{\partial}{\partial \mu^*}(1+|\mu|^2)^{2S}$$

$$= \frac{2S\mu}{1+|\mu|^2}. \tag{4.3}$$

(iii) Hence

$$\langle\mu|\hat{S}_-|\mu\rangle = \frac{2S\mu^*}{(1+|\mu|^2)^{2S}} \tag{4.4}$$

(iv)

$$\langle\lambda|\hat{p}|\mu\rangle = \frac{2S\lambda^*\mu}{1+\lambda^*\mu} \langle\lambda|\mu\rangle \tag{4.5}$$

(v)

$$\langle\lambda|\hat{S}_+|\mu\rangle = \frac{2S\mu}{1+\lambda^*\mu} \langle\lambda|\mu\rangle \tag{4.6}$$

(vi)

$$\langle\lambda|\hat{S}_-|\mu\rangle = \frac{2S\lambda^*}{1+\lambda^*\mu} \langle\lambda|\mu\rangle. \tag{4.7}$$

Note that from (4.5–4.7)

$$\langle\lambda|\hat{S}_+|\mu\rangle = \frac{1}{\lambda^*} \langle\lambda|\hat{p}|\mu\rangle \tag{4.8}$$

$$\langle\lambda|\hat{S}_-|\mu\rangle = \frac{1}{\mu} \langle\lambda|\hat{p}|\mu\rangle. \tag{4.9}$$

5. An alternative parametrization

Let us write

$$\mu = \tan(\tfrac{1}{2}\theta)e^{i\phi} \quad 0 \leqslant \theta < \pi \quad 0 \leqslant \phi < 2\pi. \tag{5.1}$$

Then the normalized states can be written

$$|\theta, \phi\rangle \equiv |\Omega\rangle = (\cos\tfrac{1}{2}\theta)^{2S} \exp\{\tan(\tfrac{1}{2}\theta)e^{i\phi}\hat{S}_-\}|0\rangle \tag{5.2}$$

and the completeness relation is

$$\frac{2S+1}{4\pi}\int d\phi\, d\theta \sin\theta|\Omega\rangle\langle\Omega| \equiv (2S+1)\int \frac{d\Omega}{4\pi}|\Omega\rangle\langle\Omega| = \mathbf{1}. \tag{5.3}$$

There is a simple geometrical construction relating μ and the variables θ, ϕ. In fact, if we write $\mu = \rho e^{i\phi'}$ and draw the μ plane as tangent plane to a sphere of unit diameter where the z axis meets the sphere, then the point μ is the projection onto the μ plane of the point (θ, ϕ) on the sphere from the opposite pole. Clearly $\phi' = \phi$, $\rho = \tan(\theta/2)$.

From equation (3.6) we find for the overlap integral between states $|\Omega\rangle$, $|\Omega'\rangle$:

$$\langle\Omega'|\Omega\rangle = \{\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta' + \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\, e^{i(\phi-\phi')}\}^{2S} \tag{5.4}$$

and so

$$|\langle\Omega'|\Omega\rangle| = \left(\frac{1+\boldsymbol{n}\,.\,\boldsymbol{n}'}{2}\right)^{S}$$

where $\boldsymbol{n}$ and $\boldsymbol{n}'$ are unit vectors in the directions specified by (θ, ϕ) and (θ', ϕ') respectively.

Finally, we calculate the expectation values of the spin components in the state $|\Omega\rangle$. From equations (4.2–4.7) we have

$$\langle\Omega|\hat{p}|\Omega\rangle = S(1-\cos\theta) \tag{5.5}$$

$$\langle\Omega|\hat{S}_+|\Omega\rangle = S\sin\theta\, e^{i\phi} \tag{5.6}$$

$$\langle\Omega|\hat{S}_-|\Omega\rangle = S\sin\theta\, e^{-i\phi} \tag{5.7}$$

from which we get the result for the expectation value of the spin vector

$$\langle\Omega|\hat{\boldsymbol{S}}|\Omega\rangle = S\boldsymbol{n} \tag{5.8}$$

where $\boldsymbol{n}$ is the unit vector specified by Ω.

