MS3(a) : SERIES SUMMATION

Note Title 15/09/2007

These are sketchy & incomplete notes on series summation, with particular reference to the perturbation/diagrammatic series involved in quantum field theory.

Some well-known stuff is reviewed first, on convergence, simple summation, integral representations of series sums, & corresponding series representation of integrals, with some standard examples relevant to QM and to QFT (eg., Euler & Borel summation). From this I go on to do Cesàro-Riesz summation.

I also give you lots of examples to test yourself on.

(A) SOME BASIC RESULTS

: Let's recall some basic stuff about convergence. For a series defined by the total sum

$$S = \sum_{n=0}^{\infty} a_n \tag{1}$$

and partial sum

$$S_n = \sum_{r=0}^{n} a_r \tag{2}$$

we have ABSOLUTE convergence to S if

$$\hat{S} = \sum_{n=0}^{\infty} |a_n| \tag{3}$$

converges to $\hat{S}$.

Thus, eg., the series

$$\left. \begin{aligned} S(z) &= \sum_{n=0}^{\infty} z^n \\ &\xrightarrow[|z|<1]{} \frac{1}{1-z} \end{aligned} \right\} \tag{4}$$

converges for $|z| < 1$, and diverges otherwise. Another example; the series

$$S = \sum_{n=0}^{\infty} 1/n \qquad \text{diverges} \tag{5}$$

but

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \longrightarrow \ln 2 \tag{6}$$

A1. TESTS for CONVERGENCE:

There are lots of these. The simplest starts from (4); we see that since (4) converges if $|z| < 1$, then if we have a series like (1), it must converge if

$$\left| \frac{a_{n+1}}{a_n} \right| \xrightarrow[n\to\infty]{} z \quad , \quad \text{s.t. } |z| < 1 \tag{7}$$

then S converges absolutely. This is d'Alembert's ratio test. Another well-known test is Cauchy's test, which states that

$$\text{if} \quad \lim_{n\to\infty} |a_n|^{1/n} < 1 \tag{8}$$

then $S = \sum_n a_n$ converges absolutely.

In the same way as with (7), we can generate convergence tests by comparing any series with the very many series that can usefully be generated by series expansion of various functions or integrals.

For example, consider the integral

$$I(x) = \int_a^\infty \frac{du}{u^x} = \left.\frac{u^{1-x}}{1-x}\right|_a^\infty \longrightarrow \text{finite if } x > 1 \tag{9}$$

Now consider the famous series $\zeta(x) = \sum_{n=0}^{\infty} \frac{1}{n^x}$

for the Rieman ζ-fn. The ratio test in (7) fails for this series, because

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left(1+\frac{1}{n}\right)^{-x} \sim 1 - \frac{x}{n} \rightarrow 1 \tag{10}$$

However since we see from (9) that the integral converges if $x > 1$, we can immediately deduce that if

$$\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow[n\to\infty]{} 1 - \frac{x}{n} \qquad \text{s.t. } x > 1 \tag{11}$$

then the series converges absolutely. One can further develop more stringent ratio tests by considering other integrals.

There is actually a large number of techniques for determining if a series converges, given in books [1]

A2. ELEMENTARY SUMMATION TRICKS

: The simplest tricks come by transforming the series into one that we already know. Thus one begins by generating a personal "library" of series. The simplest way to do this is by taking all the interesting functions one knows and writing out their Taylor-MacLaurin expansions. Thus, eg., one has

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{12}$$

so that from $e^{iz} = \cos z + i \sin z$, we have

$$\left.\begin{aligned} \cos z &= \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \\ \sin z &= \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \end{aligned}\right\} \qquad (13)$$

Then one has the binomial series, which we write as

$$(1+z)^n = \sum_{r=0}^{\infty} C^n_r \, z^r \equiv \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} z^r \qquad (14)$$

from which, eg.,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n \qquad (15)$$

$$\frac{1}{(z+1)(z+a)} = \sum_{n=0}^{\infty} (-1)^n \left[1 - \frac{1}{a^n}\right] z^n \qquad (16)$$

where we assume, without loss of generality, that $|a| > 1$. If we go on with this logical development we may generate a huge variety of series by taking all the meromorphic functions $f(z)$ we know, & generating series expansions using

$$f(z) - f(0) = \sum_j R_j \left[\frac{1}{z - z_j} + \frac{1}{z_j}\right] \qquad (17)$$

where z_j are the positions of the poles of $f(z)$, with residues R_j.

