MS5: STEEPEST DESCENTS - INTRO

Note Title 11/11/2007

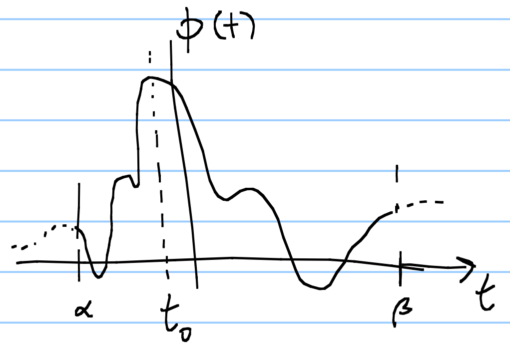
In these notes we will discuss the central core of methods used to treat the asymptotic expansion of integrals - they are also used for understanding differential equations. These methods go under the general label of "STEEPEST DESCENT" methods. Two apparent special cases of steepest descent are Laplace's method & the method of stationary phase (they are not quite special cases, because their realms of applicability are slightly different).

These notes are rather introductory - they need to be supplemented by a more thorough analysis, as well as a thorough discussion of examples.

If one is not too interested in rigour, the method is actually fairly straightforward. In its simplest form it is just a generalisation of the standard Gaussian integral. Consider an integral of the form we already discussed in eqtn (B.7) of "MS4: Asymptotic Analysis", viz., an integral

$$S(x) = \int_\alpha^\beta dt\, e^{x\phi(t)} f(t) \qquad (A.1)$$

where $f(t)$ & $\phi(t)$ are real, and we assume $x \gg 1$. In our previous discussion of this we integrated by parts, but as we saw this doesn't always work.



The basic idea is evident if we consider the behaviour of $\phi(t)$ between the limits. There will be some largest value of $\phi(t)$ for $t = t_0$, and we then make the expansion

$$\phi(t) \sim \phi_0 + \frac{\phi_0''}{2}(t-t_0)^2 + \cdots \frac{\phi_0^{(n)}}{n!}(t-t_0)^n + \cdots \qquad (A.2)$$

where $\phi_0 \equiv \phi(t_0)$ and $\phi_0^{(n)} \equiv \left.\frac{d^n\phi}{dt^n}\right|_{t=t_0}$

Now we can do the integral quite easily if we assume that $f(t)$ is sufficiently slowly varying around $t = t_0$ compared to the exponentially rapidly varying $e^{x\phi(t)}$; if x is large enough this will always be the case. We then have

$$S(x) \sim \int_\alpha^\beta dt\, e^{x(\phi_0 + \frac{\phi_0''}{2}(t-t_0)^2)} f(t)$$

$$\xrightarrow{x \gg 1} f_0 e^{x\phi_0} \int_{-\infty}^{\infty} dt\, e^{x\frac{\phi_0''}{2}(t-t_0)^2}$$

$$= \left(\frac{2}{-\phi_0'' x}\right)^{1/2} f_0 e^{x\phi_0} \int_{-\infty}^{\infty} dy\, e^{-y^2}$$

$$= \left(\frac{2\pi}{-x\phi_0}\right)^{1/2} f_0 e^{x\phi_0} \qquad (A.3)$$

where $f_0 = f(t_0)$, and $y^2 = x\phi_0''(t-t_0^2)$. Integrals of this form are known as Laplace-type integrals (the method being called Laplace's method).

If $\phi_0'' = 0$, we continue on to the first non-zero derivative. If this is $\phi_0^{(n)}$, then a repeat of the above calculation gives

$$S(x) \sim \int_\alpha^\beta dt\, e^{x(\phi_0 + \frac{\phi_0^{(n)}}{n!}(t-t_0)^n)}$$

$$\xrightarrow{x \gg 1} \frac{2}{n}\Gamma(1/n)\left(\frac{n!}{-x\phi_0^{(n)}}\right)^{1/n} f_0 e^{x\phi_0} \qquad (A.4)$$

where we use $\int_{-\infty}^{\infty} dy\, e^{-y^n} = \frac{2}{n}\Gamma(1/n)$.

<u>Example</u> : The best-known example of this is the integral leading to the Γ-function itself. Consider

$$\Gamma(1+x) = \int_0^\infty dt\, e^{-t} t^x \equiv \int_0^\infty dt\, e^{x\ln t - t} \qquad (A.5)$$

which has a maximum at $t = t_0 = x$.

