MS4: ASYMPTOTIC ANALYSIS

Note Title 21/10/2007

This is a tutorial summary of asymptotic expansions for complex functions. The material is related to other important topics, notably asymptotic & divergent series, integral transforms, perturbation theory, & non-perturbative methods like WKB, inverse scattering, boundary layer & multi-scale analysis, etc, along with various other functional methods.

The plan here is to look at the following topics:

- Intro: def^{ns} of asymptotic expansions, what we are looking at.
- Elementary asymptotic expansions of integrals

In other sets of notes we will look at the related topics of

- Steepest descent methods for evaluation of integrals.
- More general integral transforms
- Relation to asymptotic series.

(A) INTRO to ASYMPTOTIC ANALYSIS

Most of the time we will be concerned with a problem that arises all the time in doing integrals. We are faced with some integral of form

$$I_{ab}(x) = \int_a^b dt\, f(t,x) \qquad (A.1)$$

and we know that the integral possesses limiting behaviour as $x \to x_0$ and $x \to \infty$; the behaviour for the rest of x interpolates between these limits. More generally, we have some integral

$$I_{z_a z_b}(z) = \int_{z_a}^{z_b} d\xi\, f(\xi, z) \qquad (A.2)$$

in the complex plane along a contour between z_a and z_b, and we have different limiting behaviour for $|z| \to \infty$ along various "wedges", and also limiting behaviour as $z \to$ various points or lines in the finite plane (depending on the analytic form of $f(\xi, z)$ with ξ).

We want to determine this limiting behaviour, and also the form of expansions about it as we move away from the asymptotic regimes.

Often integrals like these will be thrown up when we are trying to solve differential eqtns – one can determine the limiting behaviour of these near certain points, or in the limit as $|z| \to \infty$, and these limiting behaviours are in integral form. This can happen if we convert the differential eqtn to an integral eqtn, or by carrying out an integral transform (eg., Laplace, Fourier, Mellin, etc), or in other ways. All of this part of that branch of analysis that deals with asymptotic behaviour.

So, we will in general be producing asymptotic series expansions which will give us the limiting behaviour of these integrals as we approach some point.

Let us recall here the definition of an asymptotic series (see notes on asymptotic series for more details), and also make some remarks on the general form of these in the complex plane.

A.1 Asymptotic Series: Definitions

I simply recall the basic results here, which are discussed in much more detail in my other notes on asymptotic & divergent series.

We recall the following:

Functions & Infinite Series: An infinite series of the form

$$f(z) \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (A.3)$$

is considered to represent the function $f(z)$ if we can say that

$$\left.\begin{aligned} \varepsilon_N(z) &= f(z) - \sum_{n=0}^{N} a_n (z-z_0)^n \\ &\ll (z-z_0)^N \end{aligned}\right\} z \to z_0 \qquad \forall N \qquad (A.4)$$

Notice that if we are expanding about the point $z = \infty$, the form of this relationship will instead look like:

$$f(z) \sim \sum_{n=0}^{\infty} C_n / z^n \qquad (A.5)$$

and this representation holds if

$$\left.\begin{aligned} \varepsilon_N(z) &= f(z) - \sum_{n=0}^{N} \frac{C_n}{z^n} \\ &\ll \frac{1}{z^N} \end{aligned}\right\} \quad z \to \infty \qquad \forall N \qquad (A.6)$$

Now we can compare convergent and asymptotic power series:

<u>Convergent Series</u>: The series in (A.3) and (A.5) are considered to be CONVERGENT inside the domain D if

$$\varepsilon_N(z) \to 0 \qquad N \to \infty \qquad z \in D \qquad (A.7)$$

In other words, the remainder or ERROR $\varepsilon_N(z) \to 0$ for any z inside the domain D when $N \to \infty$. An example is of course a Taylor series inside its radius of convergence.

Note that convergence as so defined is solely a property of the series. We can see this by rewriting $\varepsilon_N(z)$ as

$$\left.\begin{aligned} \varepsilon_N(z) &= \sum_{n=N+1}^{\infty} a_n (z-z_0)^n & (z_0 \text{ finite}) \\ \varepsilon_N(z) &= \sum_{n=N+1}^{\infty} C_n / z^n & (z_0 \to \infty). \end{aligned}\right\} \qquad (A.8)$$

which eliminates reference to $f(z)$. Thus convergence of the series depends ONLY on what are the a_n.