Once this is done we can fool around by integrating & differentiating these. For example, from (15) we have, by integrating, that

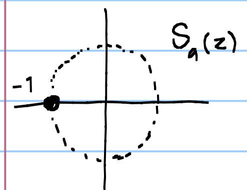
$$\ln(1+z) = \int dz \, \frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-z)^n}{n} \qquad (18)$$

and so on – clearly one can develop a very large "library" of results here.

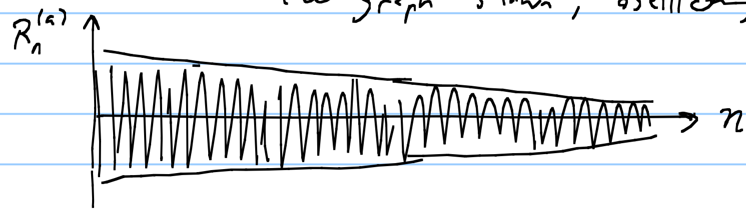
It is interesting to notice how these series approach their limiting sum. Consider, eg., the 2 series in (15) and (16), and define the remainder $R_n = S - S_n$; we have for these 2 series that

$$(a) \quad S_a = \sum_{n=0}^{\infty} (-z)^n = \frac{1}{1+z} \qquad R_n^{(a)} = \frac{(-z)^{n+1}}{1+z} \qquad (19)$$

$$
\left.\begin{aligned}
\text{(b)}\quad S_b &= \sum_{n=0}^{\infty} \left(1-a^{-n}\right)(-z)^n = \frac{1}{(1+z)(a+z)} \\
R_n^{(b)} &= \frac{1}{(1+z)(a+z)} + \frac{(-z/a)^{n+1}}{z+a} - \frac{(-z)^{n+1}}{1+z}
\end{aligned}\right\} \qquad (20)
$$



Notice how slowly the series S_a converges as z approaches the circle of convergence near $x = 1$. Suppose we plot $R_n^{(a)}$ for some value of z like $z = (0.98, 0)$; it looks like the graph shown, oscillating back & forth.



The same is true but even more so for the series $S_b(z)$. If you put this on a calculator, assuming that, e.g., $z = (0.98, 0)$, and let a take a few values like $1.5, 2, 3$, then you will see that the oscillations in $R_n^{(b)}$ fall off extremely slowly with n.

There are actually many ways that one can devise of accelerating the convergence of convergent series like this – this is a topic worth exploring, for which I have no time here.

Notice that we have now generated our "library" of series, using various ideas (Taylor series, Cauchy-Laurent series expansions, etc., of meromorphic functions, integration & differentiation of these, and so on).

This prepares us for the more difficult inverse task of summing some arbitrary series we are handed – we try & transform it to one in our library. Actually this is a huge & fascinating topic, with many interesting ideas & tricks – no time to explore it here unfortunately.

Let's now move on to series whose structure is more like those found in field theory.

(B) SIMPLE TECHNIQUES for DIVERGENT SERIES

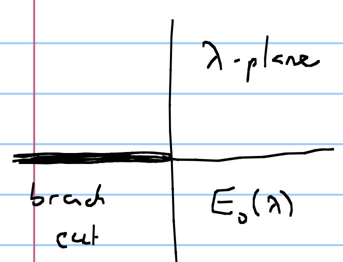
: In a typical problem in physics — not just field theory but even ordinary QM — we are faced with series which differ in key respects from the structures given in (A). In particular,

(i) They are formally divergent
(ii) We don't usually know all the terms in the series.