6. The effect of changing the ground state

At the moment we are describing spins by states of the form

$$|\mu\rangle = \hat{A}(\mu)|0\rangle \tag{6.1}$$

where

$$\hat{A}(\mu) \equiv (1+|\mu|^2)^{-S}\exp(\mu\hat{S}_-)|0\rangle \tag{6.2}$$

and $|0\rangle$ is the state such that $\hat{S}_z|0\rangle = S|0\rangle$. Consider now making a rotation to a new axis of quantization z' and write

$$|\lambda'\rangle' = \hat{A}'(\lambda')|0'\rangle \tag{6.3}$$

where $|0'\rangle$ is the state such that $\hat{S}_{z'}|0'\rangle = S|0'\rangle$ and

$$\hat{A}'(\lambda') \equiv (1+|\lambda'|^2)^{-S}\exp(\lambda'\hat{S}'_-) \tag{6.4}$$

$(\hat{S}'_- \equiv \hat{S}_{x'} - i\hat{S}_{y'})$. The problem now is to express the states $|\lambda'\rangle'$ in terms of the states $|\mu\rangle$. This question is relevant, for example, to a discussion of the structure of the density matrix (and mean values) for a pair of spins coupled by the Heisenberg interaction (which is invariant under rotations) and the Ising coupling (which is not). Explicitly, we seek the amplitude $\langle\mu|\lambda'\rangle'$ in the expansion

$$|\lambda'\rangle' = \left(\frac{2S+1}{\pi}\right)\int\frac{d^2\mu}{(1+|\mu|^2)^2}|\mu\rangle\langle\mu|\lambda'\rangle'. \tag{6.5}$$

Let a unitary rotation operator which carries $|0\rangle$ to $|0\rangle'$ be denoted by $\hat{R}$, so that

$$|0\rangle' = \hat{R}|0\rangle. \tag{6.6}$$

Then we have

$$\hat{S}_{z'} = \hat{R}\hat{S}_z\hat{R}^\dagger \tag{6.7}$$

$$(\hat{S}_\pm)' = \hat{R}\hat{S}_\pm\hat{R}^\dagger. \tag{6.8}$$

Hence we get simply

$$\begin{aligned}|\lambda'\rangle' &= (1+|\lambda'|^2)^{-S}\exp(\lambda'\hat{S}'_-)|0\rangle' \\ &= (1+|\lambda'|^2)^{-S}\hat{R}\exp(\lambda'\hat{S}_-)R^\dagger R|0\rangle \\ &= (1+|\lambda'|^2)^{-S}\hat{R}\exp(\lambda'\hat{S}_-)|0\rangle \\ &= \hat{R}|\lambda'\rangle.\end{aligned} \tag{6.9}$$

That is, one needs to evaluate

$$\langle\mu|\lambda'\rangle' = \langle\mu|R|\lambda'\rangle \tag{6.10}$$

for any calculations in which this amplitude is required explicitly. But we have (see Brink and Satchler 1968)

$$\hat{R} = \exp(-\mathrm{i}\alpha\hat{S}_z)\exp(-\mathrm{i}\beta\hat{S}_y)\exp(-\mathrm{i}\gamma\hat{S}_z) \tag{6.11}$$

where α, β, γ are the Euler angles describing the rotation. Expressing the states $\langle\mu|$, $|\lambda'\rangle$ in terms of states $\langle p|$, $|p'\rangle$, we find that

$$\begin{aligned}\langle\mu|R|\lambda'\rangle = (1+|\mu|^2)^{-S}(1+|\lambda'|^2)^{-S}\sum_{p,p'=0}^{2S}\left(\frac{(2S)!}{p!(2S-p)!}\right)^{1/2}\left(\frac{(2S)!}{p'!(2S-p')!}\right)^{1/2} \\ \times(\mu^*)^p(\lambda')^{p'}\langle p|\hat{R}|p'\rangle.\end{aligned} \tag{6.12}$$