If we now expand about $t = x$, we have

$$\Gamma(1+x) \xrightarrow{x \gg 1} \int_0^\infty dt\, e^{x\ln x - x}\, e^{-\frac{1}{2x}(t-x)^2}$$

$$\sim e^{x\ln x - x} \int_{-\infty}^{\infty} dt\, e^{-\frac{1}{2x}(t-x)^2}$$

$$= (2\pi x)^{1/2} x^x e^{-x} \equiv (2\pi x)^{1/2}\left(\frac{x}{e}\right)^x \qquad (A.6)$$

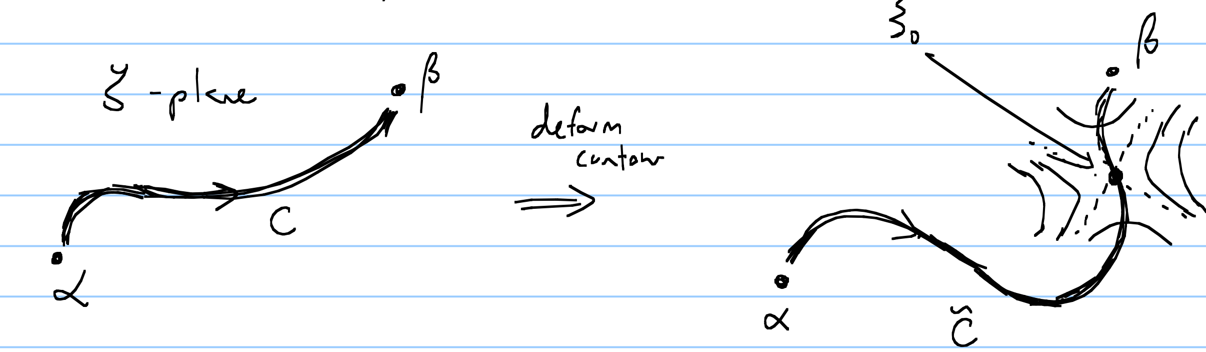
which is nothing but Stirling's formula. Note that this is just the first term in the asymptotic expansion. In fact one finds that

$$\Gamma(1+x) \sim (2\pi x)^{1/2}\left(\frac{x}{e}\right)^x \left[1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right] \qquad (A.7)$$

Now this is clearly a method that can easily be generalized. Consider the integral

$$S(z) = \int_C d\zeta\, e^{z\phi(\zeta)} f(\zeta) \qquad (A.8)$$

along some contour. Now we assume that $|\phi(\zeta)|$ has a maximum somewhere, when $\zeta = \zeta_0$



We then realise that the point $\zeta_0 = u_0 + iv_0$ must be a saddle point of the function $\phi(\zeta)$ at ζ_0, because of the Cauchy-Riemann eqtns. This means that the contours of constant $\mathrm{Re}\,\phi(\zeta)$ near ζ_0 must look like what is shown on the right picture above, and at the saddle point

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial v} = 0 \quad \Big|_{\phi=\phi(\zeta_0)=\phi_0} \qquad (A.9)$$

where $\partial\phi/\partial u = 0$ because we are at a maximum of ϕ, and then $\partial\phi/\partial v = 0$ by Cauchy-Riemann.

Provided we do not cross any singularities (poles, branch cuts) in $\phi(\xi)$, we can then deform the contour C to $\tilde{C}$ as shown above so that it follows a path, as ξ_0 is approached, such that $\mathrm{Im}\,\phi(\xi) = \text{const}$ along this path, i.e., the PHASE of the function $\phi(\xi)$ does not change as we go along the path. We can then write

$$\phi(\xi) = \gamma(\xi) + i\chi(\xi) \quad \rightarrow \quad \gamma(\xi) + i\chi_0 \qquad (A.10)$$

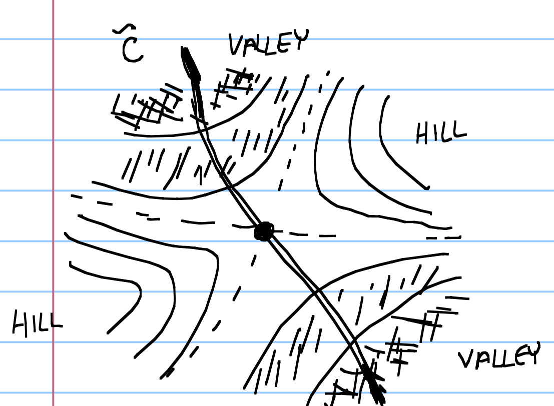
so that

$$S(z) = \int_{\tilde{C}} d\xi\, e^{z\phi(\xi)} f(\xi)$$

$$\longrightarrow e^{iz\chi_0} \int_{\tilde{C}} d\xi\, f(\xi)\, e^{z\gamma(\xi)} \qquad (A.11)$$

Now we see that the exponent is easy to treat – basically this integral is now of the Laplace type we saw before.