B1. SOME PHYSICAL BACKGROUND

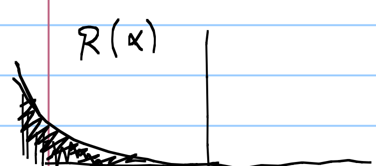
: In physics there is a rich diversity of such problems, leading to many different kinds of divergent series, for which there is no time. Let me just briefly note 2 examples, those of QED and the ϕ^4 field theory (for more details see refs [2,3]).

In QED we can ask about the structure of the ground state energy $E_0(\lambda)$, where λ is the effective coupling constant — or we can ask about, eg., the magnetic moment $g(\lambda)$. A simple argument due to Dyson shows that the analytic structure of $E_0(\lambda)$ is as shown in the diagram — there is a branch cut along the negative λ axis, whose magnitude goes like



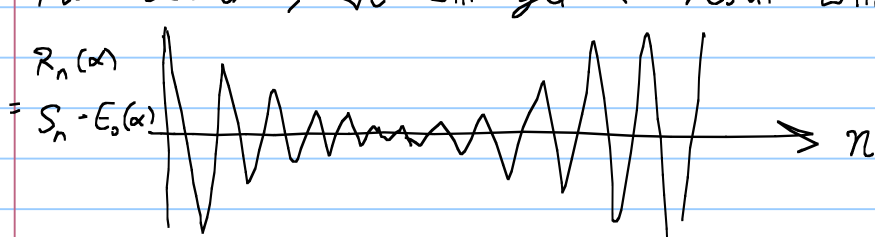
$$\Gamma(\alpha) \sim \theta(-\alpha)\, e^{-c/\alpha} \qquad (1)$$

where $\alpha = \mathrm{Re}\,\lambda$, and $\mathrm{Im}\,\lambda = 0$.



The reason is that when $\alpha < 0$, the vacuum is unstable to the production of electron-positron pairs by tunneling (through an energy barrier whose height is $\sim 2m_e c^2$), since electrons & positrons now repel each other; $\Gamma(\alpha)$ is a tunneling decay rate.

I leave it as an exercise for you to see that this means that if we expand $E_0(\alpha)$ as a power series in α for $\alpha > 0$, we will get a result which looks as follows:



ie., the series will begin by converging to some value, but it will never properly converge - when $n \sim 1/\alpha$, the remainder given by

$$R_n = S_n - E_0(\alpha) \qquad (2)$$

where

$$S_n = \sum_{r=0}^{n} c_n \alpha^n \qquad (3)$$

will start to oscillate more & more wildly away from the correct value.

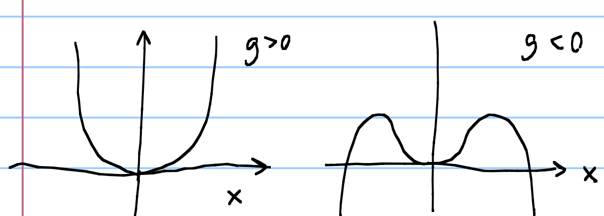
To see how this works one often studies the ϕ^4 theory, which has the Lagrangian

$$L = (\partial_\mu^2 - m^2)\phi^2 + g/_4 \phi^4 \qquad (4)$$

To understand this theory one can study the simpler 1-d problem of a particle in a quartic potential, with the Lagrangian

$$\left.\begin{aligned} L &= \tfrac{1}{2} m\dot{x}^2 - V(x) \\ V(x) &= \tfrac{1}{2}x^2 + \tfrac{g}{4}x^4 \end{aligned}\right\} \qquad (5)$$