Now consider the amplitude

$$\begin{aligned}\langle p|\hat{R}|p'\rangle &= \langle p|\exp(-\mathrm{i}\alpha\hat{S}_z)\exp(-\mathrm{i}\beta\hat{S}_y)\exp(-\mathrm{i}\gamma\hat{S}_z)|p'\rangle \\ &= \exp\{-\mathrm{i}\alpha(S-p)\}\exp\{-\mathrm{i}\gamma(S-p')\}\langle p|\exp(-\mathrm{i}\beta\hat{S}_y)|p'\rangle.\end{aligned} \tag{6.13}$$

We shall use the explicit expression for

$$\langle p|\exp(-\mathrm{i}\beta\hat{S}_y)|p'\rangle \equiv \mathrm{d}^S_{S-p,S-p'}(\beta)$$

given in Brink and Satchler (1968 p. 22), namely

$$\begin{aligned}\langle p|\exp(-\mathrm{i}\beta\hat{S}_y)|p'\rangle = \sum_t(-1)^t\frac{\{(2S-p)!p!(2S-p')!p'!\}^{1/2}}{(2S-p-t)!(p'-t)!t!(t+p-p')!} \\ \times(\cos\tfrac{1}{2}\beta)^{2S+p'-p-2t}(\sin\tfrac{1}{2}\beta)^{2t+p-p'}.\end{aligned} \tag{6.14}$$

To evaluate $\langle\mu|\hat{R}|\lambda'\rangle$ we use the alternative parametrization, writing

$$\mu \equiv \tan(\tfrac{1}{2}\theta)\mathrm{e}^{\mathrm{i}\phi} \qquad \lambda' \equiv \tan(\tfrac{1}{2}\theta')\mathrm{e}^{\mathrm{i}\phi'}.$$

Then we have, after some cancellations,

$$\begin{aligned}\langle\theta,\phi|\hat{R}|\theta',\phi'\rangle = (\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta')^{2S}\exp\{-\mathrm{i}(\alpha+\gamma)S\} \\ \times\sum_{p,p'=0}^{2S}\sum_t(-1)^t\frac{2S!}{(2S-p-t)!(p'-t)!t!(t+p-p')!} \\ \times(\tan\tfrac{1}{2}\beta)^{2t+p-p'}(\tan\tfrac{1}{2}\theta)^p(\tan\tfrac{1}{2}\theta')^{p'}\exp\{-\mathrm{i}p(\phi-\alpha)\} \\ \times\exp\{\mathrm{i}p'(\phi'+\gamma)\}.\end{aligned} \tag{6.15}$$

We write the sum in the form

$$(2S)!\sum_{p,p',t}\frac{(-1)^t x^p y^{p'} z^{2t+p-p'}}{(2S-p-t)!(p'-t)!t!(t+p-p')!} \tag{6.16}$$

where

$$x \equiv \tan(\tfrac{1}{2}\theta)\exp\{-\mathrm{i}(\phi-\alpha)\} \qquad y \equiv \tan(\tfrac{1}{2}\theta')\exp\{\mathrm{i}(\phi'+\gamma)\} \qquad z \equiv \tan(\tfrac{1}{2}\beta). \tag{6.17}$$

The limits on the sum over t are such as to ensure that no factorials shall have negative arguments, while p and p' run from 0 to $2S$. That is,

$$p' \geqslant t \qquad p+t \geqslant p' \qquad 2S-p \geqslant t \geqslant 0 \qquad 2S-p \geqslant 0. \tag{6.18}$$

We take the sums in the following sequence:

(I) sum over p' from t to $p+t$
(II) sum over t from 0 to $2S-p$
(III) sum over p from 0 to $2S$.

