PROPAGATOR for PARTICLE IN MAGNETIC FIELD

We consider problems for which the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}(\underline{p} - q A(\underline{r}))^2 + V(\underline{r}) \tag{1}$$

where $\underline{A}(\underline{r})$ is a static gauge potential, and $V(\underline{r})$ an ordinary potential. We are particularly interested in the role of $\underline{A}(\underline{r})$, and so $V(\underline{r})$ will be ignored except at the formal level.

The Lagrangian corresponding to (1) is

$$L = \tfrac{1}{2} m \dot{\underline{r}}^2 + q \underline{A}(r) \cdot \dot{\underline{r}} - V(r) \tag{2}$$

so that we have a generalized momentum $\quad p = \dfrac{\partial L}{\partial \dot{r}} = m\dot{r} + qA(\underline{r}) \quad (3)$

and the Hamiltonian and Lagrangian are related by

$$\mathcal{H} = \dot{r}\frac{\partial L}{\partial \dot{r}} - L = p\dot{r} - L \tag{4}$$

We are interested in general in problems for which $A(\underline{r})$ is some static function, so that we have a combination of static magnetic & electric fields:

$$\left.\begin{aligned} \underline{B} &= \nabla \times A(\underline{r}) \\ \underline{E} &= -\nabla V(\underline{r}) \end{aligned}\right\} \tag{5}$$

Then the propagator is given in a path integral formulation by

$$G(\underline{r}_2 \underline{r}_1; t_2 t_1) = \int_{\underline{r}(t_1) = \underline{r}_1}^{\underline{r}(t_2) = \underline{r}_2} \mathcal{D}r(t) \exp\left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left[\frac{m}{2} \dot{r}^2 + q \underline{A}(\underline{r}) \cdot \dot{\underline{r}} - V(\underline{r}) \right] \right\} \tag{6}$$

In the usual way we split this path integral into infinitesimal sections, so that

$$\left. \begin{aligned} G(\underline{r}_2 \underline{r}_1; t_2 t_1) = \lim_{N \to \infty} \prod_{j=1}^{N} \int d^3 r_j \, A_N \exp\Big\{ \frac{i}{\hbar} dt \sum_{j=0}^{N-1} \Big[\frac{m}{2} \Big(\frac{r_{j+1} - r_j}{dt} \Big)^2 - V(r_j) \\ + \frac{q}{2} \big(\underline{A}(r_{j+1}) + \underline{A}(r_j) \big) \Big(\frac{r_{j+1} - r_j}{dt} \Big) \Big] \Big\} \end{aligned} \right\} \tag{7}$$

where we take special note of the "point splitting" recipe, viz, that $A(r_j)$ is replaced by $\frac{1}{2}(\underline{A}(r_{j+1}) + \underline{A}(r_j))$ (a proof is given at the end); and the normalization factor is the usual

$$A_N = \left(\frac{m}{2\pi i \hbar \, dt} \right)^{3N/2} \qquad (3d) \tag{8}$$

Consider now what happens if we perform a gauge transformation, ie., we write

$$A(\underline{r}) \rightarrow \tilde{A}(\underline{r}) = A(r) + \nabla\phi(r) = A(r) + f(r) \tag{9}$$

where $\phi(r)$ is a scalar function without any singularities. Then we have $\nabla \times \nabla\phi(r) = \nabla \times f(r) = 0$ as usual, and if we make this transformation in the path integral (6), we have

$$\left.\begin{aligned} G(r_2 r_1; t_2 t_1) \longrightarrow \tilde{G}(r_2 r_1; t_2 t_1) &= G(r_2 r_1 t_2 t_1)\, e^{i/\hbar\, q \int_{t_1}^{t_2} dt\, \dot{\underline{r}} \cdot \underline{\nabla}\phi} \\ &= G(r_2 r_1 t_2 t_1)\, e^{i/\hbar\, q \int_{r_1}^{r_2} dr \cdot \nabla\phi} \end{aligned}\right\} \tag{10}$$