It is obvious that this system has the same kind of instability as QED when $g < 0$, just by looking at the potential. Thus one expects all of the important quantities, like $E_0(g)$, or indeed $E_\ell(g)$ (the eigenvalues)



to show branch cuts along the -ve g-axis, with decay rates $\Gamma_\ell(g)$ as with QED. In calculating all of these, as well as the partition function

$$Z(g) = \int dx\, e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4} \qquad (6)$$

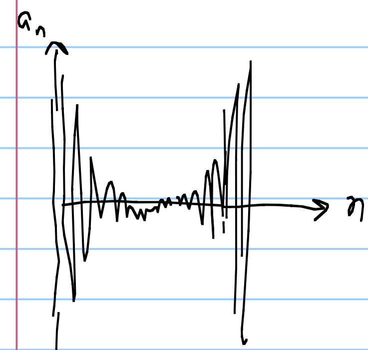
we have to evaluate an integral which goes divergent when $g<0$. Now suppose we expand $Z(g)$ as a power series - we get

$$Z(g) = \sum_{n=0}^{\infty} c_n g^n = \sum_n \frac{1}{n!}\left(\frac{-g}{4}\right)^n \int dx\, x^{4n} e^{-x^2/2}$$

$$\left(-\frac{g}{4}\right)^n \frac{(2\pi)^{1/2}}{4^n}(4n)!/n!(2n)! \qquad (7)$$

and for large n we have coefficients

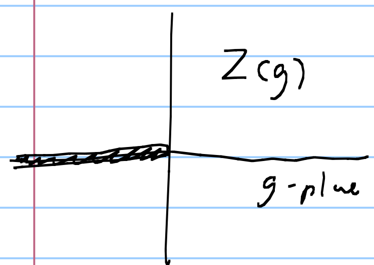
$$c_n \sim (-1)^n \pi^{1/2} \frac{4^n}{n} n! \qquad (n \gg 1) \tag{8}$$



and again the size of the $|a_n|$, as well as the remainder R_n, at first diminishes and then diverges, with

$$\min |R_n| \sim 2g^{1/2} e^{-1/4g} \tag{9}$$

for $n \sim g/4$, a sure sign of tunneling.



Thus we get the same analytic structure as before - the magnitude $\Gamma(g)$ of the branch cut in $Z(g)$ is given from WKB. The same is true for the ground state energy, given by

$$E_0(g) = \lim_{\beta \to \infty} \frac{1}{\beta} \int_{x(-\beta/2)}^{x(\beta/2)} \mathcal{D}q(\tau)\, e^{-\int_{-\beta/2}^{\beta/2} d\tau \left[\frac{m}{2}\dot{x}^2 - \frac{x^2}{2} - \frac{g}{4}x^4\right]} \tag{10}$$

and for all the higher energies, and for the Green fns.

NB: This might be a useful moment for you to look again at my course notes on (a) the perturbation expansion for a spin in a tunneling potential, and (b) instanton methods for path integrals for tunneling problems, including spin tunneling

B2 SUMMATION of SOME DIVERGENT SERIES

: Again, this is a huge topic, which is covered in treatises [4]; the following is just an introduction.

Suppose we have a divergent series of the form

$$S = \sum_{n=0}^{\infty} a_n \tag{11}$$

The remarkable thing is that such a series may have a well-defined sum — it is just not obvious from the structure of the series.

EXAMPLES Here are 5 examples, each of which has interesting features

$$\left.\begin{aligned}
&\text{(i) The series } S_1 = \sum_{n=0}^{\infty} (-1)^n \\
&\text{(ii) The series } S_2 = \sum_n n \\
&\text{(iii) The series } S_3 = \sum_{n=0}^{\infty} (-1)^n n! \\
&\qquad \text{and also } S_4 = \sum_n (-1)^n (n!)^2 \\
&\text{(iv) The series } S_5 = 1+0+0+-1+0+1+0+0+-1+0+\dots
\end{aligned}\right\} \quad (12)$$

Note the similarity of the series S_2 to the one studied for the quartic QM problem (cf eqtn (8)).