<u>Asymptotic Series</u>: In an asymptotic series we take a different limit from (A.7). Instead we say that a series is ASYMPTOTIC to $f(z)$ if

$$\left.\begin{aligned} \varepsilon_N(z) &\ll (z-z_0)^N & z \to z_0 \qquad N \text{ FIXED} \\ \varepsilon_N(z) &\ll 1/z^N & z \to \infty \qquad N \text{ FIXED} \end{aligned}\right\} \qquad (A.9)$$

This is very different from convergence. It says that FOR A GIVEN N, the error gets smaller & smaller as we approach z_0 (or ∞). However, if we fix some value of z which is removed from z_0 (or ∞), then the error is finite for any N, even if $N \to \infty$ (indeed it often gets LARGER as $N \to \infty$).

Thus an asymptotic series is always asymptotic relative to some given function $f(z)$.

Moreover a given series can be asymptotic to many different functions as $z \to z_0$. We can have

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \sim \begin{cases} f_1(z) \\ f_2(z) \end{cases} \qquad (z \to z_0) \qquad (A.10)$$

if the difference $g(z) = f_1(z) - f_2(z)$ goes to zero faster than any power $(z-z_0)$ as $z \to z_0$. An example of such a "sub-dominant" function $g(z)$ is:

$$g(z) \sim A e^{-B/(z-z_0)^2} \qquad (A.11)$$

For more details, go to the notes on asymptotic series.

A.2: The Stokes Phenomenon

: When we approximate a function by some other function or series which is suppose to represent it asymptotically, then if we extend the approximation to the complex plane, we have to worry about how the sub-dominant corrections will behave throughout the complex plane.

Consider, eg., the following asymptotic approximation:

<u>Example</u>:

$$\left.\begin{aligned} f(z) &= \sinh \tfrac{1}{z} = \tfrac{1}{2}\left(e^{1/z} + e^{-1/z}\right) \\ &\sim \tfrac{1}{2} e^{1/z} + g(z) \qquad (\mathrm{Re}\, z > 0) \end{aligned}\right\} \qquad (A.12)$$

where the sub-dominant

$$g(z) = -\tfrac{1}{2} e^{-1/z} \qquad (A.13)$$

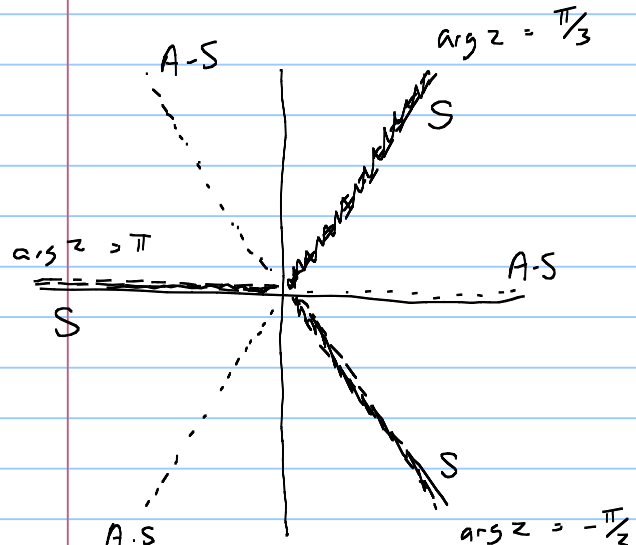
The problem here is obvious — in the half-plane $\mathrm{Re}\, z < 0$, the sub-dominant $g(z)$ is clearly dominant — the role of dominant & sub-dominant functions has reversed! This is a fairly general feature of asymptotic representations, because by their very nature, sub-dominant terms tend to be exponentials which are very small in one "wedge" of the complex plane, but blow up in another. Before giving a more general discussion of this, consider another example.

