MS3(b): SERIES SUMMATION – EXAMPLES

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The best way to get information about a series, particularly a divergent series, is to relate it to some function from which the series can be generated in some way. We've already seen a few examples of this in section I.B, in which series were generated by Taylor expansion of a function.

A proper general discussion of this requires that we deal with the general idea of an asymptotic series, & how it may be understood as being asymptotic to a function. This leads to the general problem of asymptotic expansion of integrals. A proper understanding of such expansions, and of the functions being expanded, requires examination of the analytic properties of both the expansions & of the original functions throughout the complex plane.

Before beginning the formal discussion, it is useful to have some examples; so I repeat the examples from Part I, and then add a whole lot more.

EXAMPLES : From part I we had

$$\left.\begin{aligned} S_1 &= \sum_{n=0}^{\infty} (-1)^n \\ S_2 &= \sum_{n=0}^{\infty} n \\ S_3 &= \sum_{n=0}^{\infty} (-1)^n n! \\ S_4 &= \sum_{n=0}^{\infty} (-1)^n (n!)^2 \\ S_5 &= 1+0+0-1+1+0+0-1+\cdots \end{aligned}\right\} \quad (1)$$

and here are a whole lot more; first, a few more series:

$$\left.\begin{aligned} S_6(k) &= \sum_{n=0}^{\infty} n^k \\ S_7(k) &= \sum_{n=0}^{\infty} \frac{1}{(n!)^k} \\ S_8 &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}\frac{1}{n!} \\ S_9(x) &= \sum_{n=0}^{\infty} \frac{1}{n^2+x^2} \end{aligned}\right\} \quad (2)$$

and now some functions whose asymptotic properties we seek when $x >> 1$ and/or $x << 1$:

$$\left.\begin{aligned}
S_{10}(x) &\equiv \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \\
S_{11}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{t^2} \\
S_{12}(x) &= \int_x^\infty dt\, e^{-t^4} \\
S_{13}(x) &\equiv E_1(x) = \int_x^\infty \frac{dt}{t} e^{-t} \\
S_{14}(z) &\equiv \Gamma(1+z) = \int_0^\infty dt\, e^{-t}\, t^z \\
S_{15}(z) &\equiv Ai(z) = \frac{1}{2\pi i} \int_C dt\, e^{zt - t^3/3} \\
S_{16}(k,x) &= \sum_{n=0}^{[x]} n^k \\
S_{17}(z) &= \int_0^\infty dt\, \frac{e^{-t}}{1+zt}
\end{aligned}\right\} \quad (3)$$

and finally, 2 relations of a more general nature (dealing with a general sum over terms f_n):

$$\left.\begin{aligned}
S^f(z) &\equiv \sum_{n=0}^\infty (-1) f_n z^n = \int_0^\infty dt\, \frac{f(t)}{1+zt} \\
S^f &\equiv \sum_{n=-\infty}^\infty f_n = \sum_{k=-\infty}^\infty \int_{-\infty}^\infty dt\, f(t)\, e^{2\pi i k t}
\end{aligned}\right\} \quad (4)$$

where the functions $f(t)$ are such that $f(t=n) = f_n$, but otherwise arbitrary; you need to prove the eqtns in (4), and then use them on the following examples

$$S_1^f(x) = 2\int_0^\infty dt\, \frac{K_0(t)}{1+xt} \sim \sum_{n=0}^\infty (-2)^n \left|\Gamma\left(\frac{n+1}{2}\right)\right|^2 x^n$$