Let's quickly look at a few techniques to attack these.

(a) Euler summation : This is simple and intuitively obvious. We define the function

$$f(z) = \sum_n a_n z^n \qquad (13)$$

and then if the series S in (11) is only algebraically divergent (ie, as a power law in n), then function $f(z)$ is convergent for $|z| < 1$, We then define

$$S = \lim_{x \to 1-\epsilon} f(z) \qquad (14)$$

In this way we can sum S_1 ; we have

$$S_1 = \lim_{x \to 1-\epsilon} \frac{1}{1+x} = 1/2 \qquad (15)$$

However we can't sum any of the others in any simple way, as you can quickly verify.

(b) Borel summation : This starts from a well-known theorem of Borel, which I will not prove, which states that if we define a function

$$\phi(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \qquad (16)$$

then we can define

$$S = \lim_{x \to 1-\epsilon} B(x) \qquad (17)$$

where

$$B(x) = \int_0^{\infty} dt\, e^{-t} \phi(xt) \qquad (18)$$

is the Borel transform. It is easy enough to see that (17) reproduces the series S in (11); the question of convergence is more complicated. Often this technique works when $|a_n|$ is diverging very fast, eg like a factorial of n.

Let's look, eg., at the series S_3 and S_4 in (12). We see that for the series S_3,

$$\phi_3(z) = \sum_n (-1)^n z^n = \frac{1}{1+z} \qquad (19)$$

$$\left.\begin{aligned} &\text{and so} \quad B(x) = \int_0^{\infty} dt\, e^{-t}/(1+xt) \\ &\text{and} \quad S = \int_0^{\infty} dt\, e^{-t}/1+t \end{aligned}\right\} \qquad (20)$$

and we note that a series expansion of S in (20) just reproduces the series. Note we could have done the same thing with the series given by

$$S_3' = \sum_n (-1)^n (2n)!$$

using the series expansion $\int_0^{\infty} dt \frac{e^{-xt}}{1+t^2} = \frac{1}{x}\left[1 - \frac{2!}{x^2} + \frac{4!}{x^4} + \cdots\right]$ (21)

One can also define a more general Borel summation by now defining

$$\phi_m(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n!)^m} z^m \qquad (22)$$

and a generalized Borel transform

$$B_m(z) = \int_0^{\infty} dt_m \int_0^{\infty} dt_{m-1} \cdots \int dt_1\, e^{-\sum_{j=1}^{m} t_j} \phi\left(x \prod_{j=1}^{m} t_j\right) \qquad (23)$$

As an exercise you can see that the sum S_4 in (12) converges, using this technique for $m = 2$, to the value

$$S_4 = \sum_n (-1)^n (n!)^2 = \int_0^\infty dt_1 \int_0^\infty dt_2 \; e^{-(t_1+t_2)} \frac{1}{1+t_1 t_2} \qquad (24)$$

I leave the application of all of this to field theory to later - but go to refs [2,3].

(c) Cesàro, Riesz, & Abel Summation: These are useful devices, which are in some ways rather intuitively obvious. The most elementary is the Cesàro sum, which simply averages over terms; in other words, one writes the partial sums S_n as in eqtn (2), and then defines the Cesàro sum as the limit of the "Cesàro mean" $S_c^{(N)}$:

$$S = \lim_{N\to\infty} S_c^{(N)} \; ; \qquad S_c^{(N)} = \frac{1}{N} \sum_{n=0}^{N} S_n \qquad (25)$$

This is very intuitive - it simply weights evenly all partial sums. Thus, e.g., the series S_1 in (12) has partial sums as follows:

$$S = \sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+1-1+\ldots$$

so that

$$\left.\begin{array}{ll} S_c^{(1)} = 1 & S_c^{(2)} = \tfrac{1}{2} \\ S_c^{(3)} = 2/3 & S_c^{(4)} = 1/2 \\ S_c^{(5)} = 3/5 & S_c^{(6)} = 1/2 \end{array}\right\} \qquad (26)$$

etc.