Example: Consider the Airy functions $Ai(z)$, $Bi(z)$, which are solutions to the differential eqtn

$$\frac{d^2y}{dz^2} = z\,y(z) \qquad (A.14)$$

The solutions for $|z| \gg 1$ have the asymptotic form

$$y(z) \sim e^{\pm\frac{2}{3}z^{3/2}} \qquad (A.15)$$



This then leads to the structure of wedges shown at left. The "Stokes" lines are labelled by "S", and the "anti-Stokes" lines are labelled by "A-S". These are defined as follows:

Stokes line: The line along which the dominant & sub-dominant terms become of the same order (ie one crosses or exchanges between the two). If the two functions have the form

$$\left.\begin{aligned} f_1(z) &= A_1 e^{\Gamma_1(z)} \\ f_2(z) &= A_2 e^{\Gamma_2(z)} \end{aligned}\right\} \quad (z \to z_0) \qquad (A.16)$$

then the Stokes lines is defined by $\quad \mathrm{Re}\left(\Gamma_1(z) - \Gamma_2(z)\right) = 0 \qquad (A.17)$

Anti-Stokes line: We now look for the line along which the sub-dominant term is least important. This happens when the complex phases in (A.16) are purely real – we then have

Anti-Stokes: $$\operatorname{Im}\left(\Gamma_1(z) - \Gamma_2(z)\right) = 0 \qquad (A.18)$$

Going back to the Airy differential eqtn, we have the following asymptotic behaviors:

$$Ai(z) \sim \frac{1}{2\sqrt{\pi}} \frac{1}{z^{1/4}} e^{-\frac{2}{3} z^{3/2}} \quad \left(\begin{array}{c} z \to \infty \\ |\arg z| < \pi \end{array}\right) \qquad (A.19)$$

and $$Bi(z) \sim \frac{1}{\sqrt{\pi}} \frac{1}{z^{1/4}} e^{\frac{2}{3} z^{3/2}} \quad \left(\begin{array}{c} z \to \infty \\ |\arg z| < \frac{\pi}{3} \end{array}\right) \qquad (A.20)$$

Note that in spite of the Stokes lines at $|\arg z| = \pm \pi/3$, eqtn (A.19) is still valid for the much larger region $|\arg z| < \pi$. We return to the Airy functions in more detail later.

It is useful to make a few general remarks here, without detailed discussion, or demonstration.

- If a function $f(z)$ is analytic as $z \to \infty$, then its asymptotic expansion is identical to its Taylor series, & this then converges. Then there are no Stokes lines & no Stokes phenomenon.

- If the point at $z = \infty$ is a branch point, then the asymptotic expansion of $f(z)$ must change discontinuously across the branch cut.

- If the point $z = \infty$ is an essential singularity, then there will be Stokes lines (eg., the Airy functions). The same applies for an essential singularity at any other point z_0 (eg., the sinh $1/z$ function, whose singularity is at $z = 0$),

(B) ELEMENTARY ASYMPTOTICS for INTEGRALS

Let's now look at how we can find the asymptotic properties of integrals. We begin with what can be found using elementary methods. By elementary I mean either using a Taylor expansion, or by integration by parts.

Let's start with some examples to see how it works:

<u>Example</u>: Consider the integral $S(x) = \int_x^\infty dt\, e^{-t^4}$ (B.1)

We wish to find the limiting behaviour for $x \to 0$ and $x \to \infty$. This is done as follows:

(a) <u>$x \ll 1$</u>. We do this by Taylor expansion — write

$$
\begin{aligned}
S(x) &= \left(\int_0^\infty dt - \int_0^x dt\right) e^{-t^4} \\
&= \Gamma(5/4) - \int_0^x dt\, (1 - t^4 + \tfrac{1}{2}t^8 - \tfrac{1}{6}t^{12} \dots) \\
&= \Gamma(5/4) - \left[x + \tfrac{1}{5}x^5 + \tfrac{1}{18}x^9 + \tfrac{1}{78}x^{13} + \dots\right] \\
&= \Gamma(5/4) - \sum_{n=0}^{\infty} \frac{1}{n!}\frac{x^{4n+1}}{4n+1} \qquad (B.2)
\end{aligned}
$$

where the integral from 0 to ∞ is found easily by making the substitution $u = t^4$, so that

$$
\left.\begin{aligned}
\int_0^\infty dt\, e^{-t^4} = \tfrac{1}{4}\int_0^\infty du\, \frac{e^{-u}}{u^{3/4}} &= \tfrac{1}{4}\Gamma(1/4) \\
&= \Gamma(5/4)
\end{aligned}\right\} \qquad (B.3)
$$

Note that this series in (B.2) is convergent $\forall x$, but it only converges rapidly when $x < 1$.