Sum (I):

$$\sum_{p'=t}^{p+t}\frac{y^{p'}z^{2t+p-p'}}{(p'-t)!(t+p-p')!} = y^t z^t \sum_{q=0}^{p}\frac{y^q z^{p-q}}{q!(p-q)!} = \frac{y^t z^t (y+z)^p}{p!}. \tag{6.19}$$

Sum (II):

$$\sum_{t=0}^{2S-p}(-1)^t\frac{y^t z^t}{t!(2S-p-t)!} = \frac{(1-yz)^{2S-p}}{(2S-p)!} \tag{6.20}$$

Sum (III):

$$\begin{aligned}\sum_{p=0}^{2S}&\frac{(2S)!}{p!(2S-p)!}x^p(y+z)^p(1-yz)^{2S-p}\\ &= (1+xy-yz+zx)^{2S}\\ &= [1+\tan(\tfrac{1}{2}\theta)\exp\{-\mathrm{i}(\phi-\alpha)\}\tan(\tfrac{1}{2}\theta')\exp\{\mathrm{i}(\phi'-\gamma)\}-\tan(\tfrac{1}{2}\beta)\tan(\tfrac{1}{2}\theta')\exp\{\mathrm{i}(\phi'-\gamma)\}\\ &\quad +\tan(\tfrac{1}{2}\beta)\tan(\tfrac{1}{2}\theta)\exp\{-\mathrm{i}(\phi-\alpha)\}]^{2S}.\end{aligned} \tag{6.21}$$

So, finally:

$$\begin{aligned}\langle\theta,\phi|R|\theta',\phi'\rangle &= \exp(-\mathrm{i}\alpha S)\exp(-\mathrm{i}\gamma S)(\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta')^{2S}\\ &\quad\times[1+\tan(\tfrac{1}{2}\theta)\exp\{-\mathrm{i}(\phi-\alpha)\}\tan(\tfrac{1}{2}\theta')\exp\{\mathrm{i}(\phi'-\gamma)\}-\tan(\tfrac{1}{2}\theta')\\ &\quad\times\exp\{\mathrm{i}(\phi'+\gamma)\}\tan(\tfrac{1}{2}\beta)+\tan(\tfrac{1}{2}\theta)\exp\{-\mathrm{i}(\phi-\alpha)\}\tan(\tfrac{1}{2}\beta)]^{2S}.\end{aligned} \tag{6.22}$$

As a simple check on this expression, take $\hat{R} = 1$, that is, $\alpha = \beta = \gamma = 0$. Then we find

$$\langle\theta,\phi|\theta',\phi'\rangle = [\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'+\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\exp\{\mathrm{i}(\phi'-\phi)\}]^{2S} \tag{6.23}$$

in agreement with (5.4). Another property which the amplitude $\langle\Omega|R|\Omega'\rangle$ must satisfy follows from the unitary property $\hat{R}(\hat{R}^\dagger = \hat{R}^{-1})$:

$$(\langle A|\hat{R}|B\rangle)^* = \langle B|\hat{R}^\dagger|A\rangle = \langle B|\hat{R}^{-1}|A\rangle. \tag{6.24}$$

Since $\hat{R}^{-1}$ is the rotation specified by Euler angles $(-\gamma, -\beta, -\alpha)$, we expect

$$\{\langle\theta,\phi|\hat{R}(\alpha,\beta,\gamma)|\theta',\phi'\rangle\}^* = \langle\theta',\phi'|\hat{R}(-\gamma,-\beta,-\alpha)|\theta,\phi\rangle. \tag{6.25}$$

We easily check that the condition (6.25) is indeed satisfied by the expression (6.22).

7. Some simple applications

The results obtained here with the help of the coherent-state formalism are of course well known; the point of doing the problems by this method is simply to give some extra physical insight. In particular, the connection with the classical limit comes out very clearly.

7.1. *Partition function for a single spin in a magnetic field*

With a suitable choice of the zero of energy, we can write the partition function in the form

$$Z = \mathrm{Tr}\{\exp(-\hat{p}h)\} \equiv \sum_{p=0}^{2S} \exp(-ph) \tag{7.1}$$

(where $h = \beta\gamma H$; γ is the particle's magnetic moment, H the external field and β is $1/k_B T$ as usual). It is straightforward to verify that we can write this in the form

$$\begin{aligned} Z &= (2S+1)\int \frac{\mathrm{d}\Omega}{4\pi}\{\tfrac{1}{2}(1+\mathrm{e}^{-h})+\tfrac{1}{2}(1-\mathrm{e}^{-h})\cos\theta\}^{2S} \\ &= (2S+1)\int \frac{\mathrm{d}\Omega}{4\pi}\langle\Omega|\exp(-\beta\mathscr{H})|\Omega\rangle. \end{aligned} \tag{7.2}$$