The Riesz summation is an extension of this with a different weighting function. One defines the "Riesz mean" as

$$\left.\begin{array}{l} S_R^{\delta}(\lambda) = \sum_{n=0}^{\lambda} \left(1 - n/\lambda\right)^{\delta} S_n \\ \text{and} \quad S = \lim_{\lambda\to\infty} S_R^{\delta}(\lambda) \end{array}\right\} \qquad (27)$$

I shall return to the Cesàro-Riesz summation later on. There is another nice generalization of Cesàro summation known as Abel summation, in which one defines the function

$$A_s(z) = \sum_{n=0}^{\infty} a_n e^{-nx} \equiv \sum_n a_n z^n \qquad (28)$$

where we define $z = e^{-x}$ (29)

Then we have

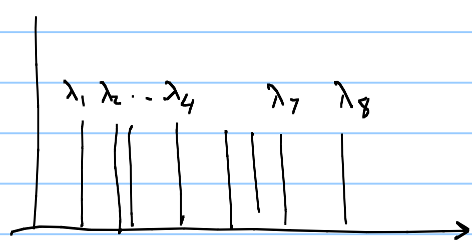
$$S = \lim_{z \to 1-\epsilon} A_s(z) \tag{30}$$

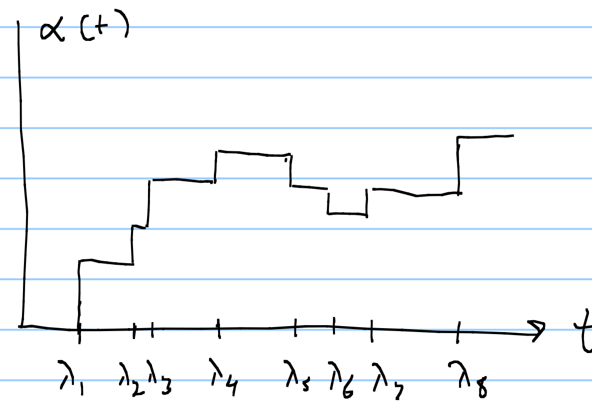
The Abel summation in this form is a special case of a more general summation technique which in mathematics is known as a generalised Dirichlet summation, & in physics as "heat-kernel regularisation"; one writes

$$\left.\begin{aligned} f(z) &= \sum_n a_n e^{-\lambda_n z} \\ \text{where} \quad & 0 < \lambda_1 < \lambda_2 < \cdots \lambda_n \end{aligned}\right\} \tag{31}$$

which is sometimes written in Laplace transformation form as

$$f(z) = \int_0^\infty d\alpha(t)\, e^{-zt} \tag{32}$$





$$\text{where} \quad \alpha(t) = \sum_{j=1}^{\infty} a_j \,\theta(t - \lambda_j) \tag{37}$$

ie, $\alpha(t)$ jumps an amount a_j when $t = \lambda_j$. In any case we have the generalised Dirichlet sum.

$$S = \lim_{z \to 0^+} f(z) \tag{38}$$

The Abel sum in (30) is produced from (38) by assuming that $\lambda_n = n$ in (31).

The Dirichlet sum is called "heat-kernel regularisation" in physics because it is often done for field theories continued to imaginary time, in which the λ_n are eigenvalues of the imaginary time Schrodinger operator (ie, the heat equation operator).

The Cesàro & Riesz summation methods are less powerful than Dirichlet / Abel summation.

This concludes the introductory discussion of series summation. To go further we need to look in more detail at the asymptotic and analytic properties of the functions that one uses to generate these series. To do this we need to look at how to relate the properties of series to asymptotic expansions of functions, usually in integral form, and to study the pole structure of these functions.

To complete this theoretical discussion it is also important to work through some examples. This is the topic of the 2nd part of this supplement (ie., MS2 (6)).