The method is called the method of steepest descents because if we hold $\chi(\xi) \rightarrow \chi_0 = \text{const}$, then the exponent $\gamma(\xi)$ varies as fast as possible along the contour $\tilde{C}$.



This is obvious from the close-up of the region round the saddle point at ξ_0; lines of steepest descent are perpendicular to contours of constant $\gamma(\xi)$, and are thus lines of constant $\chi(\xi)$. To see this formally we just write the Cauchy-Riemann eqtns

$$\left.\begin{aligned} \frac{\partial\gamma}{\partial u}\frac{\partial\chi}{\partial u} + \frac{\partial\gamma}{\partial v}\frac{\partial\chi}{\partial v} &= 0 \\ \text{or} \quad \underline{\nabla}\gamma \cdot \nabla\chi &= 0 \end{aligned}\right\} \qquad (A.12)$$

and this implies they are perpendicular.

Let's look in more detail at this saddle point region. It is useful to expand around $\xi = \xi_0$ in the form

$$\xi - \xi_0 = \rho e^{i\theta} \qquad \left.\frac{\partial^2 \phi(\xi)}{\partial \xi^2}\right|_{\xi=\xi_0} = R e^{i\psi} \qquad (A.13)$$

and also

$$\phi(\xi) = \alpha + i\beta \qquad (A.14)$$

We know that in the saddle point region

$$\phi(\xi) - \phi(\xi_0) \sim \frac{1}{2} \left.\frac{\partial^2 \phi}{\partial \xi^2}\right|_{\xi=\xi_0} (\xi - \xi_0)^2 \qquad (A.15)$$

ie

$$(\alpha - \alpha_0) + i(\beta - \beta_0) \sim \frac{1}{2} R \rho^2 e^{i(\psi + 2\theta)} \qquad (A.16)$$

Now we can see that the direction of steepest descent (ie, the angles ϕ along which we get steepest descent) are given by

$$\cos(\psi + 2\theta) = -1 \qquad (A.17)$$

so that $\psi + 2\theta = \pi, 3\pi$, and so we have

$$\text{steepest descent lines:} \qquad \theta = \begin{cases} \pi/2 - \psi/2 \\ 3\pi/2 - \psi/2 \end{cases} \qquad (A.18)$$

$$\text{steepest ascent lines:} \qquad \theta = \begin{cases} -\psi/2 \\ \pi/2 - \psi/2 \end{cases} \qquad (A.19)$$

It then follows immediately that we can write the original integral in the form

$$S(z) = \int_{\tilde{C}} d\xi \, e^{z\phi(\xi)} f(\xi)$$

$$\sim e^{z\phi_0} f_0 e^{i\theta} \int_{-\infty}^{\infty} d\rho \, e^{-\frac{z}{2} R\rho^2} = \left(\frac{2\pi}{zR}\right)^{\frac{1}{2}} e^{i\theta} f_0 e^{z\phi_0} \qquad (A.20)$$

where $\quad f_0 = f(\xi_0)$

$\phi_0 = \phi(\xi_0)$

and θ is one of the 2 angles in (A.18) (which one we take depends on which direction we follow the path).

Example : Let's look at the Γ-function again, but this time for complex argument. Then we define.

$$\Gamma(1+z) = \int_0^\infty d\xi \, e^{z \ln \xi - \xi} \tag{A.21}$$

i.e., if we write $\Gamma(1+z) = \int_0^\infty d\xi \, f(\xi) e^{z\phi(\xi)}$

then
$$\phi(\xi) = \left(\ln \xi - \xi/z\right) \tag{A.22}$$

It then follows that the saddle point is located at

$$\frac{\partial \phi}{\partial \xi} = 0 \quad \text{when} \quad \xi = \xi_0 = z \tag{A.23}$$

at which point we have
$$\left.\frac{\partial^2 \phi}{\partial \xi^2}\right|_{\xi = z} = -\frac{1}{z^2} \tag{A.24}$$

If we write
$$z = x + iy = z_0 e^{is} \tag{A.25}$$

then we easily find that the directions of steepest descent are along the directions

$$\theta = \begin{cases} s/2 \\ s/2 + \pi \end{cases} \tag{A.26}$$

and we get for the steepest descent result for $\Gamma(1+z)$ that

$$\left.\begin{aligned} \Gamma(1+z) &\sim (2\pi z_0)^{1/2} e^{is/2} e^{z \ln z - z} \\ &= (2\pi)^{1/2} z^{z+1/2} e^{-z} \end{aligned}\right\} \tag{A.27}$$