If we calculate the mean value of the operator $\hat{p}$ (equation (4.1)) by the relation $\langle\hat{p}\rangle = -Z^{-1}\partial Z/\partial H$, we find

$$\langle\hat{p}\rangle = \frac{2S(2S+1)}{Z}\int \frac{\mathrm{d}\Omega}{4\pi}\left(\frac{\sin^2(\frac{1}{2}\theta)\mathrm{e}^{-h}}{\cos^2(\frac{1}{2}\theta)+\sin^2(\frac{1}{2}\theta)\mathrm{e}^{-h}}\right)\{\tfrac{1}{2}(1+\mathrm{e}^{-h})+\tfrac{1}{2}(1-\mathrm{e}^{-h})\cos\theta\}^{2S} \tag{7.3}$$

so that in this particular case $\langle\hat{p}\rangle$ can be written in the form

$$\langle\hat{p}\rangle = (2S+1)\int \frac{\mathrm{d}\Omega}{4\pi}\langle p(\Omega)\rangle\langle\Omega|\hat{\rho}|\Omega\rangle \tag{7.4}$$

where $\hat{\rho} \equiv \exp(-\beta\mathscr{H})/Z$ is the density matrix and

$$p(\Omega) = \left(\frac{2S\sin^2(\frac{1}{2}\theta)\mathrm{e}^{-h}}{\cos^2(\frac{1}{2}\theta)+\sin^2(\frac{1}{2}\theta)\mathrm{e}^{-h}}\right). \tag{7.5}$$

In the limits $h \to 0$ and $h \gg 1$ we get, respectively, $\langle\hat{p}\rangle \to S$ and $\langle\hat{p}\rangle \to \mathrm{e}^{-h}$, as of course we must.

7.2. *Ferromagnetic spin wave*

The ground state $|0\rangle$ of the ferromagnet has $p_i = 0$ for all spins i, ($i = 1, 2, \ldots N$). In terms of the μ representation we can write

$$|0\rangle = \int \mathrm{d}M(\mu)\,|\mu\rangle\langle\mu|0\rangle \tag{7.6}$$

where $|\mu\rangle$ is shorthand for $|\mu_1, \mu_2 \ldots \mu_N\rangle$ and $\int \mathrm{d}M(\mu)$ for the expression (cf. equation (3.2))

$$\left(\frac{2S+1}{\pi}\right)^N \prod_{i=1}^{N}\int \mathrm{d}^2\mu_i(1+|\mu_i|^2)^{-2}. \tag{7.7}$$

The amplitude or 'wavefunction' of the ground state in the μ representation is

$$\langle\mu|0\rangle \equiv \Phi_0(\mu) = \prod_i (1+|\mu_i|^2)^{-S}. \tag{7.8}$$

In the p representation a state containing a single spin wave of wavevector $|\boldsymbol{k}\rangle$ is given by

$$|\boldsymbol{k}\rangle = N^{-1/2}\sum_i \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{R}_i)|0,0,\dots p_i = 1,0,\dots\rangle. \tag{7.9}$$

Therefore in μ space we have

$$|\boldsymbol{k}\rangle = \int \mathrm{d}M(\mu)|\mu\rangle\langle\mu|\boldsymbol{k}\rangle$$

$$= (2S)^{1/2}\int \mathrm{d}M(\mu)|\mu\rangle\left\{N^{-1/2}\sum_i \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{R}_i)\mu_i^*\right\}\langle\mu|0\rangle. \tag{7.10}$$

That is, the amplitude of the spin-wave state in μ space is

$$\Phi_{\boldsymbol{k}}(\mu) = (2S)^{1/2}N^{-1/2}\sum_i \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{R}_i)\mu_i^*\Phi_0(\mu) = \mu_k^*\Phi_0(\mu) \tag{7.11}$$

(where $\mu_k^* \equiv (2S/N)^{1/2}\sum_i \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{R}_i)\mu_i^*$). Thus, $\Phi_{\boldsymbol{k}}(\mu)$ is a simple algebraic multiple of $\Phi_0(\mu)$.