But this integral is unique, independent of the path taken - thus we have

$$\left.\begin{aligned} \tilde{G}(r_2 r_1 t_2 t_1) &= G(r_2 r_1 t_2 t_1)\, e^{i/\hbar\, q(\phi(r_2) - \phi(r_1))} \\ &= e^{i/\hbar\, q\phi(r_1)}\, G(r_2 r_1; t_2 t_1)\, e^{-i/\hbar\, q\phi(r_2)} \end{aligned}\right\} \tag{11}$$

which is simply what we would obtain if in the standard expansion for the Green function, in the form of an eigenfunction expansion

$$\left.\begin{aligned} G(r_2 r_1 t_2 t_1) &= \sum_n \psi_n(r_2) \langle n| e^{-i/\hbar\, \epsilon_n (t_2 - t_1)} |n\rangle \psi_n^*(r_1) \\ &\equiv \int d\omega\, e^{i\omega(t_2 - t_1)} \frac{\psi_n(r_2)\, \psi_n^*(r_1)}{\omega - \epsilon_n/\hbar} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} &\text{we make the phase transformation} && \psi_n(r) \rightarrow \hat{\psi}_n(r) = e^{-i/\hbar\, q\phi(r)}\, \psi_n(\underline{r}) \\ &\text{and indeed more generally} && \psi(r) \rightarrow \hat{\psi}(r) = e^{-i/\hbar\, q\phi(r)}\, \psi(r) \end{aligned}\right\} \tag{13}$$

as we know from elementary Q.M.

Now there are few problems of this kind that are exactly solvable. In what follows I discuss examples thoroughly, and then discuss approximations for other problems.

EXAMPLE 1 : UNIFORM FIELD

: This is a very well-known & well-studied example. We can write the gauge potential in either the symmetric gauge:

$$A(\underline{r}) = \tfrac{1}{2} B_0 (-y, x) = \frac{m}{2q} \omega_c (-y, x) \tag{14}$$

where B_0 is the field intensity, and $\qquad \omega_c = \frac{qB_0}{m}$ (15)

is the usual cyclotron frequency. Alternatively we can go to the Landau gauge, where

$$A(\underline{r}) = B_0(0, x) = \frac{m\omega_c}{q}(0, x) \tag{16}$$

We ignore the dynamics along the z-axis here, since this motion decouples from that in the xy-plane.

By far the most elegant way to solve this problem in path integral theory is to go to the symmetric gauge, and then rewrite the problem in cylindrical coordinates. Comparing the Lagrangian as written in Cartesian and cylindrical coordinates, we have

$$\left.\begin{aligned} L &= \tfrac{1}{2}m\left[\dot{\underline{r}}^2 + \omega_c(x\dot{y} - y\dot{x})\right] \\ &\equiv \tfrac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \omega_c r^2\dot{\theta}\right] \end{aligned}\right\} \tag{17}$$

since in cylindrical coordinates (r, θ) we have $A(r) = (0, r\theta)$ (18)

We now make a new gauge transformation to the "rotating frame", by writing

$$\varphi(t) = \theta + \frac{\omega_c}{2}t \tag{19}$$

so that $\qquad \dot{\theta}^2 + \omega_c\dot{\theta} = \dot{\varphi}^2 - \omega_c^2/4$ (20)

and the system now looks just like a 2-d harmonic oscillator in this new rotating frame:

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\varphi}^2 - \frac{\omega_c^2}{4}r^2\right] \tag{21}$$

in a potential well $\qquad V(r) = \frac{m\omega_c^2}{8}r^2$ (22)

From the known solution for a harmonic oscillator in 2 dimensions we may then immediately write down the solution for the problem where I now modify the normalization A_N in (8) to a 2-dimensional form, so that

$$A_N = \left(\frac{m}{2\pi i\hbar\, dt}\right)^N \qquad (2d) \tag{23}$$

Then:

$$\left.\begin{aligned} G(r_2 r_1, t_2 t_1) = {} & \left(\frac{m}{2\pi i\hbar}\right)\frac{\omega_c(t_2 - t_1)}{2\sin[\omega_c(t_2 - t_1)/2]}\,\frac{1}{t_2 - t_1} \\ & \times \exp\left\{i\frac{m\omega_c}{4\hbar}\frac{1}{\sin[\omega_c(t_2 - t_1)/2]}\left[(r_1^2 + r_2^2)\cos\left(\frac{\omega_c}{2}(t_2 - t_1)\right) \right.\right. \\ & \left.\left. - 2r_1r_2\cos(\varphi_2 - \varphi_1)\right]\right. \end{aligned}\right\} \tag{24}$$

or, substituting back using (19), we have (writing $t_2 - t_1 = \tau$):

$$\left.\begin{aligned} G(r_2, r_1; \tau) &= \left(\frac{m}{2\pi i\hbar\tau}\right)\frac{\omega_c\tau}{2\sin(\omega_c\tau/2)} \\ &\quad\times \exp\left\{\frac{i}{\hbar}\frac{m\omega_c}{4}\cot\left(\frac{\omega_c\tau}{2}\right)\left[(r_1^2 + r_2^2) - 2r_1r_2\cos(\theta_2 - \theta_1)\right]\right\} \\ &\equiv \left(\frac{m}{2\pi i\hbar\tau}\right)\frac{\omega_c\tau}{2\sin(\omega_c\tau/2)} \\ &\quad\times \exp\left\{\frac{i}{\hbar}\frac{m\omega_c}{4}\cot\left(\frac{\omega_c\tau}{2}\right)\left[|\underline{r}_1 - \underline{r}_2|^2 - \hat{\underline{z}}\cdot(\underline{r}_2\times\underline{r}_1)\right]\right\} \end{aligned}\right\} \quad (25)$$

If we now add back the 3rd dimension, we need to change A_N back to the form in (8), and multiply by the free 1-d propagator for motion along the $\hat{z}$-axis, to get

$$\left.\begin{aligned} G(r_2, r_1; z_2 z_1; \tau) &= \left(\frac{m}{2\pi i\hbar\tau}\right)^{3/2}\frac{\omega_c\tau}{2\sin(\omega_c\tau/2)}\, e^{\frac{i}{\hbar}\frac{m}{2\tau}(z_2^2 - z_1^2)} \\ &\quad\times \exp\left\{\frac{i}{\hbar}\frac{m\omega_c}{4}\cot\left(\frac{\omega_c\tau}{2}\right)\left[|\underline{r}_1 - \underline{r}_2|^2 - \hat{\underline{z}}\cdot(\underline{r}_2\times\underline{r}_1)\right]\right\} \end{aligned}\right\} \quad (26)$$

Finally, we can put this back in the form of Cartesian coordinates (x, y, z), to get

$$\left.\begin{aligned} G(r_2, r_1; z_2, z_1; \tau) &= \left(\frac{m}{2\pi i\hbar\tau}\right)^{3/2}\frac{\omega_c\tau}{2\sin(\omega_c\tau/2)}\, e^{\frac{i}{\hbar}\frac{m}{2\tau}(z_2^2 - z_1^2)} \\ &\quad\times \exp\left\{\frac{i}{\hbar}\frac{m\omega_c}{4}\cot\left(\frac{\omega_c\tau}{2}\right)\left[(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + \omega_c(x_1y_2 - x_2y_1)\right]\right\} \end{aligned}\right\} \quad (27)$$

Now we can derive the same result by a quite different and more straightforward method, viz., by starting from eqtn (12), and doing the sum over the energy eigenstates explicitly. This method (which leads to a considerable algebraic complexity) is discussed below).

Note that there is a hidden subtlety in the derivation given above, which was first noticed by SF Edwards & Y.V. Gulyaev, Proc. Roy. Soc. A279, 229 (1964). If we start directly from (7), now written in cylindrical coordinates, then we encounter the problem that whereas we know that the condition

$$d\underline{r}_N = \underline{r}_{N+1} - \underline{r}_N \sim O(dt) \quad (28)$$

can easily be implemented in Cartesian coordinates (since $dx_N \sim dy_N \sim dz_N \sim |d\underline{r}_N|$), in cylindrical coordinates it fails as $r \to 0$, because then $d\theta_N \sim dr_N/r_N$. The way to handle this is discussed at the end of these notes.