(b) <u>$x \gg 1$</u>: Now we want to develop an asymptotic expansion of $S(x)$ around $x = \infty$, i.e., in powers of x^{-1}. So lets do the following:

Write: $$S(x) = -\tfrac{1}{4}\int_x^\infty \frac{dt}{t^3}\,\frac{d}{dt}\left(e^{-t^4}\right) \tag{B.4}$$

& integrating by parts: $$= \frac{1}{4x^3}e^{-x^4} - \tfrac{3}{4}\int_x^\infty \frac{dt}{t^4}\,e^{-t^4} \tag{B.5}$$

The 2nd term is much smaller than the first for large x, because it is bounded by $S(x)/x^4$. We can continue this integration by parts systematically — you will easily find that

$$S(x) = \frac{1}{4x^3}e^{-x^4}\left[1 + \sum_{n=1}^{\infty}\prod_{j=1}^{n}(4j-1)\left(-\frac{1}{4}\right)^n \frac{1}{x^{4n}}\right] \tag{B.6}$$

which converges extremely fast for large x.

Now there are lots of examples like this. To give yourself a better understanding of them, you should try various examples, which I will give below.

Consider now the more general form

$$S_{\alpha\beta}(x) = \int_\alpha^\beta dt\; f(t)\,e^{x\phi(t)} \tag{B.7}$$

and now let's look at the asymptotic behaviour of this as $x \to \infty$. If we integrate by parts we get:

$$S_{\alpha\beta}(x) = \frac{1}{x}\int_\alpha^\beta dt\; f(t)\left(\frac{d\phi}{dt}\right)^{-1}\frac{d}{dt}\left(e^{x\phi(t)}\right)$$

$$\xrightarrow[x\to\infty]{} \frac{1}{x}\left[\frac{f(\beta)}{\phi'(\beta)}e^{x\phi(\beta)} - \frac{f(\alpha)}{\phi'(\alpha)}e^{x\phi(\alpha)}\right] \tag{B.8}$$

However (B.8) is only true if the following conditions are satisfied:

$$\left.\begin{aligned} &(i)\ \ \frac{d\phi}{dt} \neq 0,\infty\ (\beta \geq t \geq \alpha)\ ;\ \ f(\alpha) \text{ and/or } f(\beta) \neq 0,\infty \\ &(ii)\ \ \phi(t),\ \frac{d\phi}{dt},\ f(t) \text{ are continuous} \end{aligned}\right\} \tag{B.9}$$

Now typically one of the 2 terms in (B.8) will dominate. Suppose that $\mathrm{Re}\,\phi(\beta) > \mathrm{Re}\,\phi(t)$ $(t < \beta)$. Then the 1st term in (B.8) dominates, and the 2nd term is the (sub-dominant) remainder (which we can continue to integrate by parts to get an asymptotic expansion). Then we get the series

$$S_{bc}(x) \sim A \frac{e^{-x\phi(\beta)}}{x} \left[1 + \sum_{n=1}^{\infty} b_n / x^n \right] \qquad (B.10)$$

where A & b_n are found using repeated integration by parts.

<u>Example:</u> Find an asymptotic expansion for the integral

$$S(x) = \int_1^4 dt\, e^{x \cosh^2 t} \qquad (B.11)$$

This is easy using (B.8); we get

$$S(x) \xrightarrow[x \gg 1]{} \frac{1}{2x} \frac{1}{\sinh 4 \cosh 4} e^{x \cosh^2 4} + O(1/x^2) \qquad (B.12)$$

Now, if we could always use Taylor expansion & integration by parts it would be easy (but tedious). To see this is not the case, consider the following examples:

<u>Example:</u> Consider the integral $S(x) = \int_0^\infty dt\, e^{-x \sinh^2 t}$ (B.13)

Integration by parts fails (try it!). The reason is that $d\phi/dt = 0$ at $t = 0$. (cf (B.9)), giving divergent integrals.