7.3. *Two spin $\frac{1}{2}$ particles interacting via the Heisenberg Hamiltonian*

Here we have

$$\mathscr{H}' = -2J\hat{\boldsymbol{S}}_1\,.\,\hat{\boldsymbol{S}}_2. \tag{7.12}$$

It is straightforward to show that the diagonal elements of the density matrix $\hat{\rho} \equiv \exp(-\beta\mathscr{H})/\mathrm{Tr}(\exp-\beta\mathscr{H})$ are given by ($j \equiv \beta J$)

$$\langle\mu_1\mu_2|\hat{\rho}|\mu_1\mu_2\rangle = \frac{1}{3\mathrm{e}^{2j}+1}\,\frac{1}{1+|\mu_1|^2}\,\frac{1}{1+|\mu_2|^2}$$
$$\times\{\mathrm{e}^{2j}(1+\tfrac{1}{2}|\mu_1+\mu_2|^2+|\mu_1|^2|\mu_2|^2)+\tfrac{1}{2}|\mu_1-\mu_2|^2\}. \tag{7.13}$$

We notice that this expression satisfies the conditions

(i) when $j = 0$,

$$\langle\mu_1\mu_2|\hat{\rho}|\mu_1\mu_2\rangle = \tfrac{1}{4} = \frac{1}{(2S+1)^2}$$

(ii) an integration over the coordinate μ_1 of spin 1 gives

$$\int \mathrm{d}M(\mu_1)\langle\mu_1\mu_2|\hat{\rho}|\mu_1\mu_2\rangle = \tfrac{1}{2}.$$

Correlations between the two spins come out most clearly if we write equation (7.13) in the form

$$\langle\mu_1\mu_2|\hat{\rho}|\mu_1\mu_2\rangle = \frac{1}{3+\mathrm{e}^{-2j}}\left(1-\frac{(1-\mathrm{e}^{-2j})|\mu_1-\mu_2|^2}{2(1+|\mu_1|^2)(1+|\mu_2|^2)}\right). \tag{7.14}$$

This in turn can be written in terms of the angular variables (θ_1, ϕ_1) and (θ_2, ϕ_2). One finds

$$\langle\Omega_1\Omega_2|\hat{\rho}|\Omega_1\Omega_2\rangle = \tfrac{1}{4}\left(1+\frac{1-\mathrm{e}^{-2j}}{3+\mathrm{e}^{-2j}}\boldsymbol{n}_1\,.\,\boldsymbol{n}_2\right) \tag{7.15}$$

where $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ are unit vectors in the directions specified by (θ_1, ϕ_1) and (θ_2, ϕ_2) respectively.

This shows absolutely transparently that

(i) for $j > 0$, that is, ferromagnetic coupling, the spins are correlated and tend to align parallel (i.e. with $\boldsymbol{n}_1 . \boldsymbol{n}_2 > 0$).

(ii) for $j < 0$, that is, antiferromagnetic coupling, the spins tend to align anti-parallel (i.e. with $\boldsymbol{n}_1 . \boldsymbol{n}_2 < 0$).

(iii) the density matrix is clearly invariant under rotations as it should be (it contains no reference to any particular axis.)

In conclusion, although the problems treated here are basically trivial, we may hope that there are also some nontrivial problems for which the point of view developed here may be illuminating.

Note added in proof. This paper was completed in draft form by D Radcliffe before his death and was edited for publication by A. J. Leggett.

References

Brink, D. M., and Satchler, G. R., 1968, *Angular Momentum* (Oxford: Clarendon Press).
Carruthers, P., and Nieto, M. M., 1968, *Rev. mod. Phys.*, **40,** 411.
Langer, J. S., 1968, *Phys. Rev.*, **167,** 183–90.