<u>Example:</u> Consider the integral $S(x) = \int_0^\infty dt\, e^{-xt^2}$ (B.14)

Again we $d\phi/dt = 0$ at $t = 0$. If we try integration by parts, we again get divergent integrals.

These examples probably make the point. You see that one can do quite a bit with elementary methods to get the asymptotic results. This does <u>not</u> mean that these are necessarily the best methods, but they often work (even though they can be tedious at times).

To check your understanding, try some of the following examples (only some can be solved using elementary methods!):

EXAMPLES to TRY

A:E1 Find asymptotic expansion for $S(x) = \int_0^x dt\, e^{t^2}$ in limit $x \ll 1$
in limit $x \gg 1$
(2nd is non-trivial, but can be done using Int. by parts).

A:E2 Find asymptotic expansion for $S(x) = \frac{2}{\pi}\int_0^x dt\, e^{-t^2}$ in limits $x \gg 1$, $x \ll 1$
(this is the error function)

A:E3 Find expansions for $S(x) = \int_x^\infty dt\, \frac{e^{-t}}{t}$ in limits $x \gg 1$, $x \ll 1$
(the exponential integral)

A:E4 Find expansions for $S(x) = \int_0^x dt\, \frac{e^{-t}}{\sqrt{t}}$ in limits $x \gg 1$, $x \ll 1$.

A:E5 Find expansion for $S(\alpha) = \int_0^1 dt\, \frac{\sin \alpha t}{t}$ " " "

A:E6. Find expansion for $S(x) = \int_0^x dt\, e^{-t}/t^{1/4}$ " " "

A.E7: Find expansion for $S(x) = \int_0^\infty dt\, \frac{e^{-t}}{1+xt}$ " " "
(the Stieltjes function).

A.E8: Find expansion for $S(x) = \int_0^x dt\, \frac{e^{-t}}{1+t^2}$ " " " "

A.E9: " " " $S(x) = \int_0^\infty dt\, \frac{e^{-xt}}{1+t^2}$ " " "

A.E10: " " " $S(x) = \int_0^\infty dt\, \frac{e^{-xt}}{(1+t^2)^2}$ " " "

A.E11: " " " $S(x;6) = \int_0^6 dt\, \frac{e^{-xt}}{1+t^2}$ " " "

A.E12: " " " $S(x;6) = \int_0^6 dt\, \frac{e^{-xt}}{1+t}$ " " "

A. E13 : " " " $S(x;6) = \int_0^6 dt \, \frac{e^{-xt}}{1+t^6}$ " " "

A. E14 : " " " $S(x;6) = \int_0^6 dt \, \frac{e^{-xt}}{[t(t+4)]^{1/2}}$ " " "

A. E15 : " " " $S(x) = \int_0^\infty dt \, \frac{1}{1+t} e^{-x(t+4)^2}$ " "

A. E16 : " " " $S(x) = \int_1^\infty dt \, e^{-x(t^2+4)}$ " "

A. E17 : " " " $S(x) = \int_2^{10} dt \, e^{-xt} \sin t$ " "

A. E18 : " " " $S(x) = \int_2^{10} dt \, e^{-t} \sin xt$ " "

A. E19 : " " " $S(x) = \int_0^\infty dt \, \ln(1+t) e^{-x \sinh^2 t}$ " "

A. E20 " " " $S(\omega;2) = \int_0^2 dt \, \frac{e^{i\omega t}}{1+t}$ " " "

A. E21 " " " $S(\omega,2) = \int_0^2 dt \, \frac{e^{i\omega t}}{1+t^2}$ " " "

A. E23 " " " $S(\omega) = \int_0^\infty dt \, \frac{e^{i\omega t}}{1+t^2}$ " " "

A. E24 " " " $S(\omega,2) = \int_0^2 dt \, \frac{e^{i\omega t}}{\sqrt{t}}$ " " "

A. E25 " " " $S(\omega) = \int_0^\infty dt \, \frac{e^{i\omega t}}{\sqrt{t}}$ " " "